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Note on the denominators of harmonic numbers

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Abstract

Let v_n be the denominator of the n-th harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Recently, Shiu proved that there exist infinite positive integers n satisfying each of the following: (1) $v_n < v_{n+1}$; (2) $v_n = v_{n+1}$; (3) $v_n > v_{n+1}$. In this note, we extend Shiu's results and we prove the following: (1) for any $\epsilon > 0$, there exist infinite n such that $v_n < \epsilon v_{n+1}$; (2) for any $M > 0$, there exist infinite n such that $v_n > M v_{n+1}$. (3) there exists infinite n such that $v_n = v_{n+1} = v_{n+2}$.

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1 Introduction

For any positive integer n, the n-th harmonic number H_n is defined as the following

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{u_n}{v_n}, \quad (u_n, v_n) = 1,\ v_n > 0.$$

There is an old famous result about the n-th harmonic number. That is, Wolstenholme [4] (see also [1, Theorem 115]) proved that u_{p-1} can be divided by p^2 for any prime

$p \geq 5$. It is easy to see that $2 \nmid u_n$ and $2 \mid v_n$ for all $n \geq 2$, and so H_n is not an integer (see [3]).

In 2016, Shiu [2] proved that there exists infinitely many integers n satisfying each of the following: (1) $v_n < v_{n+1}$; (2) $v_n = v_{n+1}$; (3) $v_n > v_{n+1}$.

In this paper, we extend Shiu's results.

Theorem 1. *For any $\epsilon > 0$, there exist infinite positive integers n such that $v_{n+1} < \epsilon v_n$.*

Theorem 2. *For any $M > 0$, there exist infinite positive integers n such that $v_{n+1} \geq M v_n$.*

Theorem 3. *The set of positive integers n with $v_n = v_{n+1} = v_{n+2}$ has density one.*

Corollary 1. *There are infinitely many integers n such that $v_n = v_{n+1} = v_{n+2}$.*

Finally, motivated by Corollary 1, we posed the following conjecture.

Conjecture 1. *(i) There are infinitely many integers n such that $v_n < v_{n+1} < v_{n+2}$.*

(ii) There are infinitely many integers n such that $v_n > v_{n+1} > v_{n+2}$.

2 Lemmas

Lemma 1. *([1, Theorem 115]) If p is a prime greater than 3, then the numerator of the fraction*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is divisible by p^2.

For a positive integer n, we define $\nu_p(n)$ to be the integer k with $p^k \mid n$ and $p^{k+1} \nmid n$. For a rational number $\frac{a}{b}$, we define $\nu_p(\frac{a}{b}) = \nu_p(a) - \nu_p(b)$.

Lemma 2. *For a prime p, if a, b are two rational numbers with $\nu_p(a) \neq \nu_p(b)$, then*

$$\nu_p(a+b) = \min\{\nu_p(a), \nu_p(b)\}.$$

The proof of Lemma 2 is easy, we leave it to the reader.

Lemma 3. *If p is an odd prime, then the numerator of the fraction*

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-2}$$

is not divisible by p.

Lemma 3 follows from Lemmas 1 and 2, and the equality

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-2}=\frac{u_{p-1}}{v_{p-1}}-\frac{1}{p-1}$$

immediately.

Lemma 4. *([5, Theorem 1.1]) The set of positive integers n with $v_n = v_{n+1}$ has density one.*

3 Proofs

Proof of Theorem 1. Take a prme $p > \frac{1}{\epsilon}$ and let $n = (p-1)p^k - 1$. Then

$$\frac{u_n}{v_n}=1+\frac{1}{2}+\cdots+\frac{1}{(p-1)p^k-1}.$$

It is clear that

$$\frac{u_n}{v_n}=\sum_{\substack{j=1\\ p^k\nmid j}}^{(p-1)p^k-1}\frac{1}{j}+\left(\frac{1}{p^k}+\frac{1}{2p^k}+\cdots+\frac{1}{(p-2)p^k}\right).$$

By Lemma 3, we have

$$v_p\left(\frac{1}{p^k}+\frac{1}{2p^k}+\cdots+\frac{1}{(p-2)p^k}\right)=v_p\left(\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-2}\right)\cdot\frac{1}{p^k}\right)=-k.$$

Noting that

$$v_p\left(\sum_{j=1,\ p^k\nmid j}^{(p-1)p^k-1}\frac{1}{j}\right)\geq-(k-1),$$

by Lemma 2, we have $v_p\left(\frac{u_n}{v_n}\right) = -k$.

Next, we consider $v_p(\frac{u_{n+1}}{v_{n+1}})$. Since

$$\begin{aligned}\frac{u_{n+1}}{v_{n+1}} &= 1 + \frac{1}{2} + \cdots + \frac{1}{(p-1)p^k - 1} + \frac{1}{(p-1)p^k} \\ &= \frac{1}{p^k} \cdot \sum_{j=1}^{p-1} \frac{1}{j} + \frac{1}{p^{k-1}} \sum_{j=1}^{p-1} \sum_{s=1}^{p-1} \frac{1}{(j-1)p+s} + \sum_{j=1,\ p^{k-1} \nmid j}^{(p-1)p^k} \frac{1}{j}.\end{aligned}$$

By Lemma 1, it follows that $v_p\left(\frac{1}{p^k} \cdot \sum_{j=1}^{p-1} \frac{1}{j}\right) \geq -(k-2)$. Noting that

$$p \mid \sum_{s=1}^{p-1} \frac{1}{(j-1)p+s}$$

for any $j = 1, 2, \ldots, p-1$, we have

$$v_p\left(\frac{1}{p^{k-1}} \sum_{j=1}^{p-1} \sum_{s=1}^{p-1} \frac{1}{(j-1)p+s}\right) \geq -(k-2).$$

It is clear that

$$v_p\left(\sum_{j=1,\ p^{k-1} \nmid j}^{(p-1)p^k} \frac{1}{j}\right) \geq -(k-2).$$

Therefore, $v_p(\frac{u_{n+1}}{v_{n+1}}) \geq -(k-2)$. Since $\frac{u_n}{v_n} + \frac{1}{(p-1)p^k} = \frac{u_{n+1}}{v_{n+1}}$, it follows that $v_{n+1} \leq \frac{v_n}{p^2} \cdot (p-1) < \frac{v_n}{p} < \epsilon v_n$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Take a prime p with $p > M$ and let $n = p-1$. Then $\frac{u_{n+1}}{v_{n+1}} = \frac{u_n}{v_n} + \frac{1}{p} = \frac{u_n p + v_n}{p v_n}$. Since $(p, u_n p + v_n) = (p, v_n) = 1$ and $(v_n, u_n p + v_n) = (v_n, u_n p) = (v_n, p) = 1$, it follows that $v_{n+1} = p v_n > M v_n$.

This completes the proof of Theorem 2. □

Proof of Theorem 3. By Lemma 4, the set of positive integers n with $v_n = v_{n+1}$ has density one. Hence the set of positive integers n with $v_n < v_{n+1}$ (or $v_n > v_{n+1}$) has density zero. Noting that

$$\{n : \ v_n = v_{n+1} < v_{n+2}\} \subseteq \{n : \ v_{n+1} < v_{n+2}\}$$

and

$$\{n: v_n = v_{n+1} > v_{n+2}\} \subseteq \{n: v_{n+1} > v_{n+2}\},$$

we have

$$\{n: v_n = v_{n+1} < v_{n+2} \text{ or } v_n = v_{n+1} > v_{n+2}\}$$

has density zero.

Therefore, the set of positive integers n with $v_n = v_{n+1} = v_{n+2}$ has density one.

This completes the proof of Theorem 3. □

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