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Lower Bound of Bernoulli Percolation in the Critical Phase

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Abstract

In this article, we focus on *Bernoulli percolation* and mainly investigate bounds of the probability of the connectivity of 0 to the distance n. At first, we give a rough bound of the probability, and then refine our result by the high-dimensional RSW theory, which gives a nontrivial bound for a short crossing in $\mathbb{Z}^d$, as well as the renormalization method. We finish the last step of this section by coupling. Next, we give a more refined bound in $\mathbb{Z}^2$ using the dual graph. At last, we investigate the behaviors of some subgraphs of $\mathbb{Z}^2$.

Key words- Bernoulli percolation; high-dimensional RSW theory; renormalization; coupling; dual graph; subgraphs of $\mathbb{Z}^2$

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1 What's Percolation

Intuitively, percolation is a simplistic probabilistic model for a porous material. The inside of the material is described as a random maze where water can flow. The question is to investigate which part of the material will be wet when immersed in water. Mathematically, the material is modeled as a graph $\mathbb{G}$ with countable vertex-set $\mathbb{V}$ and edge-set $\mathbb{E}$ (a subset of unordered pairs of elements in $\mathbb{V}$).[5]

2 Basic Definitions

A graph $\mathbb{G}$ is made of a set $\mathbb{V}$ whose elements are called **vertices**, and a set $\mathbb{E}$ whose elements are called **edges**. In the percolation model, $\mathbb{V} := \mathbb{Z}^d$ and $\mathbb{E} := \{xy : x, y \in \mathbb{V}, \|x - y\| = 1\}$. Also, when we limit the infinite graph, we have $\Lambda_n := [-n, n]^d$ for every $n \geq 1$. We use $\Lambda_n(x)$ to represent the area of the shape Λ_n with the center point x rather than 0.

For a subgraph $G = (V, E)$ of $\mathbb{Z}^d$, the vertex boundary of G is defined by $\partial G := \{x \in V : \exists y \in \mathbb{Z}^d \text{ such that } \|x - y\| = 1 \text{ and } y \notin V\}$, and the edge boundary is defined by $\triangle G := \{xy : x \in V, y \notin V\}$

A **percolation configuration** $\omega = (\omega(e) : e \in E)$ on $G = (V, E)$ is an element of $\{(\omega(e_1), ..., \omega(e_E)) : \omega(e_i) \in \{0, 1\}\}$. When $\omega(e) = 1$, we say that the edge e is open. When $\omega(e) = 0$, the edge e is closed.

A **percolation model** is given by a distribution on percolation configurations on a graph. The simplest example of percolation model is **Bernoulli percolation**: each edge is open with probability p, and closed with probability $1 - p$, independently of the states of other edges.

We usually define Bernoulli percolation on the infinite lattice $\mathbb{Z}^d$. Therefore, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, where Ω is all the possible percolation configurations on $\mathbb{Z}^d$. $\mathcal{F}$ is the $\sigma-$algebra generated by events depending on finitely many edges, and $\mathbb{P}_p$ is the corresponding probability measure.

Define a path ζ from A to B as a chain of vertices $\zeta = (v_1, v_2, ..., v_n)$ where $v_1 = A$, $v_n = B$, $\|v_{i+1} - v_i\| = 1$ and $\omega(v_i v_{i+1}) = 1$. Define the vertices and edges of a path as $V(\zeta) = \{v : \exists 1 \leq i \leq n, v = v_i\}$ and $E(\zeta) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\}$

If there is a path from A to B, we say that A is connected with B, denoted by $A \longleftrightarrow B$. Especially, when $\{V(\zeta)\} \subseteq S$, we can denote it by $A \overset{S}{\longleftrightarrow} B$. If A and B are sets, $A \longleftrightarrow B$ means that $\exists a \in A, b \in B$ such that $a \longleftrightarrow b$.

A connected component of 0 is defined by $\mathcal{C}_0 := \{x \in \mathbb{V} : x \longleftrightarrow 0\}$.

We denote probability of the event $\{0\} \longleftrightarrow \partial\Lambda_n$ as $\theta_n(p)$, where p is the parameter of Bernoulli random variables. Besides, we define $\theta(p) := \mathbb{P}_p(0 \longleftrightarrow \infty) = \lim_{n\to\infty} \theta_n(p)$. From definition, we can know that $\theta(0) = 0$ and $\theta(1) = 1$, but the behavior of the function $\theta(p)$ is not apparent. We define the transition point $p_c := \inf\{p \in [0, 1] : \theta(p) > 0\}$. We call the situation where $p = p_c$ as the critical phase, $p < p_c$ as the subcritical phase, and $p > p_c$ as the super critical phase.

A configuration ω is **smaller** than another configuration ω' if and only if for any $e \in \mathbb{E}$, $\omega(e) \leq \omega'(e)$. An event $\mathcal{A}$ is **increasing** if and only if for any $\omega \in \mathcal{A}$, if

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$\omega' \geq \omega, \omega' \in \mathcal{A}$.

A set $I_{\mathcal{A}} \subset \mathbb{E}$ is a **witness** of an event $\mathcal{A}$ in a configuration ω if and only if $\omega \in \mathcal{A}$ and for any ω' that satisfies $\omega'(e) = \omega(e)$ where $e \in I$, then $\omega' \in \mathcal{A}$. For two events $\mathcal{A}$ and $\mathcal{B}$, if there exist $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ such that $I_{\mathcal{A}} \cap I_{\mathcal{B}} = \varnothing$, we call the situation as **disjoint occurrence** and denote it by $\mathcal{A} \circ \mathcal{B}$.

When we say that there is a horizontal crossing in a rectangle like $[0, a] \times [0, b]$, we mean that $\{0\} \times [0, b] \longleftrightarrow \{a\} \times [0, b]$. And we denote the event by $\mathcal{H}(a, b)$. If $[0, a] \times \{0\} \longleftrightarrow [0, a] \times \{b\}$, we say that there is a vertical crossing and denote it by $\mathcal{V}(a, b)$. More generally, we use $\mathcal{H}(S)$ and $\mathcal{V}(S)$ to denote the event of a horizontal or vertical crossing in S.

A dual graph $(\mathbb{Z}^2)^*$ is obtained by translating the copy of the original graph $\mathbb{Z}^2$ by the vector $(\frac{1}{2}, \frac{1}{2})$. Every edge in the dual graph is open when its corresponding edge in the original graph is closed. If the original configuration has a Bernoulli parameter p, then the dual configuration has a Bernoulli parameter $1 - p$.

Define the function $\mathbb{1}_{\mathcal{A}} : \omega \to \{0, 1\}$, where $\mathcal{A}$ is an event, as

$$\mathbb{1}_{\mathcal{A}} = \begin{cases} 0 & \omega \notin \mathcal{A} \\ 1 & \omega \in \mathcal{A} \end{cases} \tag{1}$$

3 Notations

Below are some notations we will use.

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Notations	Description
η	k-independent percolation
$\mathcal{A}_n$	lattice animals with size n
$\mathcal{C}_0$	connected component of 0
Λ_n	$[-n,n]^d$
∂S	vertex boundary of S
$\mathcal{H}(S)$	horizontal crossing in S
$\mathcal{V}(S)$	vertical crossing in S
$V(\zeta)$	vertex set of the path ζ
$E(\zeta)$	edge set of the path ζ
$\varphi(p)$	correlation length function
$\varphi_p(S)$	expectation of going out of S
$c_i(v)$	$i-th$ value function for a $n-tuple$
$C_i(v_a, v_b)$	an event about v_a, v_b in the $i-th$ direction
$\mathcal{S}_i(n, kn)$	short crossing in the $i-th$ direction

Table 1: Notations Table

4 Tools

4.1 Harris-FKG Inequality

The Harris-FKG(Fortuin, Kasteleyn, Ginibre) inequality is used to bound the probability of the intersection of two events that aren't independent.
For two increasing events $\mathcal{A}$ and $\mathcal{B}$, we have:

$$\mathbb{P}_p[\mathcal{A} \cap \mathcal{B}] \geq \mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}] \tag{2}$$

More generally, if f and g are two bounded increasing functions:

$$\mathbb{E}_p[fg] \geq \mathbb{E}_p[f]\mathbb{E}_p[g] \tag{3}$$

Notice that the FKG inequality implies that
For an increasing event $\mathcal{A}$ and a decreasing event $\mathcal{B}$:

$$\mathbb{P}_p[\mathcal{A} \cap \mathcal{B}] \leq \mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}] \tag{4}$$

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4.2 BK inequality

The BK(Berg Kesten) inequality generalizes the situation when two events $\mathcal{A}$ and $\mathcal{B}$ depend on some deterministic sets of edges.
For two increasing events $\mathcal{A}$ and $\mathcal{B}$ depending on finitely many edges , we have:

$$\mathbb{P}_p[\mathcal{A} \circ \mathcal{B}] \leq \mathbb{P}_p[\mathcal{A}]\mathbb{P}_p[\mathcal{B}] \tag{5}$$

In this article, the BK inequality is typically used to illustrate

$$\mathbb{P}_p[\{0 \longleftrightarrow A\} \circ \{0 \longleftrightarrow B\}] \leq \mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n]^2 \tag{6}$$

where $A, B \in \partial\Lambda_n$.

4.3 RSW theory

The RSW(Russo-Seymour-Welsh) theory applies to the cases when $d = 2$ and $p = p_c$. It gives a nontrivial bound for every rectangle crossing.
For two positive numbers α and β, we have:

$$c < \mathbb{P}_p[\mathcal{H}(\alpha n, \beta n)] < 1 - c \tag{7}$$

where $c > 0$ and n is any positive integer.

4.4 Exponential decay in diameter

The theorem, first presented by Aizenman and Barsky and Menshikov, is used to predict the behavior of $\theta_n(p)$ when $p < p_c$.
Fix $d \geq 2$. When $p < p_c$, there exists $c > 0$ only depending on d such that:

$$\theta_n(p) \leq \exp(-cn) \tag{8}$$

where n is any positive integer.

5 $\theta_\mathbf{n}(\mathbf{p_c}) \geq \frac{\mathbf{c}}{\mathbf{n^{d-1}}}$

This lower bound was studied by Grimmitte [7] using the correlation length function $\varphi(p)$ (11). Here we give a slightly different treatment. and show that the continuity of the function $\varphi(p)$ implies the lower bound in this section.

Denote the point $(1, 0, ..., 0)$ as e_1 and $(n, 0, ..., 0)$ as ne_1. We want to first introduce the Fekete lemma:

Lemma 1 *Let $\{a_n\}$ be a sequence of real numbers. If it satisfies $\forall$ m,n≥ 0, $a_{m+n} \leq a_m + a_n$, then the limit of $\{\frac{a_n}{n}\}$ exists and $\lim\limits_{n\to\infty}\{\frac{a_n}{n}\} = \inf\limits_{n>0}\{\frac{a_n}{n}\}$.*

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Notice that

$$\begin{aligned}\mathbb{P}_p[0 \longleftrightarrow (m+n)e_1] &\geq \mathbb{P}_p[0 \longleftrightarrow me_1]\mathbb{P}_p[me_1 \longleftrightarrow (m+n)e_1] \\ &\geq \mathbb{P}_p[0 \longleftrightarrow me_1]\mathbb{P}_p[0 \longleftrightarrow ne_1]\end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned}-\log \mathbb{P}_p[0 \longleftrightarrow (m+n)e_1] &\leq -\log \mathbb{P}_p[0 \longleftrightarrow me_1]\mathbb{P}_p[0 \longleftrightarrow ne_1] \\ &\leq -\log \mathbb{P}_p[0 \longleftrightarrow me_1] - \log \mathbb{P}_p[0 \longleftrightarrow ne_1]\end{aligned} \tag{10}$$

Let u_n be $-\log \mathbb{P}_p[0 \longleftrightarrow ne_1]$. Then u_n satisfies $u_{m+n} \leq u_m + u_n$. As a result, the limit of $\{\frac{u_n}{n}\}$ exists. We denote it by

$$\varphi(p) = \lim_{n\to\infty}\{\frac{u_n}{n}\} \tag{11}$$

Because $\varphi(p) = \lim\limits_{n\to\infty}\{\frac{u_n}{n}\} = \inf\limits_{n>0}\{\frac{u_n}{n}\}$, we have:

$$\begin{aligned}&\inf_{n>0}\{\frac{-\log \mathbb{P}_p[0 \longleftrightarrow ne_1])}{n}\} = \varphi(p) \\ &\frac{-\log \mathbb{P}_p[0 \longleftrightarrow ne_1])}{n} \geq \varphi(p) \\ &\mathbb{P}_p[0 \longleftrightarrow ne_1] \leq \exp(-n\varphi(p))\end{aligned} \tag{12}$$

Our goal is to show that

$$\frac{1}{cn^{d-1}}e^{-n\varphi(p)} \leq \theta_n(p) \leq cn^{d-1}e^{-n\varphi(p)} \tag{13}$$

If the inequality holds, then when $p > p_c$, $\varphi(p) = 0$. Consequently, we can use the continuity of the function $\varphi(p)$ to show $\varphi(p_c) = 0$ and $\theta_n(p_c) \geq \frac{1}{cn^{d-1}}$.

For the upper bound $\theta_n(p) \leq cn^{d-1}e^{-n\varphi(p)}$, we can first define a point x on $\partial\Lambda$ such that $\mathbb{P}_p[0 \longleftrightarrow x] = \max\limits_{y\in\partial\Lambda}\{\mathbb{P}_p[0 \longleftrightarrow y]\}$.

Notice that from symmetry,

$$\begin{aligned}&\mathbb{P}_p[0 \longleftrightarrow x]^2 \leq \mathbb{P}_p[0 \longleftrightarrow 2ne_1] \\ &\mathbb{P}_p[0 \longleftrightarrow x] \leq \mathbb{P}_p[0 \longleftrightarrow 2ne_1]^{\frac{1}{2}}\end{aligned} \tag{14}$$

and

$$\begin{aligned}\theta_n(p) = \mathbb{P}_p[0 \longleftarrow \partial\Lambda] &\leq |\partial\Lambda|\mathbb{P}_p[0 \longleftarrow x] \\ &\leq |\partial\Lambda|\mathbb{P}_p[0 \longleftrightarrow 2ne_1]^{\frac{1}{2}} \\ &\leq |\partial\Lambda|e^{-n\varphi(p)} \\ &\leq cn^{d-1}e^{-n\varphi(p)}\end{aligned} \tag{15}$$

Here we use the fact that $|\partial\Lambda| \leq cn^{d-1}$ for some constant c related to the dimension d.

Lemma 2

$$|\partial\Lambda| \leq 2d(2n+1)^{d-1} \tag{16}$$

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Proof:

Assume that $x \in \partial\Lambda$ in a d-dimensional graph. Then one of the coordinates of x should be n or $-n$. Arbitrarily set one coordinate $(c_1, c_2, ..., c_d)$ to be n or $-n$, and we will have $2d$ initial choices. There is no further restriction on other coordinates, so each of them can be ranged from $-n$ to n, which indicates $2n+1$ choices. There are totally $d-1$ of them.

Therefore, each point $x \in \partial\Lambda$ will belong to one of the point generated by the above method. This implies that $|\partial\Lambda| \leq 2d(2n+1)^{d-1}$.

We will then prove that

$$\theta_n(p) \geq \frac{1}{cn^{d-1}} e^{-n\varphi(p)} \tag{17}$$

Notice that

$$\begin{aligned} \theta_{m+n}(p) &\leq \theta_m(p)\mathbb{P}_p[\Lambda_m \longleftrightarrow \Lambda_{m+n}] \\ &\leq |\Lambda_m|\theta_m(p)\mathbb{P}_p[x \longleftrightarrow \Lambda_{m+n}] \\ &\leq |\Lambda_m|\theta_m(p)\theta_n(p) \\ &\leq 2d(2m+1)^{d-1}\theta_m(p)\theta_n(p) \\ &\leq 2d3^{d-1}m^{d-1}\theta_m(p)\theta_n(p) \end{aligned} \tag{18}$$

Here $\mathbb{P}_p[x \longleftrightarrow \Lambda_{m+n}] = \max\limits_{y \in \Lambda_m}\{\mathbb{P}_p[y \longleftrightarrow \Lambda_{m+n}]\}$.

Then we want to apply the Fekete's lemma again by constructing u_n such that $u_{m+n} \leq u_m + u_n$. As a result, we have:

$$2d6^{d-1}(m+n)^{d-1}\theta_{m+n}(p) \leq 2d6^{d-1}m^{d-1}\theta_m(p)2d3^{d-1}(m+n)^{d-1}\theta_n(p) \tag{19}$$

Let $a_n = 2d6^{d-1}n^{d-1}\theta_n(p)$, so we have:

$$a_{m+n} \leq a_m 2d3^{d-1}(m+n)^{d-1}\theta_n(p) \tag{20}$$

Since m and n are interchangeable, we can assume that $m \leq n$.

$$\begin{aligned} a_{m+n} &\leq a_m 2d3^{d-1}(m+n)^{d-1}\theta_n(p) \\ &\leq a_m 2d3^{d-1}2^{d-1}n^{d-1}\theta_n(p) \\ &\leq a_m a_n \end{aligned} \tag{21}$$

Therefore,

$$\log a_{m+n} \leq \log a_m + \log a_n \tag{22}$$

which means that the limit of the sequence $\{\frac{\log a_n}{n}\}$ exists and $\lim\limits_{n\to\infty}\{\frac{\log a_n}{n}\} = \inf\limits_{n>0}\{\frac{\log a_n}{n}\}$.

Suppose $\lim\limits_{n\to\infty}\{\frac{\log a_n}{n}\} = X$, then:

$$\inf_{n>0} \frac{\log 6^{d-1}n^{d-1}\theta_n(p)}{n} = X \tag{23}$$

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$$\begin{aligned} \theta_n(p) &\geq \frac{e^{nX}}{6^{d-1}n^{d-1}} \\ &\geq \frac{c}{n^{d-1}}e^{nX} \end{aligned} \tag{24}$$

Notice that

$$\mathbb{P}_p[0 \longleftrightarrow ne_1] \leq \theta_n(p) \leq cn^{d-1}e^{-n\varphi(p)} \tag{25}$$

and

$$\begin{aligned} \lim_{n\to\infty} \frac{\log 6^{d-1}n^{d-1}\mathbb{P}_p[0 \longleftrightarrow ne_1]}{n} &= \lim_{n\to\infty} \frac{\log 6^{d-1}n^{d-1} + \log \mathbb{P}_p[0 \longleftrightarrow ne_1]}{n} \\ &= \lim_{n\to\infty} \frac{\log \mathbb{P}_p[0 \longleftrightarrow ne_1]}{n} \\ &= -\varphi(p) \end{aligned} \tag{26}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{\log 6^{d-1}n^{d-1}cn^{d-1}e^{-n\varphi(p)}}{n} &= \lim_{n\to\infty} \frac{\log 6^{d-1}n^{d-1} + \log cn^{d-1}e^{-n\varphi(p)}}{n} \\ &= \lim_{n\to\infty} \frac{\log cn^{d-1}e^{-n\varphi(p)}}{n} \\ &= \lim_{n\to\infty} \frac{\log e^{-n\varphi(p)}}{n} \\ &= -\varphi(p) \end{aligned} \tag{27}$$

According to the Squeeze Theorem, $X = -\varphi(p)$.

Because $\varphi(p) = \inf_{n>0} \frac{-\log \mathbb{P}_p[0\longleftrightarrow ne_1]}{n} = \sup_{n>0} \frac{-\log cn^{d-1}\theta_n(p)}{n}$, $\varphi(p)$ is continuous according to the property of semi-continuity.

If $\varphi(p) > 0$ for some $p > p_c$, we will have:

$$\theta(p) = \lim_{n\to\infty} \theta_n(p) \leq \lim_{n\to\infty} cn^{d-1}e^{-n\varphi(p)} = 0 \tag{28}$$

From the definition of p_c, we know that this is a contradiction.

Therefore, when $p > p_c$, $\varphi(p) = 0$. Since $\varphi(p)$ is continuous, $\varphi(p_c) = 0$ and

$$\theta_n(p_c) \geq \frac{1}{c_1 n^{d-1}}e^{-n\varphi(p_c)} \geq \frac{c_2}{n^{d-1}} \tag{29}$$

6 $\theta_n(p_c) \geq \frac{c}{n^{\frac{d-1}{2}}}$

We first start from the case where $d = 2$.

According to the RSW theorem, $\mathbb{P}_p[\mathcal{H}(2n, 2n)] > c$ for some positive real number c. Then there exists some $x \in \{n\} \times [0, 2n]$ such that $\mathbb{P}_{p_c}$[horizontal crossing passes

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through $x] \geq \frac{c}{2n+1}$. Define $\mathcal{A}$ as the event $\{x \longleftrightarrow \{0\} \times [0, 2n]\}$, and $\mathcal{B}$ as the event $\{x \longleftrightarrow \{2n\} \times [0, 2n]\}$. Then, we have:

$$\mathbb{P}_{p_c}[\mathcal{A} \circ \mathcal{B}] > \frac{1}{2n+1}\mathbb{P}_p[\mathcal{H}(2n, 2n)] > \frac{c}{n} \tag{30}$$

According to the BK inequality,

$$\mathbb{P}_{p_c}[\mathcal{A}]\mathbb{P}_{p_c}[\mathcal{B}] \geq \mathbb{P}_{p_c}[\mathcal{A} \circ \mathcal{B}] \tag{31}$$

Since $\mathcal{A}, \mathcal{B} \in \{x \longleftrightarrow \partial\Lambda_n\}$,

$$\mathbb{P}_{p_c}[x \longleftrightarrow \partial\Lambda_n]^2 \geq \mathbb{P}_{p_c}[\mathcal{A}]\mathbb{P}_{p_c}[\mathcal{B}] > \frac{c}{n} \tag{32}$$

According to the invariance property of the proof, $\forall x \in \mathbb{Z}^2$, we have:

$$\theta_n(p_c) = \mathbb{P}_{p_c}[x \longleftrightarrow \partial\Lambda_n] \geq \frac{c}{\sqrt{n}} \tag{33}$$

When $d > 2$, we can apply similar reasoning for $x \in \{0\} \times [-n, n]^{d-1}$. Define $\mathcal{S}_i(n, kn)$ as the event $[0, kn]^{i-1} \times \{0\} \times [0, kn]^{d-i} \longleftrightarrow [0, kn]^{i-1} \times \{n\} \times [0, kn]^{d-i}$. When $\mathcal{S}_i(n, kn)$ happens, we say that there is a short crossing in the box $[0, kn]^{i-1} \times [0, n] \times [0, kn]^{d-i}$.

Notice that if we prove the inequality $\mathcal{S}_i(n, kn) > c$ always holds for some i, k, c when $p = p_c$, we can conclude that $\theta_{p_c}(n) \geq \frac{c}{n^{\frac{d-1}{2}}}$ by constructing a box $[-n, n] \times [-kn, kn]^{d-1}$.

We will then prove by contradiction that $\mathcal{S}_1(n, 2n) > c$ is always true when $p = p_c$.

In order to prove it, we need to know how to describe a path in detail. For a d-tuple $v = (v_1, ..., v_d)$, we define $c_i(v) = v_i$. For a path ζ in dimension d, we define $l_{min}(\zeta) = (a_1, ..., a_d)$ and $l_{max}(\zeta) = (b_1, ..., b_n)$ where $a_i = \min\limits_{v \in V(\zeta)}\{c_i(v)\}$ and $b_i = \max\limits_{v \in V(\zeta)}\{c_i(v)\}$.

We will introduce the Crossing Lemma which describes how we can have a short crossing:

Lemma 3 *For a box $\mathcal{B} = [0, 2n]^{i-1} \times [0, n] \times [0, 2n]^{d-i}$, the event $\mathcal{S}_i(n, 2n)$ happens if and only if there exists a path ζ such that $c_i(l_{min}(\zeta)) \leq 0$, $c_i(l_{max}(\zeta)) \geq n$, and $c_j(l_{min}(\zeta)), c_j(l_{max}(\zeta)) \in [0, 2n]$ for any $j \neq i$,*

Proof:

From the definition, we know that if $v_1 \longleftrightarrow v_2$ where $c_i(v_1) = 0$, $c_i(v_2) = n$, and $c_j(v_1), c_j(v_2) \in [0, 2n]$ for $j \neq i$, the event $\mathcal{S}_i(n, 2n)$ happens. Since the path ζ satisfies the condition $c_j(l_{min}(\zeta)), c_j(l_{max}(\zeta)) \in [0, 2n]$ for any $j \neq i$, for any vertex $v \in V(\zeta)$, $c_j(v) \in [0, 2n]$ where $j \neq i$. Therefore, if we prove that $\exists v_1, v_2 \in V(\zeta)$ such that $v_1 \longleftrightarrow v_2$, $c_i(v_1) = 0$, $c_i(v_2) = n$, we will complete the proof for the Crossing Lemma. Obviously, because $\|v_{i+1} - v_i\| = 1$ for any path ζ, the function

$$f_i(x) = \begin{cases} c_i(v_x) & x \in \mathbb{Z}, 1 \leq x \leq |V(\zeta)| \\ c_i(v_{\lfloor x \rfloor}) + \{x\}(c_i(v_{\lfloor x \rfloor + 1}) - c_i(v_{\lfloor x \rfloor})) & x \notin \mathbb{Z}, 1 \leq x \leq |V(\zeta)| \end{cases} \tag{34}$$

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is continuous where $\lfloor\rfloor$ is the least integer function and $\{\}$ is the decimal part function.

Assume that $v_a, v_b \in V(\zeta)$ where $b > a$, $c_i(v_a) \le 0$, and $c_i(v_b) \ge n$. According to the Intermediate Value Theorem, there exists $a \le d_1, d_2 \le b$ such that $f(d_1) = 0$ and $f(d_2) = n$. Since 0 and n are integers, d_1, d_2 are integers. Therefore, this ensures that $\exists v_{d_1}, v_{d_2}$ and $v_{d_1} \longleftrightarrow v_{d_2}$, $c_i(v_{d_1}) = 0$, $c_i(v_{d_2}) = n$. So far we have completed the proof for the lemma.

Using this lemma, we will show that

$$\mathbb{P}_p[\mathcal{S}_1(n, 4n)] \le c(d)\mathbb{P}_p[\mathcal{S}_1(n, 2n)] \tag{35}$$

where n is any integer and $c(d)$ is an integer only depending on the dimension d. Especially, we give an estimation for $c(d)$ that $c(d) = 3^{d-2}(4d-1)$.

When the event $\mathcal{S}_1(n, 4n)$ happens, there is a path $\zeta = (v_1, ..., v_k)$ such that $c_1(v_1) = 0$, $c_1(v_k) = n$, and $\forall v \in \zeta$ and $i > 1$, $c_i(v) \in [0, 4n]$.

Our goal is to show that the path will always result in a short crossing in one of the following boxes: $[0, n] \times ([0, 2n]^{d-1} + u)$, $[0, 2n] \times ([0, 2n]^f + h) \times ([0, n] + w) \times ([0, 2n]^{d-f-2} + l)$ where $u \in \{0, n, 2n\}^{d-1}$, $0 \le f \le d-2$, $h \in \{0, n, 2n\}^f$, $w \in \{0, n, 2n, 3n\}$, $l \in \{0, n, 2n\}^{d-f-2}$. The number of these boxes is just $3^{d-2}(4d-1)$.

If the path ζ doesn't result in any short crossing in those boxes, then for any subpath $\zeta' = (v_a, v_{a+1}, ..., v_b)$ where $1 \le a \le b \le k$, ζ' also doesn't result in any short crossing in those boxes.

Since ζ doesn't result in any short crossing in $[0, n] \times ([0, 2n]^{d-1} + u)$, there exists an integer $i_1 \ne 1$ such that $c_{i_1}(l_{min}(\zeta))$ and $c_{i_1}(l_{max}(\zeta))$ don't belong to one of $[0, 2n]$, $[n, 3n]$, $[2n, 4n]$ simultaneously. Denote the event $\{c_{i_1}(v_a)$ and $c_{i_1}(v_b)$ don't belong to one of $[0, 2n]$, $[n, 3n]$, $[2n, 4n]\}$ as $C_{i_1}(v_a, v_b)$

Therefore, there exist $v_a, v_b \in V(\zeta)$ where $b > a$ such that $C_{i_1}(v_a, v_b)$ happens. We cut a sub-path $\zeta^{'} = (v_a, ..., v_b)$ from the original path. Notice that ζ' doesn't result in any short crossing in $[0, 2n] \times ([0, 2n]^{i_1-1} + h) \times ([0, n] + w) \times ([0, 2n]^{d-i_1-2} + l)$. According to the Crossing Lemma, there must exist an integer $i_2 \ne i_1$ such that $c_{i_2}(l_{min}(\zeta^{'}))$ and $c_{i_2}(l_{max}(\zeta^{'}))$ don't belong to one of $[0, 2n]$, $[n, 3n]$, $[2n, 4n]$ simultaneously, which ensures that $C_{i_2}(v_c, v_d)$ happens where $a \le c < d \le b$.

So far, we have roughly established an algorithm to cut a sub-path. Notice that this algorithm can repeat itself for infinitely many steps, while the number of elements in any path is finite. If we prove that the algorithm cuts at least one element for every step, the path needs to have infinitely many elements, which is a contradiction. Therefore, we will be able to say that if $\mathcal{S}_1(n, 4n)$ happens, there will be a short crossing at least in one of $3^{d-2}(4d-1)$ boxes, whose probability is equivalent to $\mathcal{S}_1(n, 2n)$.

In the next part, we will show that for any sub-path $\zeta = (v_a, ..., v_b)$, we can easily find a vertex $v_{a'} \in V(\zeta)$ different from v_a such that the event $C_i(v_{a'}, v_b)$ happens.

Assume that we have already chosen a sub-path $\zeta = (v_a, ..., v_b)$ satisfying that $C_i(v_a, v_b)$ happens. If there exist v_c different from v_a and v_b which ensures $C_j(v_c, v_d)$ for some j, d, we can cut a sub-path $\zeta' = (v_c, ..., v_d)$ where $|V(\zeta')| < |V(\zeta)|$. If such v_c doesn't exist, the event $C_j(v_a, v_b)$ will happen for some $j \ne i$. However, since $\|v_{a+1} - v_a\| = 1$, one of $C_i(v_{a+1}, v_b)$ and $C_j(v_{a+1}, v_b)$ will happen, which indicates that such v_c exists and $v_c = v_{a+1}$.

Therefore, for every step of the algorithm, it will cut at least one element from ζ, so the algorithm will cut infinitely many elements, which is a contradiction.

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Now, we have shown that if $\mathcal{S}_1(n, 4n)$ happens, there will be a short crossing in one of $3^{d-2}(4d-1)$ boxes. Denote each event by $\mathcal{B}_i$ where $1 \leq i \leq 3^{d-2}(4d-1)$. Noticing that $\mathbb{P}_{p_c}[\mathcal{B}_i] = \mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)]$, we have

$$\begin{aligned} \mathbb{P}_{p_c}[\mathcal{S}_1(n, 4n)] &\leq \mathbb{P}_{p_c}[\bigcup_{i=1}^{3^{d-2}(4d-1)} \mathcal{B}_i] \\ &\leq 3^{d-2}(4d-1)\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)] \\ &\leq c(d)\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)] \end{aligned} \tag{36}$$

By using the BK Inequality, we get that

$$\begin{aligned} \mathbb{P}_{p_c}[\mathcal{S}_1(2n, 4n)] &\leq \mathbb{P}_{p_c}[\mathcal{S}_1(n, 4n)]^2 \\ &\leq c(d)^2\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)]^2 \end{aligned} \tag{37}$$

If $\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)] < \frac{1}{c(d)^2}$, this result will imply that

$$\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)] \leq \exp(-cn) \tag{38}$$

However, it will lead to this result

$$\begin{aligned} \theta_{p_c}(n) &\leq c_1 n^{d-1}\mathbb{P}_{p_c}[\{0\} \longleftrightarrow x] \\ &\leq c_1 n^{d-1}\mathbb{P}_{p_c}[\{0\} \longleftrightarrow \{n\} \times [0, 2n]^{d-1}] \\ &\leq c_1 n^{d-1}\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)] \\ &\leq c_1 n^{d-1}\exp(-c_2 n) \\ &\leq \exp(-c_3 n \ln n) \end{aligned} \tag{39}$$

This means that $\theta_{p_c}(n)$ is decaying exponentially fast, which is impossible because $\mathbb{E}_{p_c}[|\mathcal{C}_0|] = +\infty$(proven later in the section (8)) and if it happens

$$\begin{aligned} \mathbb{E}_{p_c}[|\mathcal{C}_0|] &= \sum_{n=1}^{\infty}\sum_{x\in\partial\Lambda_n}\mathbb{P}_{p_c}[0 \longleftrightarrow x] \\ &\leq \sum_{n=1}^{\infty}\sum_{x\in\partial\Lambda_n}\mathbb{P}_{p_c}[0 \longleftrightarrow \partial\Lambda_n] \\ &\leq \sum_{n=1}^{\infty} 2d(2n+1)^{d-1}\exp(-c_3 n \ln n) \\ &< +\infty \end{aligned} \tag{40}$$

Therefore, we have completed the proof that

$$\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)] \geq \frac{1}{c(d)^2} > c \tag{41}$$

By using this technique, we can extend our proof to any dimension. So, the inequality

$$\theta_{p_c}(n) \geq \frac{c}{n^{\frac{d-1}{2}}} \tag{42}$$

always holds for $d \geq 2$.

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7 Renormalization Method

We will use a method called renormalization for the proof of $\mathcal{S}_1(n, 2n) > c$.

The spirits of the renormalization is to amplify the effect of assumptions like $\forall\varepsilon > 0, \mathcal{S}_1(n, 2n) < \varepsilon$ by constructing a so called k-independent percolation on the original graph. Typically, the situation in the k-independent percolation will contradict some properties of the original graph, which leads to the disproof of the assumption chosen.

Assume that $\exists n_0$ such that $\mathbb{P}_{p_c}[\mathcal{S}_1(n_0, 2n_0)] < \varepsilon, \forall\varepsilon > 0$. If we have

$$\mathbb{P}_{p_c}[\partial\Lambda_{n_0} \longleftrightarrow \partial\Lambda_{3n_0}] \leq 2d\mathbb{P}_{p_c}[\mathcal{S}_1(n_0, 3n_0)] \tag{43}$$

then

$$\begin{aligned} \mathbb{P}_{p_c}[\partial\Lambda_{n_0} \longleftrightarrow \partial\Lambda_{3n_0}] &\leq 2d\mathbb{P}_{p_c}[\mathcal{S}_1(n_0, 3n_0)] \\ &\leq 2d\mathbb{P}_{p_c}[\mathcal{S}_1(n_0, 4n_0)] \\ &\leq 2dc(d)\mathbb{P}_{p_c}[\mathcal{S}_1(n_0, 2n_0)] \\ &< \varepsilon \end{aligned} \tag{44}$$

The third inequality is implied by the equation (35).

We will prove (43) by showing that the event $\{\partial\Lambda_{n_0} \longleftrightarrow \partial\Lambda_{3n_0}\}$ will result in a short crossing in one of the following boxes: $[-3n_0, 3n_0]^i \times ([-3n_0, n_0] + w) \times [-3n_0, 3n_0]^{d-i-1}$ where $0 \leq i \leq n_0 - 1$ and $w \in \{0, 4n_0\}$.

When $\{\partial\Lambda_{n_0} \longleftrightarrow \partial\Lambda_{3n_0}\}$ happens, we know there will be two points x, y such that $x \longleftrightarrow y$, $|c_{i_1}(x)| = n_0, |c_{i_2}(y)| = 3n_0$ for some i_1, i_2, and $\forall j \geq 1, |c_j(x)| \leq n_0, |c_j(y)| \leq 3n_0$. According to the Crossing Lemma, we will have a short crossing in $[-3n_0, 3n_0]^{i_2-1} \times [-3n_0, -n_0] \times [-3n_0, 3n_0]^{d-i_2}$ or $[-3n_0, 3n_0]^{i_2-1} \times [n_0, 3n_0] \times [-3n_0, 3n_0]^{d-i_2}$. So, $\mathbb{P}_{p_c}[\partial\Lambda_{n_0} \longleftrightarrow \partial\Lambda_{3n_0}] \leq 2d\mathbb{P}_{p_c}[\mathcal{S}_1(n_0, 3n_0)]$.

Now, we construct a new site percolation η on the original graph $\mathbb{Z}^d$ where $\eta(x) = \mathbb{1}_{\partial\Lambda_{n_0}(x)\longleftrightarrow\partial\Lambda_{3n_0}(x)}$ to determine whether the vertex x is open or closed. Notice that if $y \in \Lambda_{6n_0}(x) \setminus \partial\Lambda_{6n_0}(x)$, $\eta(x)$ and $\eta(y)$ aren't independent. Therefore, we call this site percolation as $6n_0$-percolation. If we only consider vertices that are independent to each other, the $6n_0$-percolation is like a Bernoulli percolation with the Bernoulli parameter $p' = \mathbb{P}_{p_c}[\partial\Lambda_{n_0}(x) \longleftrightarrow \partial\Lambda_{3n_0}(x)]$.

A path ζ in η from A to B is a chain of vertices $(v_1, ..., v_k)$ where $v_1 = A$, $v_2 = B$, $\|v_{i+1} - v_i\| = 1$, and $\eta(v_j) = 1$ where $1 \leq i \leq k-1$ and $1 \leq j \leq k$.

Define the lattice animals with size n as $\mathcal{A}_n = \{\text{all the possible } \mathcal{C}_0 : |\mathcal{C}_0| = n\}$.

For a specific element $C \in \mathcal{A}_n$, since the η percolation is $6n_0$-independent, there will be at least $\frac{n}{36n_0^2}$ independent vertices in C, so

$$\begin{aligned} \mathbb{P}[C \text{ is a connected inside}] &\leq \mathbb{P}[\frac{n}{36n_0^2} \text{ independent vertices are open}] \\ &\leq p'^{\frac{n}{36n_0^2}} \end{aligned} \tag{45}$$

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$$
\begin{aligned}
\mathbb{P}[|\mathcal{C}_0| \geq n] &\leq \sum_{C \in \mathcal{A}_n} \mathbb{P}[C \text{ is a connected inside}] \\
&\leq |\mathcal{A}_n| p'^{\frac{n}{36n_0^2}} \\
&\leq 16^{dn} p'^{\frac{n}{36n_0^2}} \\
&\leq \exp(-cn)
\end{aligned} \tag{46}
$$

Here we use the fact that $p' < (\frac{1}{e16^d})^{36n_0^2}$ and $|\mathcal{A}_n| \leq 16^{dn}$.

Notice that

$$
\begin{aligned}
\mathbb{P}_{p_c}[0 \longleftrightarrow \partial\Lambda_n] &\leq \mathbb{P}[|\mathcal{C}_0| \geq n - 3n_0] \\
&\leq \exp(-cn)
\end{aligned} \tag{47}
$$

We have $\theta_{p_c}(n)$ decaying exponentially fast, which we have known is impossible.

Therefore, the assumption $\exists n_0$ such that $\mathbb{P}_{p_c}[\mathcal{S}_1(n_0, 2n_0)] < \varepsilon, \forall \varepsilon > 0$ is false, so $\mathbb{P}_{p_c}[\mathcal{S}_1(n, 2n)] > c$ for some c. As a result, we have shown that $\theta_{p_c}(n) \geq \frac{c}{n^{\frac{d-1}{2}}}$ by the renormalization method.

8 $\mathbb{E}_{\mathbf{p_c}}[|\mathcal{C}_\mathbf{0}|] = +\infty$

We will give two different proofs to this theorem. In the second proof, we will give an inverse proportional function to estimate $|\mathcal{C}_0|$ when $p < p_c$.

Proof 1:

We start with the fuction $\varphi_p(\mathcal{S}) = \mathbb{E}_p[\sum_{xy \in \triangle\mathcal{S}} \mathbb{1}_{\{\text{xy is open}, 0 \overset{\mathcal{S}}{\longleftrightarrow} x\}}]$. It has been shown before [6] that $\forall \mathcal{S}, \varphi_{p_c}(\mathcal{S}) \geq 1$. Therefore,

$$
\begin{aligned}
\mathbb{E}_{p_c}[|\mathcal{C}_0|] &= \sum_{i=0}^{\infty} \sum_{x \in \partial\Lambda_i} \mathbb{P}_{p_c}[0 \longleftrightarrow x] \\
&\geq \sum_{i=0}^{\infty} \varphi_p(\Lambda_i) \\
&\geq \sum_{i=0}^{\infty} 1
\end{aligned} \tag{48}
$$

which indicates that $\mathbb{E}_{p_c}[|\mathcal{C}_0|] = +\infty$.

Notice that because $\forall \mathcal{S}, \varphi_{p_c}(\mathcal{S}) \geq 1$, we have $\forall n \geq 1, \varphi_{p_c}(\Lambda_n) \geq 1$. This also implies the result $\theta_{p_c}(n) \geq \frac{c}{n^{d-1}}$ since

$$
\begin{aligned}
cn^{d-1}\mathbb{P}_{p_c}[0 \longleftrightarrow \partial\Lambda_n] &\geq cn^{d-1}\mathbb{P}_{p_c}[0 \longleftrightarrow x] \\
&\geq \mathbb{E}_p[\sum_{xy \in \triangle\Lambda_n} \mathbb{1}_{\{\text{xy is open}, 0 \overset{\mathcal{S}}{\longleftrightarrow} x\}}] \\
&\geq 1
\end{aligned} \tag{49}
$$

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where $x \in \partial\Lambda_n$ such that $\mathbb{P}_{p_c}[0 \longleftrightarrow x] = \max_{y \in \partial\Lambda_n} \{\mathbb{P}_{p_c}[0 \longleftrightarrow y]\}$.

Proof 2:

Assume that $\mathbb{E}_p[|\mathcal{C}_0|] = t(p)$. We will prove that $t(p) \geq \frac{1}{4d(p_c - p)}$ for $p < p_c$ and $t(p_c) = +\infty$.

We use the technique of coupling in this part. Suppose that $\varepsilon < \frac{1}{4dt(p)}$, U is an $i.i.d.$ uniform distribution, and ω, ω' are two percolation configurations such that $\omega(e) = \mathbb{1}_{\mathrm{U}(e) \leq p}$ and $\omega'(e) = \mathbb{1}_{\mathrm{U}(e) \leq p+\varepsilon}$. Then, each edge e has the probability ε to be open in ω' but closed in ω.

Assume that there is a path $\zeta = (v_1, ..., v_k)$ connecting 0 and x, and n edges of the path are open in ω' but closed in ω. Define $\{e(\zeta)\} = \{e \in E(\zeta) : \omega(e) = 0, \omega'(e) = 1\}$ and $\mathcal{D}_n = \{\exists \zeta : |\{e(\zeta)\}| = n, 0 \overset{V(\zeta)}{\longleftrightarrow} x\}$. Given $e_i = v_j v_{j+1} \in \{e(\zeta)\}$, we define $e_i^+ = v_j$ and $e_i^- = v_{j+1}$

Therefore, for p that satisfies $t(p) < +\infty$, we have

$$\begin{aligned}
\mathbb{P}_{p+\varepsilon}[0 \longleftrightarrow x] &\leq \sum_{n=0}^{\infty} \sum_{e_i \in \{e(\zeta)\}} \mathbb{P}_p[\mathcal{D}_n] \\
&\leq \sum_{n=0}^{\infty} \sum_{e_i \in \{e(\zeta)\}} \mathbb{P}_p[0 \leftrightarrow e_1^-] ... \mathbb{P}_p[e_{n-1}^+ \leftrightarrow e_n^-] \mathbb{P}_p[e_n^+ \leftrightarrow x] \varepsilon^n
\end{aligned} \tag{50}$$

$$\begin{aligned}
t(p+\varepsilon) &= \mathbb{E}_{p+\epsilon}[|\mathcal{C}_0|] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p+\varepsilon}[0 \longleftrightarrow x] \\
&\leq \sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} \sum_{e_i \in \{e(\zeta)\}} \mathbb{P}_p[0 \leftrightarrow e_1^-] ... \mathbb{P}_p[e_{n-1}^+ \leftrightarrow e_n^-] \mathbb{P}_p[e_n^+ \leftrightarrow x] \varepsilon^n \\
&\leq \sum_{n=0}^{\infty} \sum_{e_i \in \{e(\zeta)\}} \mathbb{P}_p[0 \leftrightarrow e_1^-] ... \mathbb{P}_p[e_{n-1}^+ \leftrightarrow e_n^-] t(p) \varepsilon^n \\
&\leq \sum_{n=0}^{\infty} (2dt(p))^n t(p) \varepsilon^n \\
&< \sum_{n=0}^{\infty} \frac{t(p)}{2^n} \\
&< +\infty
\end{aligned} \tag{51}$$

Notice that $\forall x \in \mathbb{Z}^d$, $\sum_{y \in \mathbb{Z}^d} \mathbb{P}_p[x \longleftrightarrow y] = t(p)$ and there are $2d$ choices for each e_i^+ when e_i^- is fixed.

Now, if $t(p_c) < +\infty$, we will have $t(p_c + \varepsilon) < +\infty$ for some $\varepsilon > 0$. This is a contradiction because there exists an infinite cluster almost surely when $p > p_c$, which means that $\mathbb{E}_p[|\mathcal{C}_0|] = +\infty$ for $p > p_c$.

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Besides, when $p < p_c$, $t(p) \geq \frac{1}{4d(p_c-p)}$ because if $t(p) < \frac{1}{4d(p_c-p)}$, we can choose $\varepsilon = (p_c - p) < \frac{1}{4dt(p)}$ such that $t(p_c) = t(p+\varepsilon) < +\infty$, leading to a contradiction. This estimation is useful when p and p_c are close.

9 $\theta_{\mathbf{n}}(\mathbf{p_c}) \geq \frac{\mathbf{c}}{\mathbf{n}^{\frac{1}{3}}}, \mathbf{d} = \mathbf{2}$

Define Γ as the lowest left-right crossing in $[-2n, 2n]^2$. Our goal is to show that $\mathbb{P}_{p_c}[\Gamma \in [-2n, 2n] \times [-n, n]] > c$. It has been shown that when $d = 2$, $p_c = \frac{1}{2}$.[8]

According to the graph below, we define $A = [-2n, 2n] \times [-n, n]$, $A^+ = [-2n, 2n] \times [0, n]$, $A^- = [-2n - \frac{1}{2}, 2n + \frac{1}{2}] \times [-n + \frac{1}{2}, -\frac{1}{2}]$. Define $\mathcal{H}^*(S)$ as the event of having a horizontal crossing in the dual graph, and $\mathcal{V}^*(S)$ as the event of having a vertical crossing in the dual graph. As a result,

$$\begin{aligned}
\mathbb{P}_{p_c}[\Gamma \in A] &\geq \mathbb{P}_{p_c}[\mathcal{H}(A^+) \cap \mathcal{H}^*(A^-) \cap \mathcal{V}^*([-2n - \frac{1}{2}, 2n + \frac{1}{2}] \times [-2n - \frac{1}{2}, -\frac{1}{2}])] \\
&\geq \mathbb{P}_{p_c}[\mathcal{H}(A^+)]\mathbb{P}_{p_c}[\mathcal{H}^*(A^-) \cap \mathcal{V}^*([-2n - \frac{1}{2}, 2n + \frac{1}{2}] \times [-2n - \frac{1}{2}, -\frac{1}{2}])] \\
&\geq \mathbb{P}_{p_c}[\mathcal{H}(A^+)]\mathbb{P}_{p_c}[\mathcal{H}^*(A^-)]\mathbb{P}_{p_c}[\mathcal{V}^*([-2n - \frac{1}{2}, 2n + \frac{1}{2}] \times [-2n - \frac{1}{2}, -\frac{1}{2}])] \\
&\geq c
\end{aligned} \tag{52}$$

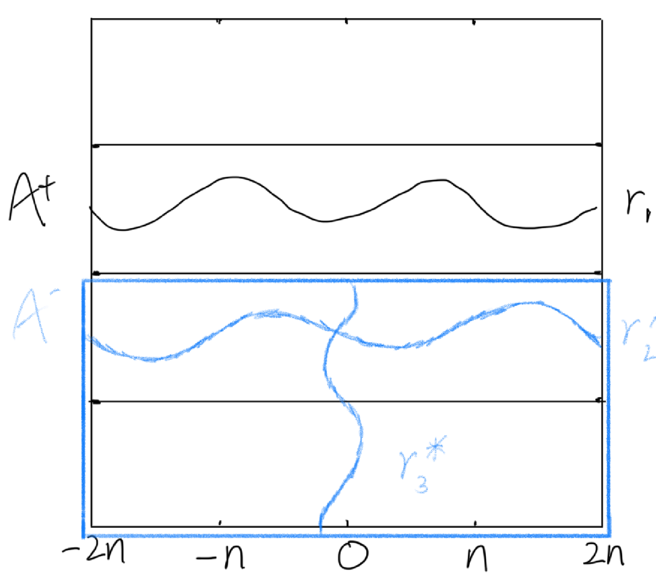

Figure 1: Lowest Crossing

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Suppose ζ is the path of the lowest crossing in A. We find a pair of points x and x^* such that $x^* = x + (0, -\frac{1}{2})$ and $\mathbb{P}_{p_c}[x \in \{\{0\} \times [-n, n]\} \cap V(\zeta)] \geq \frac{c}{2n+1}$.

Therefore,

$$\begin{aligned}
\theta_{p_c}(n)^3 &\geq \mathbb{P}_{p_c}[x \leftrightarrow \{-2n\} \times [-n,n]]\mathbb{P}_{p_c}[x \leftrightarrow \{2n\} \times [-n,n]]\mathbb{P}_{p_c}[x^* \leftrightarrow [-2n+\frac{1}{2}, 2n-\frac{1}{2}] \times \{-2n-\frac{1}{2}\}] \\
&\geq \mathbb{P}_{p_c}[x \leftrightarrow \{-2n\} \times [-n,n] \circ x \leftrightarrow \{2n\} \times [-n,n]]\mathbb{P}_{p_c}[x^* \leftrightarrow [-2n+\frac{1}{2}, 2n-\frac{1}{2}] \times \{-2n-\frac{1}{2}\}] \\
&\geq \mathbb{P}_{p_c}[x \leftrightarrow \{-2n\} \times [-n,n] \circ x \leftrightarrow \{2n\} \times [-n,n] \cap x^* \leftrightarrow [-2n+\frac{1}{2}, 2n-\frac{1}{2}] \times \{-2n-\frac{1}{2}\}] \\
&\geq \mathbb{P}_{p_c}[x \in \{\{0\} \times [-n,n]\} \cap V(\zeta)] \\
&\geq \frac{c}{2n+1}
\end{aligned} \tag{53}$$

The third inequality comes from the FKG inequality applied to one increasing event and one decreasing event.

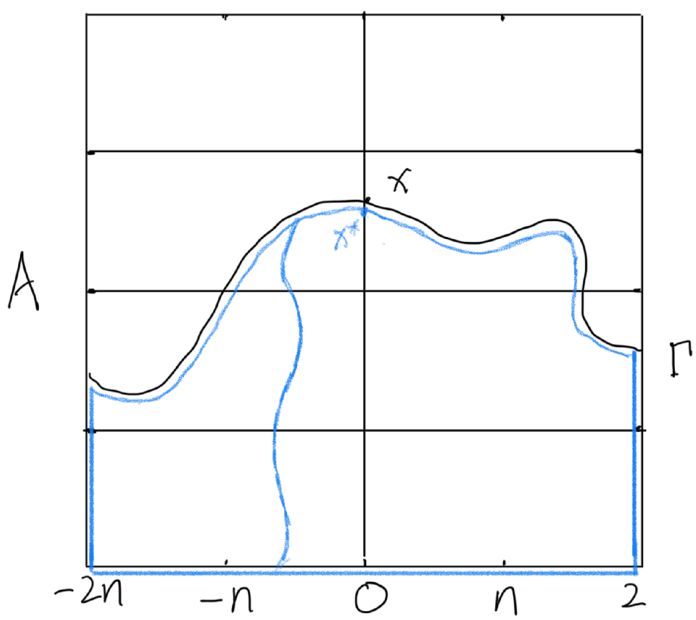

Figure 2: Dual Point

As a result,

$$\theta_n(p_c) \geq \frac{c}{n^{\frac{1}{3}}} \tag{54}$$

By using similar technique, we are able to show in further that

$$\mathbb{P}_{p_c}[\Gamma \in A] > c \tag{55}$$

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where Γ is the lowest crossing in $[-2n, 2n]^2$ and $A = [-2n, 2n] \times [-\alpha n, \beta n]$ with $0 < \alpha, \beta < 2$

10 Percolation on the subgraphs of $\mathbb{Z}^2$

In this section, we will talk about the percolation limited on some subgraphs of $\mathbb{Z}^2$. We start from the example of the graph $H_1 = \{(x, y) : 0 \leq y \leq x + 1\}$.

One natural question is that what is $p_c(H_1)$, the critical phase of the graph H_1. Define $\theta_n^{H_1}(p) = \mathbb{P}_p[0 \xleftrightarrow{H_1} \Lambda_n]$. Since $\{0 \xleftrightarrow{H_1} \Lambda_n\} \in \{0 \longleftrightarrow \Lambda_n\}$, $\theta_n^{H_1}(p) \leq \theta_n(p)$, which implies that $p_c(H_1) \geq \frac{1}{2}$.

Actually,

$$p_c(H_1) = \frac{1}{2} \tag{56}$$

We will prove the claim by contradiction.
Proof:

Assume $p_c(H_1) = \frac{1}{2} + \varepsilon$ where $\varepsilon > 0$. Consider the dual graph $(\mathbb{Z}^2)^*$ whose Bernoulli parameter is $\frac{1}{2} - \varepsilon$, indicating a subcritical percolation. Define two boxes $B(a, b) = \{(x, y) : a \leq x \leq b, 0 \leq y \leq x + 1\}$, $B^*(a, b) = \{(x, y) : a + \frac{1}{2} \leq x \leq b - \frac{1}{2}, -\frac{1}{2} \leq y \leq x + \frac{3}{2}\}$, and the event $\mathcal{K}_n$ in $(\mathbb{Z}^2)^*$ as $\mathcal{K}_n = \{(n + \frac{1}{2}, n + \frac{3}{2}) \xleftrightarrow{B^*(n,+\infty)} \{(x, y) : y = -\frac{1}{2}\}\}$ where $B(a, b)$ consists of vertices with integer coordinate and $B^*(a, b)$ consists of vertices in the corresponding dual graph $(\mathbb{Z}^2)^*$.

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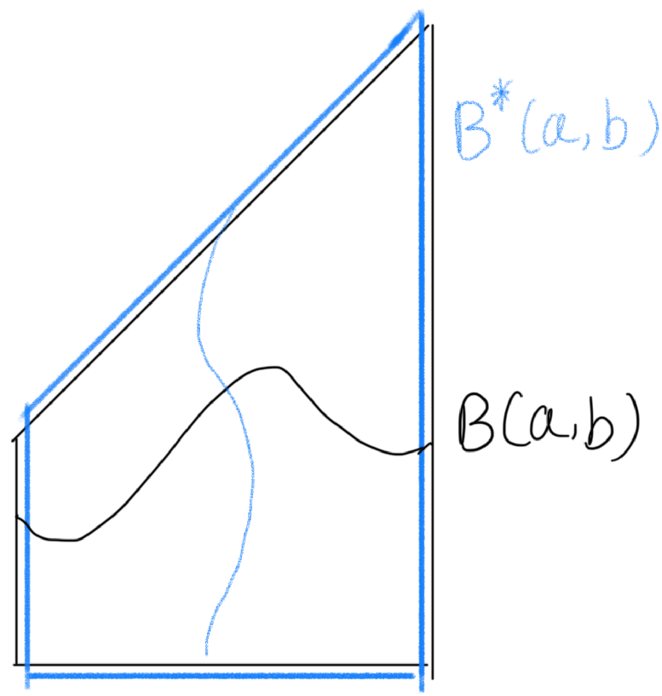

Figure 3: Boxes

Notice that $\mathcal{K}_n \in \{(n+\frac{1}{2}, n+\frac{3}{2}) \longleftrightarrow \partial\Lambda_n((n+\frac{1}{2}, n+\frac{3}{2}))\}$. Therefore,

$$\begin{aligned}\mathbb{P}_{p_c-\varepsilon}[\mathcal{K}_n] &\leq \mathbb{P}_{p_c-\varepsilon}[0 \longleftrightarrow \partial\Lambda_n] \\ &\leq \exp(-cn)\end{aligned} \tag{57}$$

which means that $\sum\limits_{n=0}^{\infty} \mathbb{P}_{p_c-\varepsilon}[\mathcal{K}_n] < +\infty$. According to the Borel–Cantelli lemma, $\mathbb{P}_{p_c-\varepsilon}[\mathcal{K}_n$ only happens for finitely many times$] = 1$.

If $\mathcal{K}_n$ only happens for finitely many times, there exists an integer N for each specific configuration such that $\forall n \geq N, \mathbb{1}_{\mathcal{K}_n} = 0$. Therefore, we can claim that there exists an integer N large enough such that

$$\mathbb{P}_{p_c-\varepsilon}[\{\forall n \geq N, \mathbb{1}_{\mathcal{K}_n} = 0\}] > 1 - \varepsilon \tag{58}$$

for any $\varepsilon > 0$. Define $\mathcal{L}_N = \{\bigcup\limits_{i=N}^{\infty} \mathcal{K}_i\}$. If $\mathcal{L}_N$ doesn't happen, the event $\{\forall n \geq$

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$N, \mathbb{1}_{\mathcal{K}_n} = 0\}$ will happen. As a result,

$$\begin{aligned}\mathbb{P}_{p_c-\varepsilon}[\{\forall n \geq N, \mathbb{1}_{\mathcal{K}_n} = 0\}] &= 1 - \mathbb{P}_{p_c-\varepsilon}[\mathcal{L}_N] \\ &\geq 1 - \sum_{i=N}^{\infty} \mathbb{P}_{p_c-\varepsilon}[\mathcal{K}_i] \\ &> 1 - \varepsilon\end{aligned} \tag{59}$$

Notice that the event $\{\forall n \geq N, \mathbb{1}_{\mathcal{K}_n} = 0\}$ is equivalent to the event $\{\forall K \geq N$, there is no vertical crossing in $B^*(N, K)\}$. Since $B^*(N, K)$ exists in the dual graph, the event is also equivalent to $\{\forall K \geq N$, there is a horizontal crossing in $B(N, K)\}$. Consequently,

$$\begin{aligned}\mathbb{P}_{p_c+\varepsilon}[0 \xleftrightarrow{H_1} \partial\Lambda_K] &\geq \mathbb{P}_{p_c+\varepsilon}[0 \xleftrightarrow{H_1} x(N)]\mathbb{P}_{p_c-\varepsilon}[\mathcal{L}_N] \\ &> c_1(1-\varepsilon) \\ &> c_2\end{aligned} \tag{60}$$

where $x(N) \in \partial\Lambda_N$ is a point of the horizontal crossing in $B(N, K)$.

Therefore, there exists a uniform lower bound $c > 0$ such that $\theta_n^{H_1}(\frac{1}{2} + \varepsilon) > c$ for all n, which means that $\forall p > \frac{1}{2}$, $\theta^{H_1}(p) > 0$. Consequently, we can conclude that $p_c(H_1) = \frac{1}{2}$.

This method can also be applied to any subgraph $H' = \{(x, y) : 0 \leq y \leq f(x)\}$. As long as $\sum\limits_{n=0}^{\infty} \exp(-cf(x))$ converges, $p_c(H') = \frac{1}{2}$. For example, given a subgraph $H' = \{(x, y) : 0 \leq y \leq \log(x+1)^2\}$, since the sequence

$$\begin{aligned}\sum_{n=0}^{\infty} \exp(-c\log(n+1)^2) &= \sum_{n=0}^{\infty} (n+1)^{-c\log(n+1)} \\ &= \sum_{n=0}^{\exp(\frac{2}{c})-1} (n+1)^{-c\log(n+1)} + \sum_{n=\exp(\frac{2}{c})}^{\infty} (n+1)^{-c\log(n+1)} \\ &\leq \sum_{n=0}^{\exp(\frac{2}{c})-1} (n+1)^{-c\log(n+1)} + \sum_{n=\exp(\frac{2}{c})}^{\infty} (n+1)^{-2} \\ &< +\infty\end{aligned} \tag{61}$$

converges, $p_c(H') = \frac{1}{2}$.

The other problem we want to discuss is the bound of $\theta_n^H(p_c)$ for some subgraphs, as we have done for $\mathbb{Z}^d$.

We again start from the subgraph $H_1 = \{(x, y) : 0 \leq y \leq x + 1\}$. We will show that

$$\theta_n^{H_1}(p_c) \geq \frac{1}{n^c} \tag{62}$$

Proof:

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We will show that if there are $k_1 \log n$ boxes of the shape $[0, 2n_0] \times [0, n_0]$ with a crossing in each box, the event $\{0 \xleftrightarrow{H_1} \Lambda_n\}$ will happen.

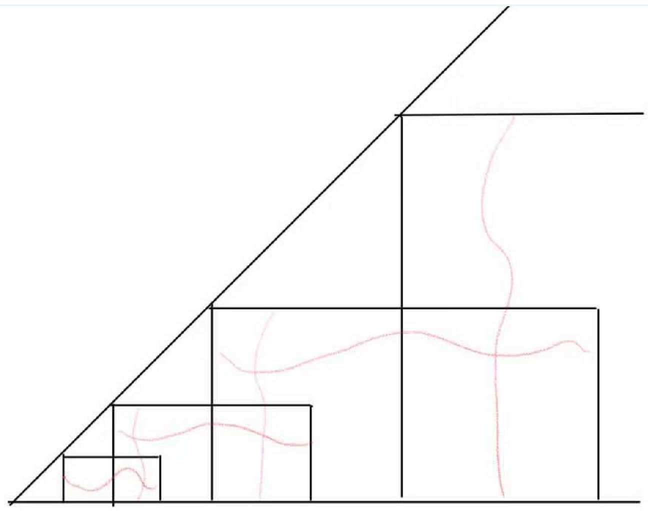

Figure 4: Exponentially Increasing

According to the graph,

$$\begin{aligned}\theta_n^{H_1}(p_c) \geq & p\mathbb{P}_p[\bigcup_{i=1}^{\lceil \log_2(n) \rceil} \mathcal{H}([2^{i-1}-1, 2^{i-1}3-1] \times [0, 2^{i-1}]) \cup \\ & \bigcup_{i=1}^{\lceil \log_2(n) \rceil - 1} \mathcal{V}([2^i - 1, 2^{i-1}3-1] \times [0, 2^i])]\end{aligned} \tag{63}$$

Notice that all those boxes are in the shape $[0, 2n_0] \times [0, n_0]$, so they have a uniform lower bound $c_0 > 0$. And then

$$\begin{aligned}\theta_n^{H_1}(p_c) &\geq pc_0^{2\lceil \log_2(n) \rceil - 1} \\ &\geq pe^{-c_1 \log(n)} \\ &\geq \frac{1}{n^c}\end{aligned} \tag{64}$$

Next, we will investigate on a series of subgraphs:

$$H_a = \{(x, y) : 0 \leq y \leq x^{\frac{1}{a}} + 1\} \tag{65}$$

The simplest example is the subgraph $H_2 = \{(x, y) : 0 \leq y \leq x^{\frac{1}{2}} + 1\}$. Our claim is that

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$$\theta_n^{H_2}(p_c) \geq \exp(-c\sqrt{n}) \tag{66}$$

Our inspiration comes from the previous proof which applies the use of exponentially increasing boxes.

We consider the integer values of $f(x) = x^{\frac{1}{2}} + 1$ and make a table:

Table 2: integer values

f(x)	2	3	4	...	n	...
x	1	4	9	...	$(n-1)^2$	...

We will show that if there $2n-1$ boxes of two shapes with a crossing in each box, the event $\{0 \xleftrightarrow{H_2} \partial\Lambda_{(n-1)^2}\}$ will happen.

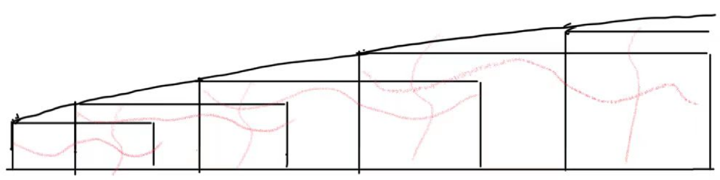

Figure 5: Arithmetically Increasing

According to the graph,

$$\begin{aligned}\theta_{(n-1)^2}^{H_2}(p_c) \geq & p\mathbb{P}_p[\bigcup_{i=1}^{n} \mathcal{H}([(i-1)^2, (i-1)^2 + 3i] \times [0, i]) \cup \\ & \bigcup_{i=1}^{n-1} \mathcal{V}([(i^2, i^2 + i + 1] \times [0, i+1])]\end{aligned} \tag{67}$$

All $2n-1$ boxes are either the shape $[0, n_0] \times [0, n_0]$ or $[0, n_0] \times [0, 3n_0]$, so, according to the RSW theory, the probability of their crossing has a lower bound $c_0, c_1 > 0$. So

$$\begin{aligned}\theta_n^{H_2}(p_c) &\geq \theta_{(\lceil\sqrt{n}\rceil)^2}^{H_2}(p_c) \\ &\geq pc_0^{\lceil\sqrt{n}\rceil+1} c_1^{\lceil\sqrt{n}\rceil} \\ &\geq \exp(-c\sqrt{n})\end{aligned} \tag{68}$$

We can extend this technique easily to H_a based on the fact that $n^a - (n-1)^a = f(n)$ where the degree of $f(n)$ is $a-1$. We can plot $c(a-1)$ boxes between n^a and $(n-1)^a$

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to make them connected. Therefore, we have this conclusion

$$\theta_n^{H_a}(p_c) \geq \exp(-cn^{\frac{a-1}{a}}) \tag{69}$$

for each subgraph H_a.

The conclusion is intuitive, as one can find out that as a goes to the infinity, H_a will just be a vertical line, which indicates that $\theta_n^H(p_c) \approx p_c^n \geq \exp(-cn)$.

Remark 1 *In fact, using the Lemma 6.1 in the article [3] given by Cerf, for any subgraph* $H = \{(x, y) : 0 \leq y \leq f(x) + 1\}$ *where* $f(x)$ *is plain enough, we can infer that*

$$\theta_n^H(p_c) \geq (\frac{c}{f(n)})^{\frac{n}{f(n)}} \tag{70}$$

but this lower bound is not as precise as the one we propose for subgraphs like H_a.

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