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论文题目： Study of the loci of configurations of points

# Study of the loci of configurations of points

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**Abstract**

We study in this work the configurations of points in projective spaces. We consider locus of configurations such that the number of linear independent hypersurfaces of a given degree is greater or equal to a given value. This locus is shown to be a projective variety. We prove some existence and connectedness results for this variety when the base field is algebraically closed. We also calculate the number of elements of this projective variety when the base field is a finite field.
**Keywords:** Configurations, Projective varieties, Zariski topology

## Contents



# 1 Introduction

We study in this article the configurations of points in projective spaces. Let $\mathbf{k}$ be a field and let $r$ be a positive integer. Let $p_1,\dots,p_d$ be $d$ points in the projective space $\mathbb{P}^r(\mathbf{k})$. It is natural to ask the number of *linearly independent* hypersurfaces of a given degree $k$ passing through these $d$ given points. Here, a series of hypersurfaces of degree $k$ is called linearly independent if the defining polynomials are linearly independent in the vector space of all degree $k$ homogeneous polynomials. To start with, let us use an example to demonstrate the subtlety of this problem.

**Example 1.** *Consider* 5 *distinct points* $p_1,\dots,p_5$ *on the real projective plane* $\mathbb{P}^2(\mathbb{R})$. *If these* 5 *points are in the* general position*, that is, no three points of them lie on the same straight line, then there is exactly* 1 *conic passing through these* 5 *points. For the other extreme, if these* 5 *points all lie on a same straight line L, then by the Bézout's theorem, any conic passing through these* 5 *points must have a component that coincides with L. The choice of the other component (which is another straight line) forms a three dimensional vector space. Hence, for these* 5 *points, there are* 3 *linearly independent conics passing through them.*

This easy example already shows that the number of independent hypersurfaces is related to the configuration of the points. Motivated by this, let us define

**Definition 1.** *A* configuration *of d points in the projective space* $\mathbb{P}^r(\mathbf{k})$ *is an unordered d-tuple* $(x_1,\dots,x_d)$*, where* $x_1,\dots,x_d$ *are points in* $\mathbb{P}^r(\mathbf{k})$ *which are not necessarily distinct.*

Let $\mathbb{P}^r(\mathbf{k})^{(d)}$ denote the set of all configurations of $d$ points in $\mathbb{P}^r(\mathbf{k})$. It is well-known [9] that $\mathbb{P}^r(\mathbf{k})^{(d)}$ is a projective variety, and that the canonical map $\mathbb{P}^r(\mathbf{k})^d \to \mathbb{P}^r(\mathbf{k})^{(d)}$, sending ordered $d$-tuples onto the corresponding unordered ones, is a morphism of algebraic varieties.

**Definition 2.** *For a given configuration* $S \in \mathbb{P}^r(\mathbf{k})^{(d)}$*, we define* $i(k,S)$ *to be the number of* linearly independent *hypersurfaces of degree k passing through the d points in the configuration S.*

**Example 2.** *Let us return to Example 1. Consider configurations* $S \in \mathbb{P}^2(\mathbb{R})^{(5)}$. *If the points in S are in the general position, then* $i(2,S)=1$. *If the* 5 *points in S are distinct and lie on the same straight line, then* $i(2,S)=3$.

The function $i(k,S)$ has the following properties.

**Proposition 1.** *(i) Fix* $k \in \mathbb{N}$. *The map*

$$\begin{array}{ccc} \mathbb{P}^r(\mathbf{k})^{(d)} & \to & \mathbb{N} \\ S & \mapsto & i(k,S) \end{array}$$

*is upper-semicontinuous with respect to the Zariski topology of* $\mathbb{P}^r(\mathbf{k})^{(d)}$. *In other words, given* $\beta \in \mathbb{N}$*, the subset* $\{S \in \mathbb{P}^r(\mathbf{k})^{(d)} : i(k,S) \geq \beta\} \subset \mathbb{P}^r(\mathbf{k})^{(d)}$ *is Zariski closed.*
*(ii) Fix* $S \in \mathbb{P}^r(\mathbf{k})^{(d)}$. *The map* $k \mapsto i(k,S)$ *is increasing. Furthermore, for k large enough,* $i(k+1,S) > i(k,S)$.

This proposition motivates us to define the following object.

**Definition 3.** *Let* $r,d,k,\beta$ *be natural numbers. Define* $R^{r,d}_{k,\beta}(\mathbf{k})$ *to be the subset*

$$\{S \in \mathbb{P}^r(\mathbf{k})^{(d)} : i(k,S) \geq \beta\}.$$

Proposition 1 (i) shows that $R^{r,d}_{k,\beta}(\mathbf{k})$ is a projective variety. The study of this geometric object for different fields occupies the main part of the article.

Our study of $R^{r,d}_{k,\beta}(\mathbf{k})$ contains to two main themes. The first one consists of studying the geometric or topological properties of $R^{r,d}_{k,\beta}(\mathbf{k})$ when $\mathbf{k}$ is algebraically closed field. The main result we obtain in this direction is

**Theorem 1.** *Let* $\mathbf{k}$ *be an algebraically closed field.*
*(i) The set* $R^{r,d}_{k,\beta}(\mathbf{k})$ *is nonempty if and only if* $\beta \le \binom{r+k}{r} - 1$.
*(ii) The projective variety* $R^{r,d}_{k,\beta}(\mathbf{k})$ *is connected with respect to the Zariski topology, if it is nonempty.*

The second theme of our study is to count the number of elements in $R^{r,d}_{k,\beta}(\mathbf{k})$ when $\mathbf{k}$ is a finite field. In the statement of the following theorems, we let $\mathbf{k} = \mathbb{F}_q$ be the finite field with $q$ elements where $q = p^n$ is a power of a prime number $p$. To be able to state the theorem, let us define a sequence $\{\alpha_d\}$ recursively as follows. For $d \le r+1$, $\alpha_d = \frac{(q^{r+1}-1)\dots(q^{r+1}-q^{d-1})}{(q-1)^d}$. For $d > r+1$, $\alpha_d := \alpha_{d-1}.\max\{\frac{q^{r+1}-1}{q-1} - \binom{d-1}{r}\frac{q^r-1}{q-1}, 0\}$.

**Theorem 2.** *Assume* $\beta \ge \max\{\binom{k+r}{r} - d + 1, \binom{k+r}{r} - kr\}$. *Then the number of elements in* $R^{r,d}_{k,\beta}(\mathbb{F}_q)$ *is less than or equal to*

$$\binom{q.\frac{q^r-1}{q-1}+d}{d} - \frac{\alpha_d}{d!}.$$

**Example 3.** *Let us use the estimate in the above theorem to do some explicit calculations. When* $r = 2$, $d = 5$, $k = 3$ *and* $q = 5$. *Our estimate gives* $|R^{2,5}_{3,6}(\mathbb{F}_5)| \le 324632$. *It means that there are less than or equal to* 324632 *configurations of* 5 *points (not necessarily distinct) in the projective plane over the field* $\mathbb{F}_5$ *that can be passed by at least* 6 *linearly independent plane quintic curves.*

Our estimate in the above theorem is far from optimal. The difficulty comes from the increasing complexity of the configurations of hyperplanes in the projective space when the number of points is large. However, it can be expected that better upper bounds can be obtained if we make finer analysis on the configurations when the dimension of the projective space and the number of points are small. Indeed, in Section 4.2, we give sharper upper bounds for the number of elements in $R^{2,d}_{k,\beta}(\mathbb{F}_q)$ for $\beta \ge \max\{\frac{(k+2)(k+1)}{2} - d + 1, \frac{(k+2)(k+1)}{2} - 2k\}$ and $d < 10$. Since the formula is quite complicated, in the Introduction let us only present the result when $k = 3$ and $q = 5$ in the following table. The general formula is given in Proposition 7 and is a function in $q$. One can compare the result when $d = 5$ in the table with Example 3.

| $d$ | $\beta$ | upper bound for $\lvert R^{2,d}_{3,\beta}(\mathbb{F}_5)\rvert$ |
|---|---|---|
| 1 | 10 | 0 |
| 2 | 9 | 31 |
| 3 | 8 | 1581 |
| 4 | 7 | 30876 |
| 5 | 6 | 306032 |
| 6 | 5 | 1944692 |
| 7 | 4 | 10295472 |
| 8 | 4 | 48903492 |

When $d$ is relatively small, we can get the explicit value of $R^{r,d}_{k,\beta}(\mathbf{k})$:

$\boxed{d=1}$

$$|R^{r,1}_{k,\beta}(\mathbb{F}_q)| = \begin{cases} 0 & \text{if } \beta > \binom{k+r}{r} - 1 \\ \frac{q^{r+1}-1}{q-1} & \text{if } \beta \le \binom{k+r}{r} - 1 \end{cases}.$$

$\boxed{d=2}$

$$|R^{r,2}_{k,\beta}(\mathbb{F}_q)| = \begin{cases} 0 & \text{if } \beta \ge \binom{k+r}{r} \\ \frac{q^{r+1}-1}{q-1} & \text{if } \beta = \binom{k+r}{r} - 1 \\ \binom{\frac{q^{r+1}-q}{q-1}+2}{2} & \text{if } \beta \le \binom{k+r}{r} - 2 \end{cases}.$$

$\boxed{d=3}$

$$|R^{r,3}_{k,\beta}(\mathbb{F}_q)| = \begin{cases} 0 & \text{if } \beta \ge \binom{k+r}{r} \\ \frac{q^{r+1}-1}{q-1} & \text{if } \beta = \binom{k+r}{r} - 1 \\ 2\cdot\binom{\frac{q^{r+1}-1}{q-1}}{2} + \frac{q^{r+1}-q}{q-1} & \text{if } \beta = \binom{k+r}{r} - 2 \text{ and } k \ge 2 \\ 2\cdot\binom{\frac{q^{r+1}-1}{q-1}}{2} + \frac{q^{r+1}-q}{q-1} + \binom{\frac{q^{r+1}-1}{q-1}}{2}(q-1) & \text{if } \beta = \binom{k+r}{r} - 2 \text{ and } k = 1 \\ \binom{\frac{q^{r+1}-1}{q-1}+2}{3} & \text{if } \beta \le \binom{k+r}{r} - 3 \end{cases}.$$

# 2 Zariski topology and Basic properties of $i(k,S)$

## 2.1 Zariski topology

We have to present a kind of topology, the Zariski topology, that we often use in algebraic geometry, in order to describe the connectedness of $R^{r,d}_{k,\beta}(\mathbf{k})$ and the continuity of some maps we will construct. We will first define the Zariski topology in affine spaces, which is the base for us to define the Zariski topology in other more complicated spaces.

**Definition 4.** *Let $I$ be a set of polynomial in $\mathbf{k}[x_1,\ldots,x_n]$, where $\mathbf{k}[x_1,\ldots x_n]$ is the set of polynomials in $x_1,\ldots x_n$ with coefficients in the field $\mathbf{k}$. Define $V(I)$ to be the set $\{x\in\mathbb{A}^n(\mathbf{k}):$ for any $f\in I,\ f(x)=0\}$. A subset $F\subset\mathbb{A}^n(\mathbf{k})$ is called Zariski closed, if there exists $I\subset\mathbf{k}[x_1,\ldots,x_n]$, satisfying $F=V(I)$.*

Then we generalize the Zariski topology on affine space to projective space using a general method which we call the process *from local to global*. This process is based on the lemma below

**Lemma 1.** *Let $X$ be a set and let $\{U_i\}_{i\in I}$ be a family of subsets of $X$, satisfying that $\bigcup_{i\in I}U_i=X$. Assume that for each $i\in I$, there is given a topology on $U_i$, such that for each pair $i,\ j\in I$, $U_i\cap U_j$ is an open subset of $U_j$. Then, there is a canonical way to endow $X$ with a topology, satisfying that each $U_i\subset X$ is open in $X$ and that the given topology of $U_i$ coincides with the induced topology from $X$.*

*Proof* We define the topology of $X$ as follows: a subset $V\subset X$ is defined to be open if and only if for each $i\in I$, $V\cap U_i\subset U_i$ is open. First, let us check that this definition indeed gives a topology of $X$.

**Claim.** *(1) $\varnothing$ and $X$ are open in $X$.*
*(2) The union of any open subsets is still open.*
*(3) The intersection of two open subsets is still open.*

*Proof* (1) is obvious. Let us prove (2). Let $\{V_i\}_{i\in J}$ be a family of open subsets in $X$. It is easy to know that for each $j\in I$,

$$(\bigcup V_i)\cap U_j=\bigcup(V_i\cap U_j)$$

With the definition, $(V_i\cap U_j)$ are open in $U_j$, so $\bigcup(V_i\cap U_j)$ is open in $U_j$, which means that $\bigcup_{i\in J}V_i\cap U_j$ is open in $U_j$ for each $j\in I$. Therefore, the union of any open subsets in $X$ is still open. Now let us prove (3). Let $V_1$, $V_2$ be two open subsets in $X$. For each $j\in I$, $(V_1\cap V_2)\cap U_j=(V_1\cap U_j)\cap(V_2\cap U_j)$. By definition, $(V_1\cap U_j)$ and $(V_2\cap U_j)$ are open in $U_j$, which implies that $(V_1\cap V_2)\cap U_j$ is open in $U_j$ for each $j$. Therefore, the intersection of two open subsets in $X$ is still open. □

Now let us prove the second statement. Each $U_i$ is open in $X$ since for each $j\in I$, $U_i\cap U_j$ is open in $U_j$. To show that the induced topology of $U_i$ coincides with its original topology, we need to check that for each open subset $V$ in $X$, $V\cap U_i$ is open in $U_i$. But this follows directly from the definition of openness of subsets of $X$. □

With lemma 1, we can naturally define the Zariski topology on projective spaces.

**Lemma 2.** *There exists a family of subsets $\{U_i\}_{i=\{0,\ldots,r\}}\subset\mathbb{P}^r(\mathbf{k})$, satisfying that $\bigcup U_i=\mathbb{P}^r(\mathbf{k})$, and $U_i\cong\mathbb{A}^r(\mathbf{k})$.*

*Proof* The elements in $\mathbb{P}^r(\mathbf{k})$ can be represented in the form $[x_0 : \ldots : x_r]$, where not all $x_i$ are 0. Let $U_i \subset \mathbb{P}^r(\mathbf{k})$ be the subset where vectors satisfy that $x_i \neq 0$. Therefore, the elements in $U_i$ can be represented in the form $[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_1}, \ldots, \frac{x_r}{x_i}]$, which is naturally morphism to $\mathbb{A}^r(\mathbf{k})$ □

**Definition 5.** *We have already known that* $\mathbb{P}^r(\mathbf{k}) = U_0 \cup \ldots \cup U_r$*, where* $U_i \cong \mathbb{A}^r(\mathbf{k})$*. Following the* from-local-to-global *process in Lemma 2, a subset* $V \subset \mathbb{P}^r(\mathbf{k})$ *is defined to be open if and only if for each* $i \in \{0, \ldots, r\}$*,* $V \cap U_i \subset U_i$ *is open, where the topology of* $U_i$ *is the same as the Zariski topology on affine space, in definition 4*

In order to make the definition of Zariski topology more complete, we will introduce the definition the product topology and the quotient topology.

**Definition 6.** *Let* $X_1, \ldots, X_s$ *be topological spaces. There is a canonical way to define a topology on* $X_1 \times \ldots \times X_s$ *as follows: a subset* $V$ *of* $X_1 \times \ldots \times X_s$ *is open if and only if* $V$ *is a union of subsets of the form* $U_1 \times \ldots \times U_s$*, where* $U_j \subset X_j$ *is open for each* $j = 1, \ldots, s$*.*

**Remark 1.** *(1) It is easy to check that the definition above does give a topology on* $X_1, \times \ldots, \times X_s$*.*
*(2) We use the same way to define the topology of* $(\mathbb{P}^r)^d = \mathbb{P}^r \times \ldots \times \mathbb{P}^r$*.*

**Lemma 3.** *Let* $X, Y_1, \ldots, Y_r$ *be topological spaces and let* $f : X \to Y_1 \times \ldots \times Y_r$ *be a map. The map* $f$ *is continuous, if and only if for each* $i \in \{1, \ldots r\}$*, the map* $f_i := pr_i \circ f : X \to Y_i$ *is continuous, where* $pr_i : Y_1 \times \ldots \times Y_r \mapsto Y_i$ *where* $U_i \subset Y_i$ *is the projection map to the i-th component.*

*Proof* First, we will prove that if $f$ is continuous, the maps $f_i$ are continuous. Let $i$ be fixed. Let $V \subset Y_i$ be an open subset. We can find that $f_i^{-1}(V) = f^{-1}(Y_1 \times \ldots \times V \times \ldots \times Y_r)$. Since $f$ is continuous, $f^{-1}(Y_1 \times \ldots \times V \times \ldots \times Y_r)$ is open, which means that $f_i^{-1}(V)$ is open, implying that $f_i$ is continuous. Since $i$ is chosen freely, for any $i \in \{1, \ldots r\}$, $f_i$ is continuous. Then, we will prove that if for each $i \in \{1, \ldots r\}$, $f_i$ is continuous, $f$ is continuous. Let $V \subset Y_1 \times \ldots \times Y_r$ be open, then

$$V = \bigcup_i U_{i1} \times \ldots \times U_{ir},$$

where $U_{ij} \subset Y_j$ are open set. We can find that

$$f^{-1}(U_{i1} \times \ldots \times U_{ir}) = \bigcap_{j=1}^{n} f_j^{-1}(U_{ij}).$$

Since $f_j$ are continuous, $f_j^{-1}(U_{ij})$ are open, and then $\bigcap_{j=1}^{n} f_j^{-1}(U_{ij})$ is open. Hence, $f^{-1}(U_{i1} \times \ldots \times U_{ir})$ is open. Therefore,

$$f^{-1}(V) = \bigcup_i f^{-1}(U_{i1} \times \ldots \times U_{ir})$$

is open, suggesting that $f$ is continuous. □

**Definition 7.** *Let $X$ be a topological space, $Y$ be a set and $f : X \to Y$ be a surjective map. Then, we can define canonically a topology on $Y$, called the quotient topology of $Y$ with respect to $f : X \to Y$, as follows: a subset $V \subset Y$ is defined to be open if and only $f^{-1}(V) \subset X$ is open.*

**Remark 2.** *We have already known the product and quotient topology, we can get the Zariski topology on $(\mathbb{P}^r(\mathbf{k}))^{(d)}$. There is a canonical surjective map $\pi : (\mathbb{P}^r(\mathbf{k}))^d \to (\mathbb{P}^r(\mathbf{k}))^{(d)}$. We define the topology on $(\mathbb{P}^r(\mathbf{k}))^{(d)}$ to be the quotient topology with respect to $\pi$.*

**Proposition 2.** *Let $g : \mathbb{A}^m(\mathbf{k}) \to \mathbb{A}^n(\mathbf{k})$ be a polynomial map. Then $g$ is continuous with respect to the Zariski topology.*

*Proof* Let $g : \mathbb{A}^m(\mathbf{k}) \to \mathbb{A}^n(\mathbf{k})$ be a polynomial map, $F \subset \mathbb{A}^n(\mathbf{k})$ be a closed subset of $\mathbb{A}^n(\mathbf{k})$, i.e. there are $f_i \in \mathbf{k}[x_1, \dots, x_n]$, where $i \in I$, satisfying that $F = \{f_i(x) = 0, i \in I\}$. Then, it is easy to know that $g^{-1}(F) = \{f_i \circ g(x) = 0, i \in I\}$, which means that $g^{-1}(F)$ is closed in $\mathbb{A}^m(\mathbf{k})$. Therefore, $g$ is continuous. □

## 2.2 Basic properties of the function $i(k,S)$

In this section, we want to do some researches about the topology and other geometry properties of the function $i(k,S)$ defined in Definition 1 in the Introduction and prove Proposition 1 in several steps.

**Definition 8.** *Let $i(k,S)$ be the number of linear independent hypersurfaces of degree $k$ in $\mathbb{P}^n$ passing through $S$*

$$i : \mathbb{N} \times (\mathbb{P}^r)^{(d)} \to \mathbb{N}$$

**Proposition 3.** *Fix a configuration $S \in \mathbb{P}^r(\mathbf{k})^{(d)}$, the function $i(k,S)$ increases with respect to $k$, and when $k$ is large enough, it strictly increases with respect to $k$.*

*Proof.* Let $n = i(k,S)$. Let $\{f_j\}_{j=\{1,\dots,n\}}$ be polynomials of degree $k$ that are linearly independent and pass through $S$. Let us consider $i(k+1,S)$. Let $S = (p_1, \dots, p_d) \in \mathbb{P}^r(\mathbf{k})^{(d)}$ and we may regard $p_1, \dots, p_d$ as nonzero vectors in $\mathbf{k}^{r+1}$. The homogeneous polynomials $\{x_0 \cdot f_j\}_{j \in \{1,\dots,n\}}$ are of degree $k+1$, linearly independent and pass through $S$. Therefore, $i(k+1,S) \geq n = i(k,S)$. Now let us show that this inequality is strict when $k$ is large enough. To see this, first notice that $i(k,S) = \dim \ker \psi_{\mathrm{S,k}} = \dim \mathbf{k}_{\mathrm{k}}[\mathrm{x}_0, \dots \mathrm{x}_{\mathrm{r}}] - \operatorname{rank} \psi_{S,k}$, where

$$\begin{array}{rccc} \psi_{S,k} : & \mathbf{k}_k[x_0, \dots x_r] & \to & \mathbf{k}^d \\ & f & \mapsto & (f(p_1), \dots, f(p_d)). \end{array}$$

One needs to remark that the definition of $\psi_{S,k}$ depends on the choice of homogeneous coordinates of each point $p_i$ in the configuration $S$, making the definition of $\psi_{S,k}$ apriori not well-defined. However, as far as we are concerned in this article, only the rank of $\psi_{S,k}$ is needed for us, and this rank is independent of the choice of the homogeneous coordinates of each point. In fact, no matter the choice of homogeneous coordinates chosen for each point, the kernel of the linear map $\psi_{S,k}$ in $\mathbf{k}_k[x_0, \dots, x_r]$ is the same. Since $\operatorname{Im} \psi_{S,k} \cong \mathbf{k}_k[x_0, \dots, x_r] / \ker \psi_{S,k}$, the dimension of $\operatorname{Im} \psi_{S,k}$ does not depend on the choice of the homogeneous coordinates. Hence, the rank of $\psi_{S,k}$ is well-defined.

**Claim.** *The rank of $\psi_{S,k}$ increases with respect to k.*

*Proof* We may assume that $\mathbf{k}$ is an algebraically closed field, since if $\mathbf{k}$ is not algebraically closed field, we can take the field extension of $\mathbf{k}$ as the field of $\psi_{S,k}$, which has not influence on the rank of $\psi_{S,k}$. Since $p_1, \dots, p_d \in \mathbb{P}^r(\mathbf{k})$ are finitely many points, we may find a hyperplane passing through none of these points. Let $L$ be the line function defining this hyperplane. Then $L(p_i) \neq 0$, for each $i$. Since the coordinate of each $p_i$ can be chosen up to a nonzero constant without changing the rank of $\psi_{S,k}$ or $\psi_{S,k+1}$. As shown in the above remark, we may choose the homogeneous coordinate of each $p_i$ with care to be able to assume $L(p_i) = 1$ for each $i$. Let $I_k$ be the image of $\psi_{S,k} : \mathbf{k}_k[x_0, \dots x_r] \to \mathbf{k}^d$ and similarly let $I_{k+1}$ be the image of $\psi_{S,k+1}$. We want to show that $I_k \subset I_{k+1}$. Let $(y_i, \dots, y_d) \in I_k$, which means that there exists $f \in \mathbf{k}_k[x_0, \dots x_r]$, satisfying that $y_i = f(p_i)$ for each $i$. Then, for $g = L \cdot f \in \mathbf{k}_{k+1}[x_0, \dots x_r]$, we have $(1 \cdot y_1, \dots, 1 \cdot y_d) = (g(p_1), \dots, g(p_d)) \in I_{k+1}$. Therefore, $I_k \subset I_{k+1}$, which means that the rank of $\psi_{S,k}$ increases with respect to $k$. □

By the above claim, the rank of $\psi_{S,k}$ increases with respect to $k$, and $\operatorname{rank}\psi_{S,k} \leq d$, so there exists a $K \in \mathbb{N}$ satisfying for any $k > K$, $\operatorname{rank}\psi_{S,k+1} = \operatorname{rank}\psi_{S,k}$. But $\dim \mathbf{k}_{\mathrm{k}}[\mathrm{x}_0, \dots \mathrm{x}_\mathrm{r}] = \binom{k+r+1}{r}$ is strictly increasing. Therefore, when $k > K$, $i(k+1,S) > i(k,S)$. □

Here is an example that $i(k+1,S) = i(k,S)$: let $S \in (\mathbb{P}^2)^{(10)}$, $S$ can be general enough that $i(2,S) = i(1,S) = 0$

Here is also an interesting topological property of $i(k,S)$.

**Proposition 4.** *Fix $k \in \mathbb{N}$, the map $S \mapsto i(k,S)$ is upper semi-continuous with respect to the Zariski topology of $(\mathbb{P}^r)^{(d)}$*

*Proof.* Let $V, W$ be two vector spaces with dimension $n$ and $m$. By choose base of $V$ and $W$, $Hom(V,W) \cong Mat_{n \times m}(\mathbf{k}) \cong \mathbb{A}^{nm}(\mathbf{k})$. Let

$$Hom_{rank \geq r}(V,W) = \{\psi : V \to W \quad \text{is a linear map, satisfying } \operatorname{rank}\psi \geq r\}$$

For any $n \times m$ matrix whose rank$\geq r$, there exists $r \times r$ submatrix of it whose determinant is not 0. Therefore, $Hom_{rank \geq r}(V,W)$ is open in $Hom(V,W)$ with respect to the Zariski topology.

$$\begin{array}{cccc} \varphi : & (\mathbb{A}^{r+1})^d & \to & Hom(\mathbf{k}_k[x_0, \dots, x_r], \mathbf{k}^d) \\ & S & \mapsto & \psi_{S,k} \end{array}$$

Via $\varphi$, $(\mathbb{A}^{r+1})^d$ can be regarded as a subset of $Hom(\mathbf{k}_k[x_0, \dots, x_r], \mathbf{k}^d)$. Hence, the topology of $(\mathbb{A}^{r+1})^d$ can be generated from $Hom(\mathbf{k}_k[x_0, \dots, x_r], \mathbf{k}^d)$, which means that $\{S = (v_1, \dots, v_d) \in (\mathbb{A}^{r+1} - \{0\})^d : \operatorname{rank}\psi_{S,k} \geq \text{a constant}\}$ is open. Assume that $i(k,S) \leq n, n \in \mathbb{Z}$, $\operatorname{rank}\psi_{S,k} = \dim \mathbf{k}_{\mathrm{k}}[\mathrm{x}_0, \dots \mathrm{x}_\mathrm{r}] - \dim \ker \psi_{\mathrm{S,k}} \geq \binom{k+r+1}{r} - n$. Hence, $\{S \in (\mathbb{A}^{r+1} - \{0\})^d : i(k,S) \leq n\}$ is open. $(\mathbb{P}^r)^{(d)} \cong (\mathbb{A}^{r+1} - \{0\})^d$. Therefore, for any $n$, $\{S \in (\mathbb{P}^r)^{(d)} : i(k,S) \leq n\}$ is open, which means that $i(k,S)$ is upper-semi-continuous.

□

# 3 Existence and connectedness results

In this section, we will try to explore the existence and connectedness results of the projective variety $R_{k,\beta}^{r,d}(\mathbf{k})$. First, we will introduce some basic definitions and tools we would use in the proof. We assume in this section that the field $\mathbf{k}$ is an algebraically closed field.

## 3.1 Existence result

Before talking about the properties of $R_{k,\beta}^{r,d}(\mathbf{k})$, we want to decide whether the $R_{k,\beta}^{r,d}(\mathbf{k})$ is an empty set or nonempty set.

**Theorem 3.** *$R_{k,\beta}^{r,d}(\mathbf{k}) \neq \varnothing$, if and only if $\beta \leq \binom{k+r}{r} - 1$.*

*Proof.* First, we will prove that if $R_{k,\beta}^{r,d}(\mathbf{k}) \neq \varnothing$, $\beta \leq \binom{k+r}{r} - 1$ by reduction to absurdity. We assume that $\beta \geq \binom{k+r}{r}$. Since $R_{k,\beta}^{r,d}(\mathbf{k}) \neq \varnothing$, there exists a configuration $S \in R_{k,\beta}^{r,d}(\mathbf{k})$. By definition, $i(k,S) \geq \beta \geq \binom{k+r}{r}$. However, $i(k,S) = \dim\ker(\psi_{k,S} : \mathbf{k}_k[x_0,\ldots,x_r] \to \mathbf{k}^d) = \dim\mathbf{k}_k[x_0,\ldots,x_r] - \operatorname{rank}(\psi_{k,S} : \mathbf{k}_k[x_0,\ldots,x_r] \to \mathbf{k}^d)$. Hence, $\operatorname{rank}(\psi_{k,S} : \mathbf{k}_k[x_0,\ldots,x_r] \to \mathbf{k}^d) \leq \binom{k+r}{r} - i(k,S) \leq 0$. Therefore, $\psi_{k,S}$ is a zero map, which is impossible. In fact, for any point $p \in \mathbb{P}^r(\mathbb{C})$ appeared in $S$, there exists a polynomial $f \in \mathbf{k}_k[x_0,\ldots,x_r]$, satisfying $f(p) \neq 0$. Hence, $\psi_{k,S}(f) \neq 0$, which is contradicting to the fact that $\psi_(k,S)$ is a zero map. Therefore, if $R_{k,\beta}^{r,d}(\mathbf{k}) \neq \varnothing$, $\beta \leq \binom{k+r}{r} - 1$. Then, we will prove that if $\beta \leq \binom{k+r}{r} - 1$, $R_{k,\beta}^{r,d}(\mathbf{k}) \neq \varnothing$. There exists configurations $S = (p,p,\ldots,p,p)$, where $p \in \mathbb{P}^r(\mathbf{k})$. In this case, $\operatorname{rank}\psi_{k,S} = 1$, so $i(k,S) = \binom{k+r}{r} - 1$. Therefore, if $\beta \leq \binom{k+r}{r} - 1$, $S = (p,p,\ldots,p,p)$ will be in $R_{k,\beta}^{r,d}(\mathbf{k})$, which means that $R_{k,\beta}^{r,d}(\mathbf{k}) \neq \varnothing$. Therefore, we can say that $R_{k,\beta}^{r,d}(\mathbf{k}) \neq \varnothing$, if and only if $\beta \leq \binom{k+r}{r} - 1$. □

## 3.2 Connectedness result

Now, we have already had the tools and conditions we need. We will then explore the topological proposition of $R_{k,\beta}^{r,d}(\mathbf{k})$ with respect to the Zariski topology. The method for us to prove the connectedness of $R_{k,\beta}^{r,d}(\mathbf{k})$ is to construct a connected structure. In this subsection, the topology we use is the Zariski topology. First, we will construct the structure we use.

**Definition 9.** *Let $X$ be a projective variety over $\mathbf{k}$. The variety $X$ is called continuously rational-chain connected, if for any points $x$, $y \in X$, there is a sequence of points $x = x_0$, $x_1, \ldots x_{r-1}$, $x_r = y$ in $X$, satisfying that each pair of adjacent points in the sequence can be connected by a rational curve, i.e., for each $i \in \{1,\ldots,r\}$, there is a continuous map $\varphi_i : \mathbb{A}^1(\mathbf{k}) \to X$, satisfying that $\varphi_i(0) = x_{i-1}$, $\varphi_i(1) = x_i$.*

With the definition of continuously rational-chain connected projective variety, we will explore the proposition of it, which is essential for our proof of the connectedness of $R_{k,\beta}^{r,d}(\mathbf{k})$.

**Proposition 5.** *A continuously rational-chain connected projective variety is connected.*

*Proof* We will prove this proposition by contradiction. Assume that $X$ is continuously rational-chain connected, but $X$ is not connected. There exists two nonempty open subsets $U$, $V \subset X$, satisfying that $U \cup V = X$, and $U \cap V = \varnothing$. Take $x \in U$, $y \in V$. By the assumption, there exists a sequence of points $x = x_0, x_1, \dots x_{r-1}, x_r = y$, satisfying that for each $i \in \{1, \dots, r\}$, there is a continuous map $\varphi_i : \mathbb{A}^1(\mathbf{k}) \to X$, satisfying that $\varphi_i(0) = x_{i-1}$, $\varphi_i(1) = x_i$. Let $C_i$ be $\mathrm{Im}\varphi_i \subset X$. Let $i \in \{1, \dots, r\}$ be the maximal satisfying that $p_{i-1} \in U$, so $p_i \in V$. Therefore, $U \cap C_i \neq \varnothing$, $V \cap C_i \neq \varnothing$. Since $\varphi_i$ is continuous, $\varphi_i^{-1}(U)$, $\varphi_i^{-1}(V) \subset \mathbb{A}^1(\mathbf{k})$ are nonempty open subsets and $\varphi_i^{-1}(U) \cap \varphi_i^{-1}(V) = \varnothing$, and $\varphi_i^{-1}(U) \cup \varphi_i^{-1}(V) = \mathbb{A}^1(\mathbf{k})$, which contradicts the connectedness of $A^1(\mathbf{k})$. Therefore, a continuously rational-chain connected projective variety is connected. □

Now, we will prove the final theorem of connectedness of $R^{r,d}_{k,\beta}(\mathbf{k})$.

**Theorem 4.** *If $R^{r,d}_{k,\beta}(\mathbf{k})$ is nonempty, then it is continuously rational-chain connected.*

*Proof* Let $S$, $S_1$ be configurations in $R^{r,d}_{k,\beta}(\mathbf{k})$. Let $p'$ be a point lying neither in $S$ nor in $S_1$. Let $P = \{p', \dots, p'\} \in \mathbb{P}^r(\mathbf{k})^{(d)}$. Then, $P \in R^{r,d}_{k,\beta}(\mathbf{k})$, since $R^{r,d}_{k,\beta}(\mathbf{k})$ is nonempty.

**Claim.** *There exist a continuous map $f_i : \mathbf{A}^1(\mathbf{k}) \to R^{r,d}_{k,\beta}(\mathbf{k})$, satisfying that $f_i(0) = S_i$, $f_i(1) = P$.*

*Proof* We will first construct a map

$$\begin{array}{cccc} f: & \mathbb{A}^1(\mathbf{k}) & \to & \mathbb{P}^r(\mathbf{k})^{(d)} \\ & t & \mapsto & S_t \end{array}$$

$S_t$ will be defined later in this part. The map $f$ factors through the following commutative diagram defining $\tilde{f}$:

$$\begin{array}{ccc} \mathbb{A}^1(\mathbf{k}) & \xrightarrow{\tilde{f}} & \mathbb{P}^r(\mathbf{k})^d \\ \downarrow f & \swarrow \pi & \\ \mathbb{P}^r(\mathbf{k})^{(d)} & & \end{array},$$

which means that $f = \pi \circ \tilde{f}$. We will still use the configuration $S$ defined above in the proof. $S = (p_1, \dots, p_d) \in \mathbb{P}^r(\mathbf{k})^{(d)}$, where $(p_1, \dots, p_d)$ is an unordered $d$-uple of points in $\mathbb{P}^r(\mathbf{k})$. Let $\tilde{S} = (p_1, \dots, p_d) \in \mathbb{P}^r(\mathbf{k})^d$ be an ordered $d$-uple corresponding to the same points of $S$. There are $d!$ ways of arrangement for $\tilde{S}$, and we just need to randomly choose one of them. Let $S_t = (p_1(t), \dots, p_d(t)) \in \mathbb{P}^r(\mathbf{k})^{(d)}$, where $p_1(0) = p_i$, $p_i(1) = p'$, for each $i \in \{1, \dots, d\}$. Then, we can similarly define $\tilde{S}_t$ as $S_t = (p_1(t), \dots, p_d(t)) \in \mathbb{P}^r(\mathbf{k})^d$ which corresponds the points of $S$. Then, we will get the new map:

$$\begin{array}{cccc} \tilde{f}: & \mathbb{A}^1(\mathbf{k}) & \to & \mathbb{P}^r(\mathbf{k})^d \\ & t & \mapsto & \tilde{S}_t \end{array}$$

Then, we want to see the relationship between $f$ and $\tilde{f}$. For any open sets $V \subset \mathbb{P}^r(\mathbf{k})^{(d)}$, $f^{-1}(V) = (\pi \circ \tilde{f})^{-1}(V) = \tilde{f}^{-1}(\pi^{-1}(V))$. With the respect to definition 7, $\pi^{-1}(V)$ is open. Therefore, we can say that $f$ is continuous if and only if $\tilde{f}$ is continuous. Then, we will start to show that $\tilde{f}$ is continuous. Due to lemma 2.1, it suffices to show the following map is continuous:

$$\begin{array}{rccc} \tilde{f}_i: & \mathbb{A}^1(\mathbf{k}) & \to & \mathbb{P}^r(\mathbf{k}) \\ & t & \mapsto & p_i(t) \end{array}$$

Let $U = \mathbb{P}^r(\mathbf{k}) - H$, where $H$ is a hyperplane in $\mathbb{P}^r(\mathbf{k})$, satisfying that $\mathbb{P}^r(\mathbf{k}) - H$ contains all the points of $S$, $S_1$ and $P$. Up to a projective transformation, we may assume that the defining equation of $H$ is $\{x_0 = 0\}$, so here is a canonical isomorphism that $U \cong \mathbb{A}^r(\mathbf{k})$:

$$[x_0 : \ldots : x_r] \mapsto (\frac{x_1}{x_0}, \ldots \frac{x_r}{x_0}).$$

Hence, it suffices to show the following map is continuous:

$$\begin{array}{rccc} \tilde{f}_i: & \mathbb{A}^1(\mathbf{k}) & \to & U \cong \mathbb{A}^r(\mathbf{k}) \\ & t & \mapsto & tp' + (1-t)p_i \end{array}$$

Since $p'$ is chosen arbitrarily, we can let $p'$ be the original point in $U$ after the isomorphism above. Therefore, the map $\tilde{f}_i$ can be represented as following:

$$\begin{array}{rccc} \tilde{f}_i: & \mathbb{A}^1(\mathbf{k}) & \to & U \cong \mathbb{A}^r(\mathbf{k}) \\ & t & \mapsto & (1-t)p_i \end{array}$$

We can see clearly that $\tilde{f}_i$ is a polynomial map from an affine space to affine space. Due to Lemma 2.1, the map $\tilde{f}_i$ is continuous. Therefore, $f$ is a continuous map, satisfying that $f(0) = S$, $f(1) = P$.

**Claim.** *For each $t \in \mathbb{A}^1(\mathbf{k})$, $S_t \in R^{r,d}_{k,\beta}(\mathbf{k})$.*

*Proof* We first prove that when $t \neq 1$, $i(k,S) = i(k,S_t)$. In fact, for $t \neq 1$, there is a projective transformation

$$\begin{array}{rccc} T_t: & \mathbb{P}^r & \to & \mathbb{P}^r \\ & [x_0 : \ldots : x_r] & \mapsto & [x_0 : (1-t)x_1 : \ldots : (1-t)x_r] \end{array}$$

that sends $S$ onto $S_t$. Hence, $S$ and $S_t$ are projectively equivalent. Therefore, $i(k,S) = i(k,S_t)$, which implies that when $t \neq 1$, $S_t \in R^{r,d}_{k,\beta}(\mathbf{k})$. When $t = 1$, $S_t = P$. Due to the proof of Theorem 3, it is easy to see that $P$ is in $R^{r,d}_{k,\beta}(\mathbf{k})$, Therefore, for each $t \in \mathbb{A}^1(\mathbf{k})$, $S_t \in R^{r,d}_{k,\beta}(\mathbf{k})$. □

In fact, there exists a continuous map $f_i : \mathbf{A}^1(\mathbf{k}) \to R^{r,d}_{k,\beta}(\mathbf{k})$, satisfying that $f_i(0) = S_i$, $f_i(1) = P$, which is a segment of $f$. □

With the claim, for $S$ and $S_1$, there is a continuous map $f_1 : \mathbb{A}^1(\mathbf{k}) \to R^{r,d}_{k,\beta}(\mathbf{k})$, satisfying that $f_1(0) = S$, $f_1(1) = P$, and a continuous map $f_2 : \mathbb{A}^1(\mathbf{k}) \to R^{r,d}_{k,\beta}(\mathbf{k})$, satisfying that $f_2(0) = P$, $f_2(1) = S_1$. Therefore, $R^{r,d}_{k,\beta}(\mathbf{k})$ is continuously rational-chain connected. □

**Corollary 1.** *If $R^{r,d}_{k,\beta}(\mathbf{k})$ is nonempty, it is connected.*

*Proof* It is a direct consequence of Proposition 5 and Theorem 4 □

# 4 Counting results

In this section, we estimate and calculate the number of elements in $R^{r,d}_{k,\beta}(\mathbb{F}_q)$ for a finite field $\mathbb{F}_q$ with $q$ elements.

## 4.1 Points in the general position

In this section, we will first introduce a general estimate of the number of points in general position. Then, we will show the accurate number of points in the general position in some special cases.

**Definition 10.** *Let $S = (p_1, \ldots, p_d) \in \mathbb{P}^r(\mathbb{F}_q)^{(d)}$ be a configuration. S is called* in the general position*, if the points $p_1, \ldots, p_d$ are distinct, and that for any $k \leq r+1$, any k of the d points do* not *lie on a linear subspace of dimension $k-2$.*

Let us first estimate the number of configurations in $\mathbb{P}^r(\mathbb{F}_q)^{(d)}$ that are in the general position. In order to show the result more clearly, we define two sequences of natural numbers $\{\alpha_d\}$ and $\{\beta_d\}$ recursively as follows. For $d \leq r+1$, $\alpha_d = \beta_d = \frac{(q^{r+1}-1)\ldots(q^{r+1}-q^{d-1})}{(q-1)^d}$. For $d > r+1$, $\alpha_d := \alpha_{d-1}.\max\{\frac{q^{r+1}-1}{q-1} - \binom{d-1}{r}\frac{q^r-1}{q-1}, 0\}$ and $\beta_d := \beta_{d-1}.(\frac{q^{r+1}-1}{q-1} - \binom{d-1}{r}\frac{q^r-1}{q-1} + \binom{\binom{d-1}{r}}{2}\frac{q^{r-1}-1}{q-1})$.

**Proposition 6.** *Let $r \geq 2$ and $d \geq 1$ be natural numbers. Let $\gamma_d$ be the number of configurations $S \in \mathbb{P}^r(\mathbb{F}_q)^{(d)}$ that are in the general position. Then we have*

$$\frac{\alpha_d}{d!} \leq \gamma_d \leq \frac{\beta_d}{d!}.$$

*Proof*

We can consider the points in $\mathbb{P}^r(\mathbb{F}_q)$ as vectors in $\mathbb{F}^{r+1}$. We first need to count the numner of ordered $d$-uple of vectors $\{v_1, \ldots, v_d\}$ that are in the general position. When $d \leq r+1$, for the first vector can be chosen freely, i.e. $(q^{r+1}-1)$ choices, and for the second vector, to keep the principal of generality, there are $(q^{r+1}-q)$ choices. Similarly, there are $(q^{r+1}-q^{d-1})$ choices for $d$-th chosen vector. Since in projective vector space, there are $(q-1)$ non-zero constants, we have to divide $(q-1)^d$. Therefore, for $d \leq r+1$, $\frac{\alpha_d}{d!} = \gamma_d = \frac{\beta_d}{d!}$. When $d > r+1$, for any subset $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, d\}$ of cardinal $r$, we define $H_I$ as the hyperplane generated by $\{v_{i_1}, \ldots v_{i_r}\}$. Suppose that the bound of $\gamma_{d-1}$ is known, let us estimate $\gamma_d$. Let $\{v_1, \ldots, v_{d-1}\}$ be a configuration in the general position that are chosen and fixed. We consider the choices of $v_d$. The assumption that $v_1, \ldots, v_{d-1}, v_d$ are in the general position forces that $v_d \in \mathbb{P}^r(\mathbb{F}_q)^{(d)} - \bigcup H_I$. Hence, we need to estimate the number of elements in $\mathbb{P}^r(\mathbb{F}_q)^{(d)} - \bigcup H_I$. By the inclusion-exclusion principle,

$$\sum_I |H_I| - \sum_{I \neq J} |H_I \cap H_J| \leq |\bigcup_I H_I| \leq \sum_I |H_I|.$$

Since $v_i$ are in general position, for different $I, J$, $H_I \neq H_J$, which implies that the dimension of $H_I \cap H_J$ is $r-2$. The number of elements in each $H_I \cap H_J$ is $\frac{q^{r-1}-1}{q-1}$ and the number of elements in each $H_I$ is $\frac{q^r-1}{q-1}$. Hence, we have

$$\binom{d-1}{r}\frac{q^r-1}{q-1} - \binom{\binom{d-1}{r}}{2}\frac{q^{r-1}-1}{q-1} \leq |\bigcup_I H_I| \leq \binom{d-1}{r}\frac{q^r-1}{q-1}.$$

Therefore, the number of choices of $v_d$ is greater than or equal to $\frac{q^{r+1}-1}{q-1} - \binom{d-1}{r}\frac{q^r-1}{q-1}$ and less than or equal to $\frac{q^{r+1}-1}{q-1} - \binom{d-1}{r}\frac{q^r-1}{q-1} + \binom{\binom{d-1}{r}}{2}\frac{q^{r-1}-1}{q-1}$. Our desired result follows. $\square$

Since it is easy to follow the sequence $\alpha$ and $\beta$, we will just show few of them in the chart for convenience. Readers can compare the estimates we make and the accurate result (Theorem 5) calculated in [3], [4], [7], [8]. In following case, we suppose $\alpha_d \neq 0$.

| $d$ | lower bound($\frac{\alpha_d}{d!}$) | upper bound($\frac{\beta_d}{d!}$) |
|---|---|---|
| 1 | $q^2+q+1$ | $q^2+q+1$ |
| 2 | $\frac{1}{2}(q^2+q+1)(q^2+q)$ | $\frac{1}{2}(q^2+q+1)(q^2+q)$ |
| 3 | $\frac{1}{6}(q^2+q+1)(q^2+q)q^2$ | $\frac{1}{6}(q^2+q+1)(q^2+q)q^2$ |
| 4 | $\frac{1}{24}(q^2+q+1)(q^2+q)q^2(q^2-3q-3)$ | $\frac{1}{24}(q^2+q+1)(q^2+q)q^2(q^2-3q+3)$ |

As can be already seen from the proof, the estimate in Proposition 6 is quite coarse. The main difficulty that prevents us to give better estimations is the increasing complexity of the configurations of hyperplanes in $\mathbb{P}^r(\mathbb{F}_q)$ when the number of the hyperplanes increases. However, explicit expression of the number of configurations in $P^r(\mathbb{F}_q)^{(d)}$ that are in the general position is conceivable when $r$ and $d$ are relatively small. This is indeed the case. Here we collect the results in [3], [4], [7], [8] that calculate the number of configurations in the general position in $\mathbb{P}^2(\mathbb{F}_q)^{(d)}$ for $d < 10$. To state the theorem, let us first introduce some notation.

$$a(q) = \begin{cases} 1 & \text{if } q \text{ is a power of } 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\begin{aligned} b(q) &= |\{x \in \mathbb{F}_q | x^2+x+1=0\}| \\ &= \begin{cases} 0 & \text{if } p \equiv -1 \bmod 3 \text{ and } n \text{ odd} \\ 1 & \text{if } p = 3 \\ 2 & \text{otherwise} \end{cases}, \end{aligned}$$

$$c(q) = \begin{cases} 1 & \text{if } q \text{ is a power of } 3 \\ 0 & \text{otherwise} \end{cases},$$

$$d(q) = |\{x \in \mathbb{F}_q | x^2 + x - 1 = 0\}| = \begin{cases} 0 & \text{if } p \equiv \pm 2 \bmod 5 \text{ and } s \text{ is odd} \\ 1 & p=5 \\ 2 & \text{otherwise} \end{cases},$$

$$e(q) = |\{x \in \mathbb{F}_q | x^2 + 1 = 0\}| = \begin{cases} 0 & \text{if } p \equiv -1 \bmod 4 \text{ and } s \text{ is odd} \\ 1 & p=2 \\ 2 & \text{otherwise} \end{cases}.$$

The counting results go as follows.

**Theorem 5** ([3, 4, 7, 8])**.** *Let $\gamma_d$ be the number of points $S \in \mathbb{P}^2(\mathbb{F}_q)^{(d)}$ which are in the general position. Then*
*(1)* $\gamma_1 = q^2 + q + 1$.
*(2)* $\gamma_2 = \frac{1}{2}(q^2+q+1)(q^2+q)$.
*(3)* $\gamma_3 = \frac{1}{6}(q^2+q+1)(q^2+q)q^2$
*(4)* $\gamma_4 = \frac{1}{4!}(q^2+q+1)(q^2+q)q^2(q-1)^2$
*(5)* $\gamma_5 = \frac{1}{5!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q^2-5q+6)$
*(6)* $\gamma_6 = \frac{1}{6!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q^2-5q+6)(q^2-9q+21)$
*(7)* $\gamma_7 = \frac{1}{7!}(q^2+q+1)(q^2+q)q^2(q-1)^2((q-3)(q-5)(q^4-20q^3+148q^2-468q+498)-30a(q))$
*(8)* $\gamma_8 = \frac{1}{8!}(q^2+q+1)(q^2+q)q^2(q-1)^2((q-5)(q^7-43q^6+788q^5-7937q^4+47097q^3-162834q^2+299280q-222960)-240(q^2-20q+78)a(q)+840b(q))$
*(9)* $\gamma_9 = \frac{1}{9!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q^{10}-75q^9+2530q^8-50466q^7+657739q^6-5835825q^5+35563770q^4-146288034q^3+386490120q^2-588513120q+389442480-1080(q^4-47q^3+807q^2-5921q+15134)a(q)+840(9q^2-243q+1684)b(q)+30240(-9c(q)+9d(q)+2e(q)))$.

## 4.2 Estimates of the number of elements in $R^{r,d}_{k,\beta}(\mathbb{F}_q)$

In order to estimate the number of elements in $R^{r,d}_{k,\beta}(\mathbb{F}_q)$, we should find the condition to use our method to take the estimate.

Let $\mathbf{k}$ be a field, let $V$ be vector space of dimension $r+1$. Let $S = \{v_1, v_2, \ldots, v_d\}$ be a subset of $V$. Let $k$ be a positive integer. Define a map:

$$\begin{array}{rccc} \psi_S : & Sym^k V^* & \to & \mathbf{k}^d \\ & f & \mapsto & (f(v_1), f(v_2), \ldots, f(v_d)), \end{array}$$

where $f \in Sym^k V^*$ is viewed as a degree $k$ homogeneous polynomial on $V$ and $f(v_i)$ is the value of the polynomial $f$ at $v_i \in V$.

**Theorem 6.** *Assume that S is in the general position. Then rank* $\psi_S \geq min\{d, kr+1\}$

*Proof* First assume that $d \leq kr+1$. We need to show that $\psi_s$ is surjective in this case.

**Lemma 4.** *There exists* $f_1 \in Sym^k V^*$ *such that* $f_1(v_1) = 1$ *and* $f_1(v_i) = 0$ *for any* $i = 2, 3, \dots, d$.

*Proof of Lemma 4.* Since the number of elements of $\{v_2, v_3, \dots, v_d\}$ is less than or equal to $kr$ by our assumption on $d$, we can form $k$ groups of $r$ vectors $G_1, \dots, G_k \subset \{v_2, \dots, v_d\}$ satisfying: every vector of $\{v_2, \dots, v_d\}$ lies in at least one of $G_1, \dots, G_k$. Then the the assumption that $S$ is in the general position says that any $r+1$ vectors in $\{v_1, \dots, v_d\}$ are linearly independent, which implies that $v_1$ is *not* in the span of $G_i$ for any $i = 1, 2, \dots, k$. Therefore, for any $i$, there exists a linear map $L_i : V \to k$ such that $L_i(v_1) = 1$ and $L_i | G_i = 0$. Take $f_1 = L_1 L_2 \dots L_k \in Sym^k V^*$. Then $f_1(v_1) = L_1(v_1) L_2(v_1) \dots L_k(v_1) = 1$ and $f_1(v_i) = L_1(v_i) L_2(v_i) \dots L_k(v_i) = 0$ for $i = 2, 3 \dots, d$. □

Similarly, for any $i = 1, 2, \dots, d$, there exists $f_i \in Sym^k V^*$ satisfying $f_i(v_i) = 1, f_i(v_j) = 0$ for $j \neq i$. Now we can show that $\psi_S : Sym^k V^* \to \mathbf{k}^d$ is surjective. Let $(y_1, \dots, y_d) \in \mathbf{k}^d$ be an arbitrary element. We define $f := y_1 f_1 + y_2 f_2 + \dots + y_d f_d$. Then we have

$$f(v_i) = y_1 f_1(v_i) + \dots + y_d f_d(v_i) = y_i \text{ for any } i = 1, \dots, d.$$

Hence, we have proved the surjectivity of $\psi_S$ when $d \leq kr+1$. For $d > kr+1$, we need to show that rank$\psi_S \geq kr+1$ , in this case, take $S' = \{v_1, \dots, v_{kr+1}\} \subset S$ and

$$\begin{array}{rccc} \psi_S : & Sym^k V^* & \to & \mathbf{k}^d \\ & f & \mapsto & (f(v_1), f(v_2), \dots, f(v_d)). \end{array}$$

$$\begin{array}{rccc} \psi_{S'} : & Sym^k V^* & \to & \mathbf{k}^{kr+1} \\ & f & \mapsto & (f(v_1), f(v_2), \dots, f(v_{kr+1})). \end{array}$$

Therefore, we have rank$\psi_S \geq$ rank$\psi_{S'} = kr+1$, where the last equality follows from the first case of the proof. □

**Corollary 2.** *Let* $S = (p_1, \dots, p_d) \in \mathbb{P}^r(\mathbf{k})^d$ *be a configuration that are in the general position (i.e. any* $r+1$ *of them are* not *in a hyperplane). The number of linearly independent hypersurfaces of degree k passing through S is less than or equal to* $\max\{\binom{k+r}{r} - d, \binom{k+r}{r} - kr - 1\}$, *which means that* $i(k,S) \leq \max\{\binom{k+r}{r} - d, \binom{k+r}{r} - kr - 1\}$.

*Proof.* Let $\mathbf{k}_k[x_0, \dots x_r]$ be the set of degree $k$ homogeneous polynomials in $x_0, \dots x_r$ with coefficients in the field $\mathbf{k}$. The dimension of $\mathbf{k}_k[x_0, \dots x_r]$ is $\binom{k+r}{r}$. Let $Y = V(f)$ be the hypersurface defined by a degree $k$ polynomial $f$. We can consider the the points $p_1, \dots, p_d$ as vectors $v_1, \dots, v_d$ discussed in Theorem 6. Obviously, $p_1, \dots, p_d \in Y$, if and only if the corresponding $v_i$, satisfies that $f(v_i) = 0$ for each $i$. Then define:

$$\begin{array}{rccc} \psi_{\{v_1, \dots, v_d\}} : & \mathbf{k}_k[x_0, \dots x_r] & \to & \mathbf{k}^d \\ & f & \mapsto & (f(v_1), f(v_2), \dots, f(v_d)). \end{array}$$

Then $Y = V(f)$ passes through $p_1, \ldots, p_d$, if and only if $f \in \ker\psi_{\{v_1,\ldots,v_d\}}$. Therefore, the number of linearly independent hypersurfaces of degree $k$ is equal to $\dim\ker\psi_{\{v_1,\ldots,v_d\}} = \dim \mathbf{k}_k[x_0,\ldots x_d] - \dim \operatorname{Im} \psi_{\{v_1,\ldots,v_d\}} = \binom{k+r}{r} - \operatorname{rank} \psi_{\{v_1,\ldots,v_d\}}$, which is less or equal to $\max\{\binom{k+r}{r} - d, \binom{k+r}{r} - kr - 1\}$.

□

Now we can prove readily Theorem 2 stated in the Introduction.

*Proof of Theorem 2.* It is a direct consequence of Proposition 6 and Corollary 2. □

When $r = 2$, we have the following better estimations.

**Proposition 7.** *Assume $\beta \geq max\{\frac{(k+2)(k+1)}{2} - d + 1, \frac{(k+2)(k+1)}{2} - 2k\}$.*
*(1)* $|R^{2,4}_{k,\beta}(\mathbb{F}_q)| \leq \binom{q(q+1)+4}{4} - \frac{1}{4!}(q^2+q+1)(q^2+q)q^2(q-1)^2$
*(2)* $|R^{2,5}_{k,\beta}(\mathbb{F}_q)| \leq \binom{q(q+1)+5}{5} - \frac{1}{5!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q^2-5q+6)$
*(3)* $|R^{2,6}_{k,\beta}(\mathbb{F}_q)| \leq \binom{q(q+1)+6}{6} - \frac{1}{6!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q^2-5q+6)(q^2-9q+21)$
*(4)* $|R^{2,7}_{k,\beta}(\mathbb{F}_q)| \leq \binom{q(q+1)+7}{7} - \frac{1}{7!}(q^2+q+1)(q^2+q)q^2(q-1)^2((q-3)(q-5)(q^4-20q^3+148q^2-468q+498)-30a(q))$
*(5)* $|R^{2,8}_{k,\beta}(\mathbb{F}_q)| \leq \binom{q(q+1)+8}{8} - \frac{1}{8!}(q^2+q+1)(q^2+q)q^2(q-1)^2((q-5)(q^7-43q^6+788q^5-7937q^4+47097q^3-162834q^2+299280q-222960)-240(q^2-20q+78)a(q)+840b(q))$
*(6)* $|R^{2,9}_{k,\beta}(\mathbb{F}_q)| \leq \binom{q(q+1)+9}{9} - \frac{1}{9!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q^{10}-75q^9+2530q^8-50466q^7+657739q^6-5835825q^5+35563770q^4-146288034q^3+386490120q^2-588513120q+389442480-1080(q^4-47q^3+807q^2-5921q+15134)a(q)+840(9q^2-243q+1684)b(q)+30240(-9c(q)+9d(q)+2e(q)))$.

*The functions $a(q)$, $b(q)$, $c(q)$, $d(q)$ and $e(q)$ are defined before Theorem 5.*

*Proof* It follows directly from Theorem 5 and Corollary 2. □

## 4.3 Explicit values

In fact, when $d$ is relatively small, we can calculate the explicit value of $|R^{r,d}_{k,\beta}(\mathbb{F}_q)|$. In this section, we calculate the value of $|R^{r,d}_{k,\beta}(\mathbb{F}_q)|$ when $d = 1, 2, 3$. We will still use the following map:

$$\begin{array}{rccc} \psi_{\{v_1,\ldots,v_d\}}: & \mathbf{k}_k[x_0,\ldots x_r] & \to & \mathbf{k}^d \\ & f & \mapsto & (f(v_1), f(v_2), \ldots, f(v_d)). \end{array}$$

$\boxed{d = 1}$

It is easy to know that for any $S$, $i(k,S) = \binom{k+r}{r} - 1$. Therefore

$$|R^{r,1}_{k,\beta}(\mathbb{F}_q)| = \begin{cases} 0 & \text{if } \beta > \binom{k+r}{r} - 1 \\ \frac{q^{r+1}-1}{q-1} & \text{if } \beta \leq \binom{k+r}{r} - 1 \end{cases}.$$

$\boxed{d=2}$

We can divide the problem into two possibilities–that the two points in the configuration are the same point, and that the two points in the configuration are distinct. First, we will talk about the condition that the two points are the same points, which means that configuration $S=\{p,\ p\}$, which is the same condition as when $d=1$. Thus, we have $i(k,S)=\binom{k+r}{r}-1$. Then, we consider the condition that the two points are distinct, which means that $S=\{p,\ q\}$ with $p\neq q$. In this case, it is easy to find that rank $\psi_{\{v_1,\ldots,v_d\}}$ is 2, which means that $i(k,S)=\binom{k+r}{r}-2$. Therefore, if $\beta\geq\binom{k+r}{r}$, there is no points in $R^{r,2}_{k,\beta}(\mathbb{F}_q)$. If $\beta=\binom{k+r}{r}-1$, $R^{r,2}_{k,\beta}(\mathbb{F}_q)$ contains the configurations of the form $\{p,p\}$ with $p\in\mathbb{P}^r(\mathbb{F}_q)$ and there are $\frac{q^{r+1}-1}{q-1}$ choices. If $\beta\leq\binom{k+r}{r}-2$, the set $R^{r,2}_{k,\beta}(\mathbb{F}_q)$ contains any possible configurations of two points. We have to choose 2 unordered points (not necessarily distinct) among $\frac{q^{r+1}-1}{q-1}=|\mathbb{P}^r(\mathbb{F}_q)|$. There are $\binom{\frac{q^{r+1}-q}{q-1}+2}{2}$ of them. In conclusion,

$$|R^{r,2}_{k,\beta}(\mathbb{F}_q)|=\begin{cases}0 & \text{if } \beta\geq\binom{k+r}{r}\\ \frac{q^{r+1}-1}{q-1} & \text{if } \beta=\binom{k+r}{r}-1\\ \binom{\frac{q^{r+1}-q}{q-1}+2}{2} & \text{if } \beta\leq\binom{k+r}{r}-2\end{cases}.$$

$\boxed{d=3}$

We will divide the situations into three parts–that the three points are the same, that two points are the same and one is different, and that the three points are distinct.

1. There are $\frac{q^{r+1}-1}{q-1}$ configurations representing three same points. In this case, as in the case of $d=1$, $i(k,S)=\binom{k+r}{r}-1$.
2. The case where two points are the same and one is different is the same with the case of $d=2$. In this case, there are $2\cdot\binom{\frac{q^{r+1}-1}{q-1}}{2}$ configurations whose $i(k,S)=\binom{k+r}{r}-2$.
3. Now we consider the case when the three points are distinct. In this case, we have the following lemma.

   **Lemma 5.** *If $k\geq 2$, then the map*

   $$\begin{array}{rccc}\psi_{\{v_1,v_2,v_3\}}: & \mathbf{k}_k[x_0,\ldots x_r] & \to & \mathbf{k}^3\\ & f & \mapsto & (f(v_1),f(v_2),f(v_3))\end{array}$$

   *is surjective. In particular, $i(k,S)=\binom{k+r}{r}-3$.*

   *Proof* When $k\geq 2$, for each $v_i\in S$, we have a hyperplane $L_i$, satisfying that $L_i(v_i)=0$, but $L_i(v_j)\neq 0$, for each $i\neq j$. Take $f_i=\Pi_{i\neq j}L_j$, which is a degree 2 homogeneous polynomial. Then, $\psi_{\{v_1,v_2,v_3\}}(f_1)=(a_1,0,0)$, $\psi_{\{v_1,v_2,v_3\}}(f_2)=(0,a_2,0)$, $\psi_{\{v_1,v_2,v_3\}}(f_3)=(0,0,a_3)$, where $a_1,a_2,a_3\neq 0$. It means that rank$\psi_{\{v_1,v_2,v_3\}}=3$. □

Hence, it remains to consider the case $k = 1$ and that the three points are disctinct. We need to divide this question into two subcases.

(a) If the three points lie on a same straight line $\ell$, then by Bézout's theorem, any hyperplane containing the three points must contain $\ell$. The number of linearly independent hyperplanes containing $\ell$ is $r-1$. To determine the number of configurations in this case, it is easy to know the choice of two distinct points in $\mathbb{P}^r(\mathbf{k})$ is $\binom{\frac{q^{r+1}-1}{q-1}}{2}$, and choice of another distinct point lying on the line determined by the two chosen points is $q+1-2 = q-1$, which means that the number of configuration in this condition is $\binom{\frac{q^{r+1}-1}{q-1}}{2}(q-1)$.

(b) If the three points are not on one line, we denote $P$ the plane containing these points. Let $H$ be a hyperplane containing these three points, we claim that $H$ contains $P$. In fact, if $P$ is not contained in $H$, then $P \cap H$ is a line. But these three points, which are in the intersection of $P$ and $H$, are not on one line. Hence, any hyperplane containing these three points must contain $P$. Therefore, we need to calculate the number of linearly independent hyperplanes which contain $P$, which is $i(k,S) = r-2$. The number of configurations in $\mathbb{P}^r(\mathbf{k})$ is $\binom{\frac{q^{r+1}-1}{q-1}}{3} - \binom{\frac{q^{r+1}-1}{q-1}}{2}(q-1)$.

By the discussion above, when $\beta \geq \binom{k+1}{r}$, there is no points in $R^{r,3}_{k,\beta}(\mathbb{F}_q)$. When $\beta = \binom{k+r}{r} - 1$, there are only one situation where $i(k,S) \geq \beta$, namely, the Case 1. There are $\frac{q^{r+1}-1}{q-1}$ configurations in this case. When $\beta = \binom{k+r}{r} - 2$, there are several cases where $i(k,S) \geq \beta$, namely, the Case 1, Case 2 and the Case 3a. Hence, $|R^{r,3}_{k,\beta}(\mathbb{F}_q)|$ is $\frac{q^{r+1}-1}{q-1} + 2 \cdot \binom{\frac{q^{r+1}-1}{q-1}}{2}$ when $k \geq 2$ and is $\frac{q^{r+1}-1}{q-1} + 2 \cdot \binom{\frac{q^{r+1}-1}{q-1}}{2} + \binom{\frac{q^{r+1}-1}{q-1}}{2}(q-1)$ when $k = 1$. Finally, for any configuration, we have $i(k,S) \geq \binom{k+r}{r} - 3$, by our discussion above. Hence, when $\beta \leq \binom{k+r}{r} - 3$, we have $|R^{r,3}_{k,\beta}(\mathbb{F}_q)| = \binom{\frac{q^{r+1}-1}{q-1}+2}{3}$. In conclusion,

$$|R^{r,3}_{k,\beta}(\mathbb{F}_q)| = \begin{cases} 0 & \text{if } \beta \geq \binom{k+r}{r} \\ \frac{q^{r+1}-1}{q-1} & \text{if } \beta = \binom{k+r}{r} - 1 \\ 2 \cdot \binom{\frac{q^{r+1}-1}{q-1}}{2} + \frac{q^{r+1}-q}{q-1} & \text{if } \beta = \binom{k+r}{r} - 2 \text{ and } k \geq 2 \\ 2 \cdot \binom{\frac{q^{r+1}-1}{q-1}}{2} + \frac{q^{r+1}-q}{q-1} + \binom{\frac{q^{r+1}-1}{q-1}}{2}(q-1) & \text{if } \beta = \binom{k+r}{r} - 2 \text{ and } k = 1 \\ \binom{\frac{q^{r+1}-1}{q-1}+2}{3} & \text{if } \beta \leq \binom{k+r}{r} - 3 \end{cases}.$$

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