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Research Report

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On The Coprime Product Series and Its Divergence and Bounds

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Abstract

It has been shown by L. Euler [1] that the sum of the reciprocals of all prime numbers diverges; and its growth is proportional to $\log\log n$. Others [2, 3] have shown that, in particular:

$$\sum_{p\leq n}\frac{1}{p}=\log\log n+M+O\left(\frac{1}{\log^2 n}\right),$$

where $M=0.261497...$ is a constant.
We estimate and prove an asymptotic formula on the reciprocals of the coprime series

$$\sum_{s_1 s_2}\frac{1}{s_1 s_2}$$

where the sum runs over natural numbers s_1 and s_2 that satisfy $(s_1, s_2)=1$ diverges, specifically,

$$\sum_{s_1 s_2\leq n}\frac{1}{s_1 s_2}=\frac{6}{\pi^2}\log^2 n+O(\log n)$$

and in general we conjecture that,

$$\sum_{s_1\cdots s_k\leq n}\frac{1}{s_1\cdots s_k}=\frac{1}{\zeta(k)}\log^k n+O(\log^{k-1} n)$$

where the sum runs over $s_1,\cdots,s_k$ with $(s_1,\cdots,s_k)=1$. This implies that that the density of coprime k-tuples over $\mathbb{N}^k$ is $1/\zeta(k)$.
Keywords: analytic number theory, series, coprime, divergence, bound, density, reciprocals

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On The Coprime Product Series and Its Divergence and Bounds

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1 Introduction

Prime numbers have been in the spotlight in Number Theory historically. In fact, the properties of the reciprocals of prime numbers are well known and thoroughly studied, such as their asymptotic growth and density. Another mathematically similar concept is coprimality (Two or more integers whose greatest common divisor is 1). As of date, however, not much attention is received on the reciprocals of products of coprime pairs, which seems like a "general" version of primes.
In this paper, we ask ourselves: How does the series of reciprocals products of coprime numbers asymptotically behave like? Precisely, can we derive an expression for $\sum_{(s_1,s_2)=1,\ 1\le s_1s_2\le n,\ s_1,s_2\in\mathbb{N}} 1/s_1s_2$, based on some known facts such as the probability of 2 randomly chosen integers being coprime is $1/\zeta(2)=6/\pi^2$? To answer this, we will start from some basic facts in prime numbers, and then discuss an asymptotic formula for the series of reciprocal products of coprimes, which holds several implications on the density of coprimes.

At the time of Euclid, the fact that there are an infinite number of prime numbers has been known. Since then, mathematicians have developed new approaches to tackle the problem.

In the 1730s, Leonhard Euler discovered a way [1] to prove that the sum of the reciprocals of all prime numbers diverges, with the key observation that the harmonic series, later recognized as $\lim_{s\to 1}\zeta(s)$, can be factored into a product of primes.

Take $\mathbb{P}$ to be the set of prime numbers.

To see why $\sum_{p\in\mathbb{P}} 1/p$ diverges, we begin with the Euler Product of the Riemann Zeta function,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\in\mathbb{P}}^{\infty}(1-p^{-s})^{-1}$$

Taking log on both sides, we have

$$\begin{aligned}\log\sum_{n=1}^{\infty}\frac{1}{n^s}&=-\sum_p\log(1-p^{-s})\\&=\sum_p\sum_{m=1}^{\infty}m^{-1}p^{-ms}\\&=\sum_p p^{-s}+\sum_p\sum_{m=2}^{\infty}m^{-1}p^{-ms}\end{aligned}$$

Note that

$$\sum_p\sum_{m=2}^{\infty}m^{-1}p^{-ms}<\sum_p\sum_{m=2}^{\infty}p^{-m}=\sum_p\frac{1}{p(p-1)}<1$$

It suffices to show that

$$\lim_{s\to 1}\sum_p p^{-s} \to \infty$$

since $\zeta(s) \to \infty$ as $s \to 1$.

Now, informally, since the harmonic series grow in the order of $\log n$, for very large n we have

$$\log \sum_{\substack{i\le n\\ i\in\mathbb{N}}} \frac{1}{i} \sim \log\log n$$

and so we expect $\sum_{p\le n} 1/p \sim \log\log n$ for large n.

In fact, there is an asymptotic formula proven by Mertens [2] that for the summation of reciprocals of primes up to n, we have

$$\sum_{p\le n}\frac{1}{p} = \log\log(n) + M + O\left(\frac{1}{\log n}\right) \tag{1}$$

where $M = \gamma + \sum_p(\log(1-1/p)+1/p) \approx 0.261497212847643$ is a constant and $\gamma \approx 0.577$ is the Euler-Mascheroni Constant.
The error term was improved by E. Landau [3] in 1909, in which by exploiting the prime number theorem he found that it can be further bounded to $O(e^{-(\log n)^{1/14}})$. Others [4] estimated it as

$$\left|\sum_{p\le n}\frac{1}{p} - (\log\log(n)+M)\right| \le \frac{1}{10\log^2 n} + \frac{4}{15\log^3 n}$$

Compared with primality, a closely related concept is coprimality. We say the **greatest common divisor** of a and b is d, if d is the largest integer d such that $d|a$ and $d|b$. We write this as $gcd(a,b) = d$, or if no ambiguity is present, $(a,b) = d$. We say two integers a and b are **coprime** if $d = 1$. It is natural to ask whether similar results can be established for a summation of the form

$$\sum_{\substack{1\le s_1,s_2\le n\\ (s_1,s_2)=1}} \frac{1}{s_1 s_2}, \tag{2}$$

Or further, for k integers $s_1, s_2, \cdots, s_k$ which are coprime (their greatest common divisor is 1), we are tempted to make the generalization:

$$\sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\ (s_1,s_2,\cdots,s_k)=1}} \frac{1}{s_1 s_2\cdots s_k} \tag{3}$$

Establishing such a formula for equation (2) will be one of the main goals in this paper.

An interesting interpretation of the asymptotic formula of the prime numbers is the concept of density, or loosely speaking, how "frequent" prime numbers appear in a set.

Definition 1. *The* **density** *of a set S (whose items are k-tuples of the form $(s_1, s_2, ..., s_k)$ where each s_i is a positive integer) over $\mathbb{N}^n$ is written as $\boldsymbol{\rho}_{\mathbb{N}^n} S$ (or simply $\boldsymbol{\rho} S$) and defined as*

$$\lim_{n\to\infty} \frac{\#\{s \in S : s_i \leq n \text{ for all } 0 < i \leq k\}}{n^k}.$$

Example 1. *The density of even numbers over $\mathbb{N}$ is $1/2$, and the density of $\mathbb{P}$ over $\mathbb{N}$ is 0.*

By the explicit formula proved in Merten's Second Theorem, a straightforward implication is that the prime numbers has density 0 in the natural numbers, since

$$\lim_{n\to\infty} \frac{\log\log n}{n} = 0.$$

2 Asymptotic Formula of the Sum of Reciprocal Products of Two Coprime Numbers

In this section, our goal is to make an educated guess on the rate of divergence (growth of partial sums) of the sum of reciprocals of coprime series, an analog for the reciprocals of primes, and explain the motivation behind.

2.1 Divergence of the Series

The divergence of

$$\sum_{\substack{1\leq s_1,s_2\leq n\\(s_1,s_2)=1}} \frac{1}{s_1 s_2}$$

can be seen as a corollary to the divergence of the harmonic series.

Proposition 1.

$$\sum_{\substack{1\leq s_1,s_2\leq n\\(s_1,s_2)=1}} \frac{1}{s_1 s_2} \to \infty \text{ as } n \to \infty.$$

Proof.

$$\sum_{\substack{1\leq s_1,s_2\leq n\\(s_1,s_2)}} \frac{1}{s_1 s_2} \geq \sum_{1\leq n} \frac{1}{n}$$

by letting $s_2 = 1$. But the sum on the right diverges as $n \to \infty$, so the sum on the left diverges as well. □

2.2 An Estimate of the Series

Now that we have shown that the above series diverges, we can set off to make our estimation based on a few observations.

The probability that a number is divisible by any prime p is $1/p$. Hence, the probability for two numbers to be both divisible by p is $1/p^2$, and $1 - 1/p^2$ for at least one to not be divisible by p. Because two numbers are coprime if they share no common divisors, and on top of that we know the probability that any finite collection of divisibility events associated with distinct primes is mutually independent, we have

$$\Pr(\text{two integers are coprime}) = \prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^2}\right) = \left(\prod_{p\in\mathbb{P}}(1-p^{-2})^{-1}\right)^{-1} = \zeta(2)^{-1}$$

$$\Pr(\text{two integers are coprime}) = \frac{1}{\zeta(2)} \tag{4}$$

Another observation is that the harmonic series $\sum_{i=1}^{n} 1/i \approx \log n + \gamma$, where $\gamma = 0.577...$ is the Euler-Mascheroni constant, so combining the two, it is reasonable to have the following estimate:

$$\begin{aligned}\sum_{\substack{1\le s_1,s_2\le n\\(s_1,s_2)=1}} \frac{1}{s_1 s_2} &\approx \frac{1}{\zeta(2)}\sum_{s_1=1}^{n}\frac{1}{s_1}\sum_{s_2=1}^{n}\frac{1}{s_2}\\ &\approx \frac{1}{\zeta(2)}(\log^2 n + 2\gamma\log n + \gamma^2)\\ &\approx \frac{1}{\zeta(2)}\log^2 n + \text{constant}\cdot\log n\end{aligned}$$

Therefore, we conjecture that

Theorem 1.
$$\sum_{\substack{1\le s_1,s_2\le n\\(s_1,s_2)=1}} \frac{1}{s_1 s_2} = \frac{1}{\zeta(2)}\log^2 n + O(\log n).$$

After checking our estimate using direct computation, it further suggests that the error term is $O(\log n)$. Refer to Section 5 for detailed figures.

In the next section, we attempt to prove this conjecture rigorously.

3 Proof of the Asymptotic Formula

Before we show the following results, we shall define the Möbius μ and Euler φ functions.

Definition 2. *Let $n = p_1^{a_1}p_2^{a_2}\dots p_k^{a_k}$ be the prime factorization of n.*
(a) The **Möbius function***, $\mu(n)$, is defined as*

$$\mu(1) = 1$$
$$\mu(n) = \begin{cases}(-1)^k & \text{if } a_i = 1 \text{ for all } 1\le i\le k\\ 0 & \text{otherwise.}\end{cases}$$

(b) The **Euler totient function***, $\varphi(n)$, is defined as the number of positive integers not exceeding n which are coprime to n.*

To begin with, we first establish a few theorems and lemmas, taken from M. Ram Murty's *Problems in Analytic Number Theory* [4].

Theorem 2. *For all $n \geq 1$,*

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1. \end{cases}$$

In other words, the sum is a characteristic function of the number 1.

Proof. When $n = 1$, the statement is clearly true. However, when $n > 1$, let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ as above. Notice that $\mu(d) \neq 0$ if and only if $d = 1$ or a product of distinct primes. Therefore

$$\begin{aligned} \sum_{d|n} \mu(d) &= \mu(1) + \mu(p_1) + \cdots + \mu(p_k) + \mu(p_1p_2) + \cdots + \mu(p_{k-1}p_k) \\ &\quad + \cdots + \mu(p_1p_2 \cdots p_k) \\ &= 1 + \binom{k}{1}(-1) + \binom{k}{2}(-1)^2 + \cdots + \binom{k}{k}(-1)^k \\ &= (1-1)^k \\ &= 0. \end{aligned}$$

□

Lemma 1. *For all $n \geq 1$,*

$$\sum_{d|n} \varphi(d) = n.$$

Proof. Let S be the set $\{1, 2, \cdots, n\}$, and for each d that divides n, let

$$A(d) = \{k \mid (k, n) = d, 1 \leq k \leq n\}.$$

All such $A(d)$ are disjoint, and their union is precisely S. Now $\#A(d) = \varphi(n/d)$, which is the number of k's such that $(k/d, n/d) = 1$. Therefore

$$\sum_{d|n} \varphi(n/d) = \sum_{d|n} \varphi(d) = n.$$

□

Lemma 2.

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

Proof. Suppose

$$f(n) = \sum_{d|n} \mu(d)(1/d).$$

Then

$$\begin{aligned}
\sum_{d|n} f(d) &= \sum_{d|n}\sum_{e|d}(d/e)\mu(e) \\
&= \sum_{est=n} s\mu(e) \\
&= \sum_{s|n} s \sum_{e|\frac{n}{s}} \mu(e) \\
&= n \qquad\qquad \text{(by Theorem 2)}.
\end{aligned}$$

By Lemma 1, letting f be φ gives us the required formula. ☐

Lemma 3.

$$\sum_{\substack{n\le x \\ (n,k)=1}} \frac{1}{n} \sim \frac{\varphi(k)}{k}\log x, \text{ for } x \to \infty.$$

Proof. By Theorem 2,

$$\begin{aligned}
\sum_{\substack{n\le x \\ (n,k)=1}} \frac{1}{n} &= \sum_{n\le x} \frac{1}{n} \sum_{d|(n,k)} \mu(d) \\
&= \sum_{d|k} \mu(d) \sum_{\substack{n\le x \\ d|n}} \frac{1}{n} \\
&= \sum_{d|k} \frac{\mu(d)}{d} \sum_{t\le x/d} \frac{1}{t} \\
&= \sum_{d|k} \frac{\mu(d)}{d} \left(\log\frac{x}{d} + O(1)\right),
\end{aligned}$$

by the asymptotic formula for the harmonic series. Therefore,

$$\begin{aligned}
\sum_{d|k} \frac{\mu(d)}{d}\left(\log\frac{x}{d} + O(1)\right) &= \sum_{d|k} \frac{\mu(d)}{d}\log x - \sum_{d|k}\frac{\mu(d)}{d}(\log d + O(1)) \\
&= \sum_{d|k} \frac{\mu(d)}{d}\log x + O(1) \\
&= \frac{\varphi(k)}{k}\log x + O(1) \qquad \text{(by Lemma 2)}
\end{aligned}$$

☐

Lemma 4.

$$\sum_{k=1}^{n} \varphi(k) = \frac{3n^2}{\pi^2} + O(n\log n).$$

Proof. See [6]. ☐

Lemma 5. *(Abel's summation formula) Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $f(t)$ is a continuously differentiable function on $[1, x]$. If*

$$A(t) = \sum_{n \leq t} a_n,$$

then

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

Proof. See [5], pp. 17-18. □

We are equipped with enough tools to prove Theorem 1.

Proof. (Theorem 1)

$$\begin{aligned}
\sum_{\substack{1 \leq s_1, s_2 \leq n \\ (s_1, s_2)=1}} \frac{1}{s_1 s_2} &= \sum_{s_1=1}^{n} \sum_{\substack{s_2 \leq n \\ (s_1, s_2)=1}} \frac{1}{s_1 s_2} \\
&= \sum_{s_1=1}^{n} \frac{1}{s_1} \sum_{\substack{s_2 \leq n \\ (s_1, s_2)=1}} \frac{1}{s_2} \\
&\sim \sum_{s_1=1}^{n} \frac{1}{s_1} \frac{\varphi(s_1)}{s_1} \log n \\
&= \log n \sum_{s_1=1}^{n} \frac{\varphi(s_1)}{s_1^2}
\end{aligned}$$

By Lemmas 4 and 5,

$$\begin{aligned}
\sum_{s=1}^{n} \frac{\varphi(s)}{s^2} &= \frac{3n^2}{\pi^2 n^2} - \int_1^n \left(\frac{3t^2}{\pi^2} + O(t \log t)\right) \left(\frac{-2}{t^3}\right) dt + O\left(\frac{\log n}{n}\right) \\
&= \frac{3}{\pi^2} + \int_1^n \left(\frac{6}{\pi^2 t} + O\left(\frac{\log t}{t^2}\right)\right) dt + O\left(\frac{\log n}{n}\right) \\
&= \frac{3 + 6\log(n)}{\pi^2} + O\left(\frac{\log n}{n}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{\substack{1 \leq s_1, s_2 \leq n \\ (s_1, s_2)=1}} \frac{1}{s_1 s_2} &= \log n \left(\frac{3 + 6\log(n)}{\pi^2} + O\left(\frac{\log n}{n}\right)\right) \\
&= \frac{6}{\pi^2} \log^2 n + \frac{3}{\pi^2} \log n + O\left(\frac{\log^2 n}{n}\right) \\
&= \frac{1}{\zeta(2)} \log^2 n + O(\log n)
\end{aligned}$$

□

See Figure 5.3 for a computed upper bound for the constant of proportionality of the $O(\log n)$ term.

This theorem immediately implies a generalization of the condition on $(s_1, s_2) = m$:

Corollary 1.

$$\sum_{\substack{1\le s_1,s_2\le n\\(s_1,s_2)=m}} \frac{1}{s_1 s_2} = \frac{1}{\zeta(2)}\frac{\log^2 n}{m^2} + O(\log n)$$

Proof. Writing $s_1 = mr_1$ and $s_2 = mr_2$,

$$\begin{aligned}
\sum_{\substack{1\le s_1,s_2\le n\\(s_1,s_2)=m}} \frac{1}{s_1 s_2} &= \frac{1}{m^2} \sum_{\substack{1\le r_1,r_2\le n/m\\(r_1,r_2)=1}} \frac{1}{r_1 r_2} \\
&= \frac{1}{m^2}\left(\frac{1}{\zeta(2)}\log^2\frac{n}{m} + O\left(\log\frac{n}{m}\right)\right) \\
&= \frac{1}{\zeta(2)}\frac{1}{m^2}\left(\log^2 n - \log m \log n + \log^2 m\right) + O(\log n) \\
&= \frac{1}{\zeta(2)}\frac{\log^2 n}{m^2} - \text{constant}\cdot\log n + \text{constant} + O(\log n) \\
&= \frac{1}{\zeta(2)}\frac{\log^2 n}{m^2} + O(\log n)
\end{aligned}$$

□

4 Discussion and Conclusion

4.1 Sum of Reciprocals of k-tuple Coprime Numbers

In view of Proposition 1, it is easy to see that sum of k-tuples of reciprocals of coprime integers $s_1, s_2, \cdots, s_k$ also diverges, since the summands in

$$\sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\(s_1,s_2,\cdots,s_k)=1}} \frac{1}{s_1 s_2\cdots s_k} \text{ coincide with } \sum_{\substack{1\le s_1,s_2\le n\\(s_1,s_2)=m}} \frac{1}{s_1 s_2}, \text{ by letting } s_3, \cdots, s_k = 1.$$

However, determining the rate of divergence is not as easy. Based on the results proven above, it is natural to conjecture on the formula of the general case of Theorem 1 for the k-tuple sum.

The argument of the probability of coprimality can be extended to k integers similar to Equation (4), and is shown [7] to be

$$\Pr(\text{k integers are coprime}) = \frac{1}{\zeta(k)} \tag{5}$$

Therefore, we propose that

Conjecture 1.

$$\sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\(s_1,s_2,\cdots,s_k)=1}} \frac{1}{s_1 s_2\cdots s_k} = \frac{1}{\zeta(k)}\log^k n + O(\log^{k-1} n)$$

It is not hard to see why this conjecture is possibly true. Informally, we have

$$\begin{aligned}\sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\(s_1,s_2,\cdots,s_k)=1}} \frac{1}{s_1 s_2\cdots s_k} &\approx \frac{1}{\zeta(k)}\prod_{i}^{k}\sum_{s_i=1}^{n}\frac{1}{s_i}\\ &\approx \frac{1}{\zeta(k)}(\log n+\gamma)^k\\ &= \frac{1}{\zeta(k)}\left(\sum_{r=0}^{k}\binom{k}{r}\gamma^{k-r}\log^r n\right)\\ &\approx \frac{1}{\zeta(k)}\log^k n + O(\log^{k-1} n)\end{aligned}$$

Numerically, we also check that for the $k=3$ case, the plots seem to match the prediction that the sum has an error of magnitude $\log^2 n$. For details see Section 5.

4.2 Consequences of the Asymptotic Formula

Some interesting consequences arise if we infer that equation (6) is true. For instance, if we generalize the notion of greatest common divisors, we have

$$\sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\(s_1,s_2,\cdots,s_k)=m}} \frac{1}{s_1 s_2\cdots s_k} = \frac{1}{\zeta(k)}\frac{\log^k n}{m^k} + O(\log^{k-1} n) \tag{6}$$

It sheds light on why $\frac{1}{\zeta(k)}$ seems to be the only suitable candidate to be the coefficient of Conjecture 1. If we know that for some constant A,

$$\sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\(s_1,s_2,\cdots,s_k)=m}} \frac{1}{s_1 s_2\cdots s_k} = A\cdot\frac{\log^k n}{m^k} + O(\log^{k-1} n)$$

The product of k-harmonic sums gives the sum of all possible combinations of $\frac{1}{s_1 s_2\cdots s_k}$. By grouping them based on their greatest common divisors $(s_1,s_2,\cdots,s_k)$, we have

$$\begin{aligned}\log^k n \approx \prod_{i=1}^{k}\left(\sum_{s_i}\frac{1}{s_i}\right) &= \sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\(s_1,s_2,\cdots,s_k)=1}} \frac{1}{s_1 s_2\cdots s_k} + \sum_{\substack{1\le s_1,s_2,\cdots,s_k\le n\\(s_1,s_2,\cdots,s_k)=2}} \frac{1}{s_1 s_2\cdots s_k} + \cdots\\ &\approx A\left(\frac{\log^k n}{1^k}+\frac{\log^k n}{2^k}+\cdots\right)\\ &= A\log^k n\cdot\zeta(k)\end{aligned}$$

Comparing the coefficients, we have $A \cdot \zeta(k) = 1$, meaning that $A = \frac{1}{\zeta(k)}$, and we are done.

Finally, to speak a few words on the density of k-tuples of coprimes in $\mathbb{N}^k$, notice that from the asymptotic form of the harmonic series,

$$\begin{aligned}\sum_{1 \le s_1, s_2, \cdots, s_k \le n} \frac{1}{s_1 s_2 \cdots s_k} &= \prod_{i}^{k} \sum_{s_i=1}^{n} \frac{1}{s_i} \\ &\approx (\log n + \gamma)^k \\ &= \log^k(n) + O(\log^{k-1} n)\end{aligned}$$

Hence, we can deduce that the required density is equal to

$$\lim_{n\to\infty} \frac{\frac{1}{\zeta(k)} \log^k(n) + O(\log^{k-1} n)}{\log^k(n) + O(\log^{k-1} n)} = \frac{1}{\zeta(k)}.$$

Note that $\frac{1}{\zeta(k)}$ increases monotonically as k increases. This is because the density of primes less than n decreases to 0 when n is very large. As such, the probability that 2, 3, $\cdots, k$ numbers be coprime would be large as well.

4.3 Future Work

In this paper, we have come up and given a rigorous proof on the sum of the reciprocals of two coprime numbers. However, it remains open to show rigorously that Conjecture 1 holds.

For the complete proof, it is suggested that one can apply the following result [8],

$$\sum_{k=1}^{n} \frac{\varphi(k)}{k^s} = \frac{\zeta(s-1)}{\zeta(s)} \tag{7}$$

and Theorem 1 to approach this conjecture. For instance, in the case of reciprocals of triple coprime product series, we have

$$\sum_{\substack{1 \le s_1, s_2 \le n \\ (s_1, s_2)=1}} \frac{1}{s_1 s_2} = \frac{1}{\zeta(2)} \log^2 n + O(\log n)$$

$$\left(\sum_{s_3=1}^{n} \frac{\varphi(s_3)}{s_3^3}\right) \left(\sum_{s_3} \frac{1}{s_3}\right) \sum_{\substack{1 \le s_1, s_2 \le n \\ (s_1, s_2)=1}} \frac{1}{s_1 s_2} = \left(\frac{\zeta(2)}{\zeta(3)}\right) (\log n) \left(\frac{1}{\zeta(2)} \log^2 n + O(\log n)\right)$$

$$\sum_{s_3=1}^{n} \frac{\varphi(s_3)}{s_3^3} \sum_{\substack{1 \le s_1, s_2, s_3 \le n \\ (s_1, s_2)=1}} \frac{1}{s_1 s_2 s_3} = \frac{1}{\zeta(3)} \log^3 n + O(\log^2 n)$$

Future work remains to show that left hand side is asymptotic to

$$\sum_{\substack{1 \le s_1, s_2, s_3 \le n \\ (s_1, s_2, s_3)=1}} \frac{1}{s_1 s_2 s_3}$$

so the $k = 3$ case of our conjecture can be proved. (See Figure 5.6 for a computed upper bound for the constant of proportionality of the $O(\log^2 n)$ term.) Furthermore, we can inductively use this trick to prove its generalization for reciprocal product series of k-tuple coprimes.

5 Figures of Numerical Computations

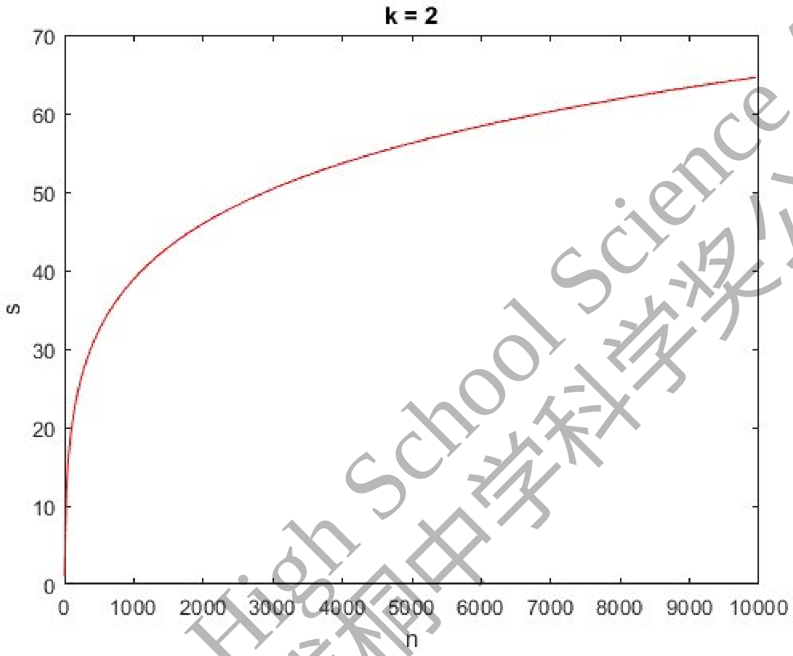

Figure 5.1: $S_2(n) = \sum_{\substack{1 \le s_1, s_2 \le n \\ (s_1, s_2)=1}} \frac{1}{s_1 s_2}$.

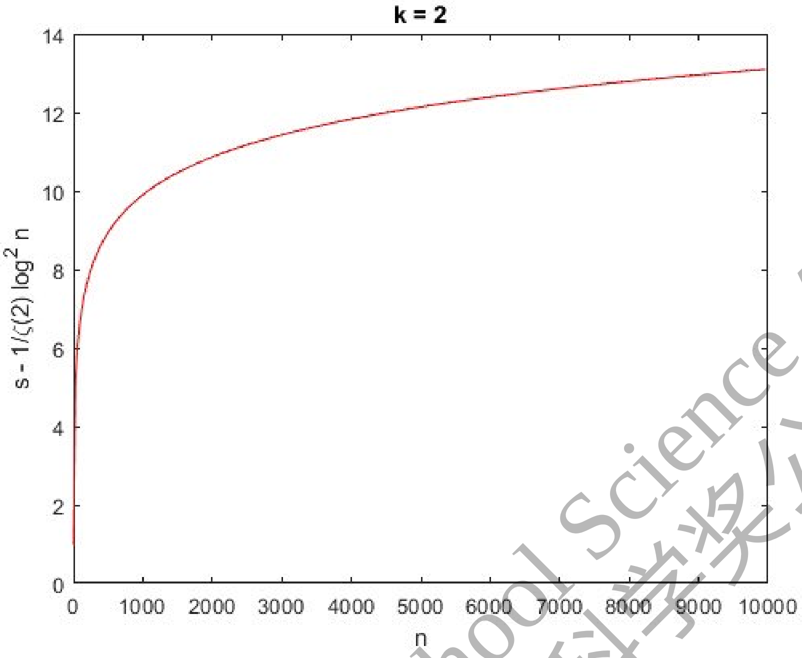

Figure 5.2: The error of $S_2(n)$ from the estimation $\frac{1}{\zeta(2)}\log^2 n$.

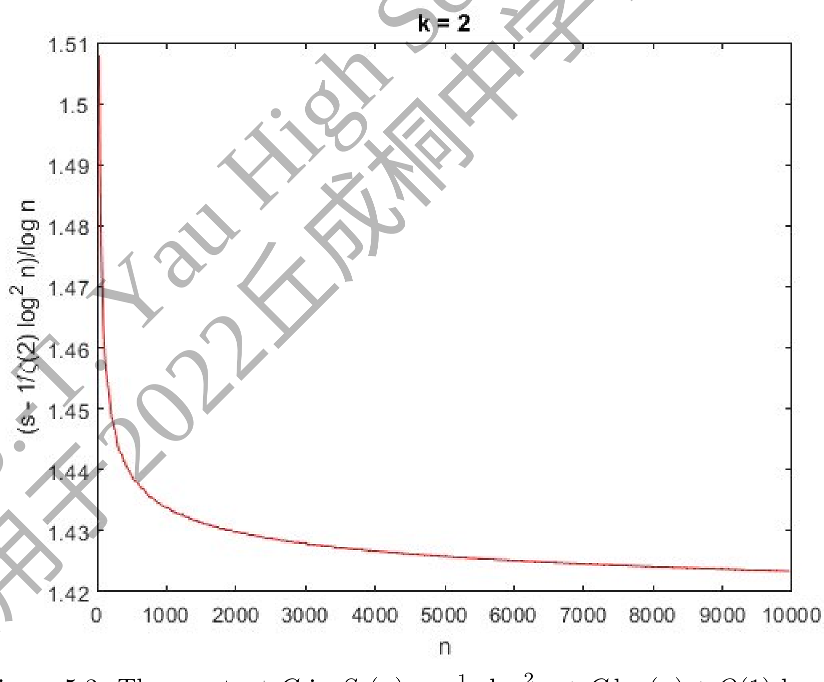

Figure 5.3: The constant C in $S_2(n) = \frac{1}{\zeta(2)}\log^2 n + C\log(n) + O(1)$ has an upper bound of 1.43 as suggested by the computational result.

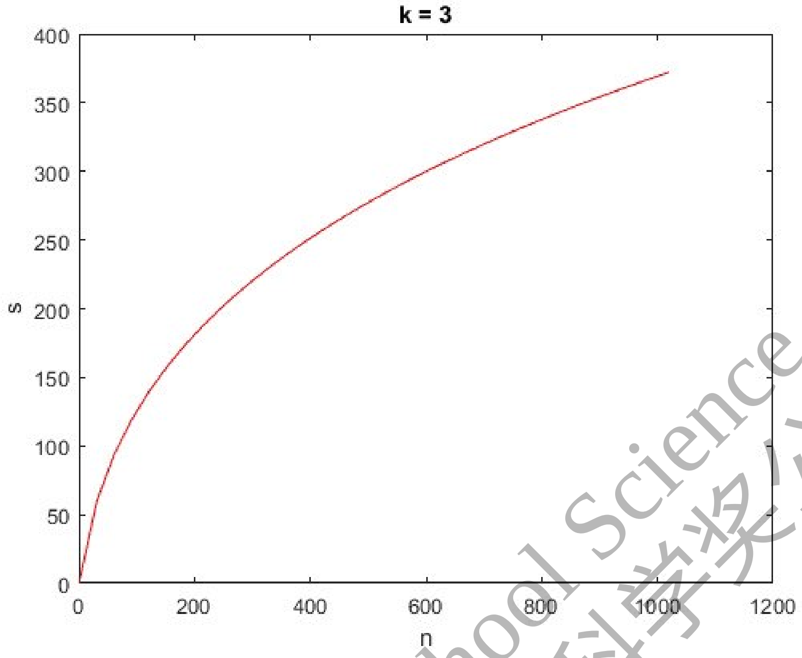

Figure 5.4: $S_3(n) = \sum_{\substack{1 \le s_1, s_2, s_3 \le n \\ (s_1,s_2,s_3)=1}} \frac{1}{s_1 s_2 s_3}$.

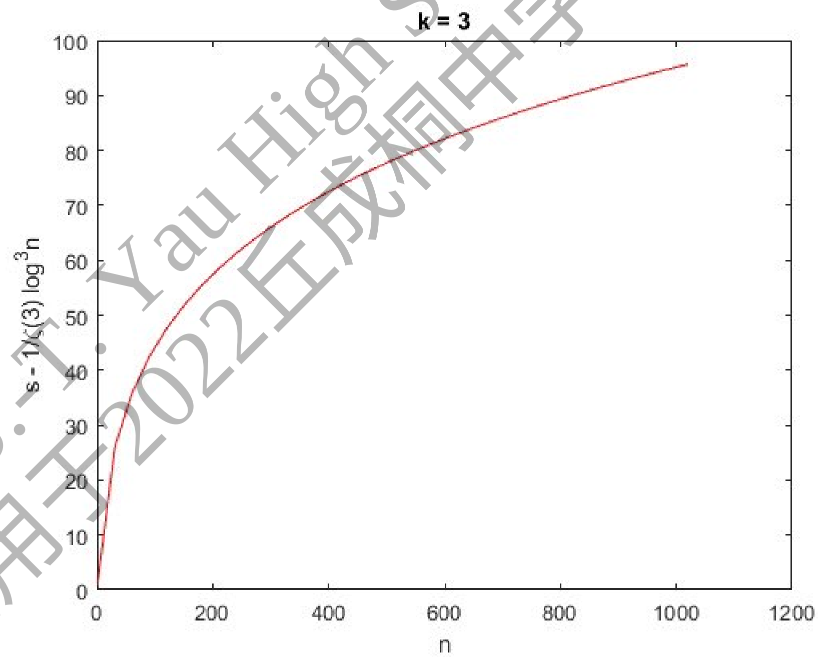

Figure 5.5: The error of $S_3(n)$ from the estimation $\frac{1}{\zeta(3)} \log^3 n$.

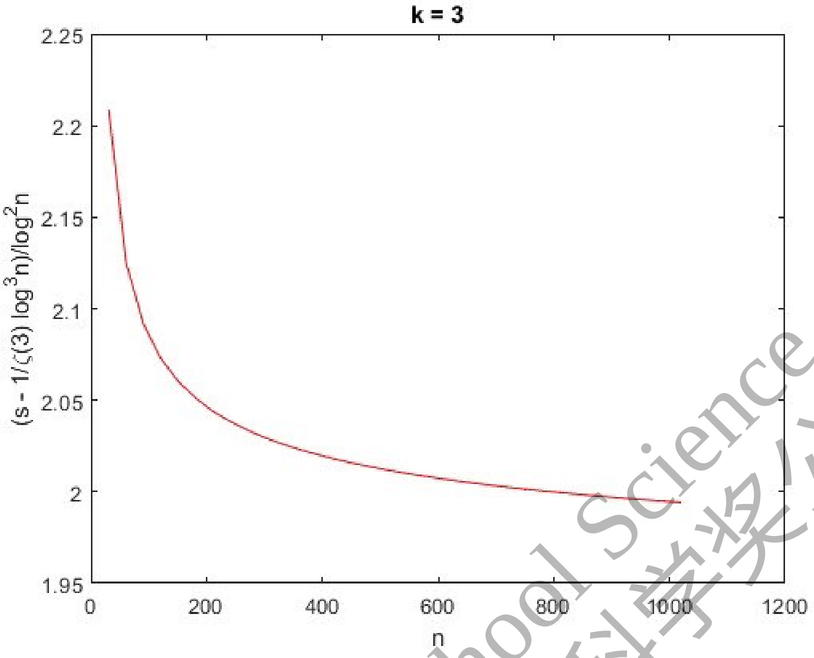

Figure 5.6: The constant C in $S_3(n) = \frac{1}{\zeta(3)} \log^3 n + C \log(n^2) + O(\log n)$ has an upper bound of 2.00 as suggested by the computational result.

6 References

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