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论文题目：A prime number estimation based on elementary mathematics

A prime number estimation based on elementary mathematics

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Abstract

The elementary proof of the prime number theorem given by Selberg[1] and Erdös[2] is one of the most famous examples of using simple methods to solve complex problems. Based on Levinson's[3] method, this paper goes a step further: while using only elementary mathematics and some basic calculus knowledge, it proves the strict form of the prime number theorem and then gives an estimation of the value of the nth prime number:

$$n \ln n < p_n < n \ln n \, (1 + \frac{4.02}{\sqrt[6]{\ln \ln n}})$$

Keywords: prime number theorem, elementary method, prime number estimation.

Symbol Description

These letters refer to real numbers: x, y, s

These letters refer to positive integers: n, m, k, d, r

Floor function and ceiling function: $[x]$, $\lceil x \rceil$

Möbius function: $\mu(n)$

Mangoldt function: $\Lambda(n)$

Chebyshev function: $\theta(x)$, $\psi(x)$

$\Lambda_2(n) = \Lambda(n)\ln n + \sum_{mk=n} \Lambda(m)\Lambda(k)$

$R(x) = \psi(x) - x \ (x \geq 2),\ 0 \ (x < 2)$

$S(x) = \int_2^x \frac{R(s)}{s} ds,\ W(x) = \frac{S(e^x)}{e^x}$

Prime-counting function: $\pi(x)$

Natural constant: $e = 2.7182818 \cdots$

Euler constant: $\gamma = 0.57721566 \cdots = 1 - \int_1^{+\infty} \frac{s-[s]}{s^2} ds$

Contents

1 Abel transformation and Euler summation formula

Theorem 1
(a) Given $A(s) = \sum_{m \le s} a(m)$ and a function $f(s)$ which is derivable on $[y, x]$ ($[y] < [x]$). Then,

$$\sum_{y<m\le x} a(m)f(m) = A(x)f(x) - A(y)f(y) - \int_y^x A(s)f'(s)ds$$

(b) Follows (a), given $A(s) = H(s) + r(s)$, and $H(s)$ is also derivable on $[y, x]$. Then,

$$\sum_{y<m\le x} a(m)f(m) = \int_y^x f(s)H'(s)ds + r(x)f(x) - r(y)f(y) - \int_y^x r(s)f'(s)ds$$

(c) Follows (a),

$$\sum_{y<m\le x} f(m) = \int_y^x f(s)ds - (x-[x])f(x) + (y-[y])f(y) + \int_y^x (s-[s])f'(s)ds$$

Proof.

$$\begin{aligned}
\sum_{y<m\le x} a(m)f(m) &= \sum_{m=[y]+1}^{[x]} f(m)(A(m) - A(m-1)) \\
&= A([x])f([x]) - A([y])f([y]+1) - \sum_{m=[y]+1}^{[x]-1} A(m)(f(m+1) - f(m)) \\
&= A(x)f([x]) - A(y)f([y]+1) - \int_{[y]+1}^{[x]} A(s)f'(s)ds \\
&= A(x)f(x) - A(y)f(y) - \int_y^x A(s)f'(s)ds
\end{aligned}$$

Now (a) is proven. Since $Hf' = (Hf)' - fH'$,

$$\begin{aligned}
\int_y^x A(s)f'(s)ds &= \int_y^x H(s)f'(s)ds + \int_y^x r(s)f'(s)ds \\
&= H(x)f(x) - H(y)f(y) - \int_y^x f(s)H'(s)ds + \int_y^x r(s)f'(s)ds
\end{aligned}$$

Plug it into the result of (a), then (b) is proven.
Set $a(m) = 1$ and $H(s) = s$ in the result of (b), we get (c).

2 Möbius inversion formula

Lemma 2.1

$$\sum_{m|n} \mu(m) = [\frac{1}{n}]$$

Proof.

When $n = 1$, the proposition is evident.
When $n > 1$, suppose the prime factorization of n is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then,

$$\sum_{m|n} \mu(m) = \prod_{i=1}^{k} \sum_{j=0}^{\alpha_i} \mu(p_i^j) = 0 = [\frac{1}{n}]$$

Theorem 2

(a)

$$g(n) = \sum_{m|n} f(m) \iff f(k) = \sum_{d|k} \mu(d) g(\frac{k}{d})$$

(b)

$$G(x) = \sum_{m \le x} F(\frac{x}{m}) \ (x \ge 1) \iff F(y) = \sum_{d \le y} \mu(d) G(\frac{y}{d}) \ (y \ge 1)$$

Proof.

If $g(n) = \sum_{m|n} f(m)$, then

$$\sum_{d|k} \mu(d) g(\frac{k}{d}) = \sum_{d|k} \mu(d) \sum_{m|\frac{k}{d}} f(m) \xlongequal{order-exchange} \sum_{m|k} f(m) \sum_{d|\frac{k}{m}} \mu(d) \xlongequal{Lemma\ 2.1} f(k)$$

If $f(k) = \sum_{d|k} \mu(d) g(\frac{k}{d})$, then

$$\begin{aligned}
\sum_{m|n} f(m) = \sum_{m|n} \sum_{d|m} \mu(d) g(\frac{m}{d}) &\xlongequal{d \to \frac{m}{d}} \sum_{m|n} \sum_{d|m} \mu(\frac{m}{d}) g(d) \\
&\xlongequal{order-exchange} \sum_{d|n} g(d) \sum_{d|m, m|n} \mu(\frac{m}{d}) \\
&\xlongequal{r = \frac{m}{d}} \sum_{d|n} g(d) \sum_{r|\frac{n}{d}} \mu(r) \\
&\xlongequal{Lemma\ 2.1} g(n)
\end{aligned}$$

Now (a) is proven.

If $G(x) = \sum_{m\leq x} F(\frac{x}{m})$, then

$$\sum_{d\leq y}\mu(d)G(\frac{y}{d}) = \sum_{d\leq y}\mu(d)\sum_{m\leq \frac{y}{d}} F(\frac{y}{md}) \xlongequal{r=md} \sum_{d\leq y}\mu(d)\sum_{d|r, r\leq y} F(\frac{y}{r})$$

$$\xlongequal{order-exchange} \sum_{r\leq y} F(\frac{y}{r})\sum_{d|r}\mu(d)$$

$$\xlongequal{Lemma\ 2.1} F(y)$$

If $F(y) = \sum_{d\leq y}\mu(d)G(\frac{y}{d})$, then

$$\sum_{m\leq x} F(\frac{x}{m}) = \sum_{m\leq x}\sum_{d\leq \frac{x}{m}}\mu(d)G(\frac{x}{dm}) \xlongequal{order-exchange} \sum_{d\leq x}\mu(d)\sum_{m\leq \frac{x}{d}} G(\frac{x}{dm})$$

$$\xlongequal{r=dm} \sum_{d\leq x}\mu(d)\sum_{d|r, r\leq x} G(\frac{x}{r})$$

$$\xlongequal{order-exchange} \sum_{r\leq x} G(\frac{x}{r})\sum_{d|r}\mu(d)$$

$$\xlongequal{Lemma\ 2.1} G(x)$$

Hence (b) is also proven.

3 The estimation of $\psi(x)$

Lemma 3.1

$$\sum_{m|n} \Lambda(m) = \ln n$$

Proof.

When $n = 1$, the proposition is evident.
When $n > 1$, suppose the prime factorization of n is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then,

$$\sum_{m|n} \Lambda(m) = \sum_{i=1}^{k} \alpha_i \ln p_i = \ln n$$

Lemma 3.2

$$\sum_{m \le x} \psi(\frac{x}{m}) = \ln([x]!) \; (x \ge 1)$$

Proof.

$$\sum_{m \le x} \psi(\frac{x}{m}) = \sum_{mk \le x} \Lambda(k) \overset{r=mk}{=\!=\!=} \sum_{r \le x} \sum_{k|r} \Lambda(k) \overset{Lemma\ 3.1}{=\!=\!=\!=\!=} \sum_{r \le x} \ln r = \ln([x]!)$$

Theorem 3

$$0.43x < \psi(x) < 1.65x \; (x \ge 8)$$
$$0.69x < \psi(x) < 1.39x \; (x \ge e^8)$$

Proof.

For every positive integer n,

$$\begin{aligned} \ln \frac{(2n)!}{(n!)^2} \overset{Lemma\ 3.2}{=\!=\!=\!=\!=} \sum_{m \le 2n} \psi(\frac{2n}{m}) - 2 \sum_{m \le n} \psi(\frac{n}{m}) &= \sum_{m \le 2n} \psi(\frac{2n}{m}) - 2 \sum_{m \le n} \psi(\frac{2n}{2m}) \\ &= \sum_{m \le 2n} (-1)^{m-1} \psi(\frac{2n}{m}) \end{aligned}$$

Since $\psi(x)$ is non-decreasing,

$$\psi(2n) - \psi(n) \le \ln \frac{(2n)!}{(n!)^2} \le \psi(2n)$$

Notice $\frac{2^{2n-1}}{\sqrt{n}} \le \frac{(2n)!}{(n!)^2} \le 2^{2n-1}$ (It can be easily proved by induction) , then we get

$$\psi(2n) - \psi(n) \le (2n - 1) \ln 2$$
$$\psi(2n) \ge (2n - 1) \ln 2 - \ln \sqrt{n}$$

$$\begin{aligned}
\Rightarrow \psi(x) \le \psi(2\lceil \frac{x}{2} \rceil) &\le \psi(2\lceil \frac{x}{2} \rceil) + \sum_{i=1}^{s-1} (\psi(2\lceil \frac{x}{2^{i+1}} \rceil) - \psi(\lceil \frac{x}{2^i} \rceil)) \ (s = [\log_2 x]) \\
&= \sum_{i=1}^{s-1} (\psi(2\lceil \frac{x}{2^i} \rceil) - \psi(\lceil \frac{x}{2^i} \rceil)) + \ln 2 \\
&\le (2 \sum_{i=1}^{s-1} \lceil \frac{x}{2^i} \rceil - (s-1) + 1) \ln 2 \\
&\le (2 \sum_{i=1}^{s-1} \frac{x}{2^i} + s) \ln 2 \le (2x + \log_2 x) \ln 2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \psi(x) &\le (2x + \frac{\log_2 8}{8} x) \ln 2 < 1.65x \ (x \ge 8) \\
\psi(x) &\le (2x + \frac{\log_2 e^8}{e^8} x) \ln 2 < 1.39x \ (x \ge e^8)
\end{aligned}$$

On the other hand,

$$\psi(x) \ge \psi(2[\frac{x}{2}]) \ge (2[\frac{x}{2}] - 1) \ln 2 - \ln \sqrt{[\frac{x}{2}]} \ge (x-2) \ln 2 - \ln \sqrt{\frac{x}{2}}$$

$$\begin{aligned}
\Rightarrow \psi(x) &\ge (\frac{8-2}{8} x) \ln 2 - \frac{\ln \sqrt{\frac{8}{2}}}{8} x > 0.43x \ (x \ge 8) \\
\psi(x) &\ge (\frac{e^8 - 2}{e^8} x) \ln 2 - \frac{\ln \sqrt{\frac{e^8}{2}}}{e^8} x > 0.69x \ (x \ge e^8)
\end{aligned}$$

4 The estimation of $\sum_{m\le x}\frac{\Lambda(m)}{m}$

Lemma 4.1

$$x\ln x - x - \frac{1}{2}\ln x + 0.84 \le \ln([x]!) \le x\ln x - x + \ln x + 1 \ (x \ge 1)$$

Proof.

On the one hand,

$$\ln([x]!) = \sum_{m\le [x]-1}\ln m + \ln[x] \le \sum_{m\le [x]-1}\int_m^{m+1}\ln s\,ds + \ln x \le \int_1^x \ln s\,ds + \ln x$$
$$= x\ln x - x + \ln x + 1$$

On the other hand,

$$\ln([x]!) \ge \sum_{m\le [x]}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\ln s\,ds \ge \int_{\frac{1}{2}}^{x-\frac{1}{2}}\ln s\,ds$$
$$= (x-\frac{1}{2})(\ln(x-\frac{1}{2})-1) - \frac{1}{2}(\ln\frac{1}{2}-1)$$
$$= x\ln x - x - \frac{1}{2}\ln x - (x-\frac{1}{2})\ln\frac{x}{x-\frac{1}{2}} + 1 + \frac{1}{2}\ln 2$$
$$\ge x\ln x - x - \frac{1}{2}\ln x + \frac{1+\ln 2}{2}$$
$$\ge x\ln x - x - \frac{1}{2}\ln x + 0.84$$

Theorem 4

$$\ln x - 1.04 < \sum_{m\le x}\frac{\Lambda(m)}{m} < \ln x + 1.04 \ (x \ge 1)$$

$$\ln x - 1.01 < \sum_{m\le x}\frac{\Lambda(m)}{m} < \ln x + 0.40 \ (x \ge e^8)$$

Proof.

It is easy to prove

$$\ln(n+1) - 1.04 < \sum_{m\le n}\frac{\Lambda(m)}{m} < \ln n + 1.04 \ (1 \le n \le 7)$$

Hence,

$$\ln x - 1.04 < \sum_{m\le x}\frac{\Lambda(m)}{m} < \ln x + 1.04 \ (1 \le x < 8)$$

When $x \geq 8$, we have

$$0 \leq \sum_{m \leq x} \Lambda(m)(\frac{x}{m} - [\frac{x}{m}]) \leq \psi(x) \overset{Theorem\ 3}{\leq} 1.65x$$

and

$$\sum_{m \leq x} \Lambda(m)[\frac{x}{m}] = \sum_{mk \leq x} \Lambda(m) = \sum_{k \leq x} \psi(\frac{x}{k})$$

$$\overset{Lemma\ 3.2}{=\!=\!=} \ln([x]!)$$

$$\overset{Lemma\ 4.1}{\in} [x \ln x - x - \frac{1}{2} \ln x + 0.84, x \ln x - x + \ln x + 1]$$

$$\Rightarrow x \ln x - x - \frac{1}{2} \ln x + 0.84 \leq x \sum_{m \leq x} \frac{\Lambda(m)}{m} \leq x \ln x + 0.65x + \ln x + 1$$

$$\Rightarrow \ln x - 1.04 < \sum_{m \leq x} \frac{\Lambda(m)}{m} < \ln x + 1.04$$

Similarly, when $x \geq e^8$, we have

$$0 \leq \sum_{m \leq x} \Lambda(m)(\frac{x}{m} - [\frac{x}{m}]) \leq \psi(x) \overset{Theorem\ 3}{\leq} 1.39x$$

and

$$\sum_{m \leq x} \Lambda(m)[\frac{x}{m}] \in [x \ln x - x - \frac{1}{2} \ln x + 0.84, x \ln x - x + \ln x + 1]$$

$$\Rightarrow x \ln x - x - \frac{1}{2} \ln x + 0.84 \leq x \sum_{m \leq x} \frac{\Lambda(m)}{m} \leq x \ln x + 0.39x + \ln x + 1$$

$$\Rightarrow \ln x - 1.01 < \sum_{m \leq x} \frac{\Lambda(m)}{m} < \ln x + 0.40$$

5 The estimation of $\psi(x)\ln x + \sum_{m\le x}\Lambda(m)\psi(\frac{x}{m})$

Lemma 5.1

$$\ln x - 1 + \gamma + \frac{[x]}{x} < \sum_{m\le x}\frac{1}{m} < \ln x - 1 + \gamma + \frac{[x]+1}{x}\ (x\ge 1)$$

Proof.

Set $f(s)=\frac{1}{s}$ and $y=1$ in the result of Theorem 1(c), we get

$$\sum_{1<m\le x}\frac{1}{m} = \int_1^x \frac{1}{s}ds - 1 + \frac{[x]}{x} - \int_1^x \frac{s-[s]}{s^2}ds$$

$$\Rightarrow \sum_{m\le x}\frac{1}{m} = \ln x - 1 + \gamma + \frac{[x]}{x} + \int_x^{+\infty}\frac{s-[s]}{s^2}ds$$

Since

$$\int_x^{+\infty}\frac{s-[s]}{s^2}ds < \int_x^{\infty}\frac{1}{s^2}ds = \frac{1}{x}$$

we complete the proof.

Lemma 5.2

$$-\frac{1}{2}\ln x - 0.74 < \sum_{m\le x}(\psi(\frac{x}{m}) - (\frac{x}{m} - 1 - \gamma)) < \ln x + 1\ (x\ge 1)$$

Proof.

We have

$$\sum_{m\le x}\psi(\frac{x}{m}) \overset{Lemma\ 3.2}{=\!=\!=} \ln([x]!) \overset{Lemma\ 4.1}{\in} [x\ln x - x - \frac{1}{2}\ln x + 0.84, x\ln x - x + \ln x + 1]$$

and

$$\sum_{m\le x}\frac{x}{m} = x\sum_{m\le x}\frac{1}{m} \overset{Lemma\ 5.1}{\in} (x\ln x - x + \gamma x + [x], x\ln x - x + \gamma x + [x] + 1)$$

$$\Rightarrow \sum_{m\le x}(\psi(\frac{x}{m}) - (\frac{x}{m} - 1 - \gamma)) \in (-\frac{1}{2}\ln x - 0.16 - \gamma(x-[x]), \ln x + 1 - \gamma(x-[x]))$$

Notice $0\le\gamma(x-[x]) < 0.58$, then the proof is done.

Lemma 5.3
For $x \geq 1$,

$$\sum_{m\leq x}\mu(m)\ln\frac{x}{m}\sum_{k\leq\frac{x}{m}}(\psi(\frac{x}{mk})-(\frac{x}{mk}-1-\gamma)) < 5.34x-2.16\sqrt{x}+\frac{1}{2}\ln x-0.84$$

$$\sum_{m\leq x}\mu(m)\ln\frac{x}{m}\sum_{k\leq\frac{x}{m}}(\psi(\frac{x}{mk})-(\frac{x}{mk}-1-\gamma)) > -2.91x+1.08\sqrt{x}-0.37\ln x+0.62$$

Proof.
Based on

$$\sum_{m\leq x}\ln\frac{x}{m} = [x]\ln x-\ln([x]!) \overset{Lemma\ 4.1}{\leq} x+\frac{1}{2}\ln x-0.84-(x-[x])\ln x \leq x+\frac{1}{2}\ln x-0.84$$

and

$$\begin{aligned}\sum_{m\leq x}\ln^2\frac{x}{m} \leq \sum_{m\leq x}(\frac{4}{e}\sqrt[4]{\frac{x}{m}})^2 = \frac{16}{e^2}\sqrt{x}\sum_{m\leq x}\sqrt{\frac{1}{m}} &\leq \frac{16}{e^2}\sqrt{x}\,(1+\sum_{2\leq m\leq[x]}\int_{m-1}^{m}\sqrt{\frac{1}{s}}ds)\\ &= \frac{16}{e^2}\sqrt{x}\,(1+\int_1^{[x]}\sqrt{\frac{1}{s}}ds)\\ &= \frac{16}{e^2}\sqrt{x}(2\sqrt{[x]}-1)\\ &\leq \frac{32}{e^2}x-\frac{16}{e^2}\sqrt{x}\end{aligned}$$

we have

$$\begin{aligned}\sum_{m\leq x}\mu(m)\ln\frac{x}{m}\sum_{k\leq\frac{x}{m}}(\psi(\frac{x}{mk})-(\frac{x}{mk}-1-\gamma)) &\overset{Lemma\ 5.2}{\leq} \sum_{m\leq x}\ln^2\frac{x}{m}+\sum_{m\leq x}\ln\frac{x}{m}\\ &< 5.34x-2.16\sqrt{x}+\frac{1}{2}\ln x-0.84\end{aligned}$$

and

$$\begin{aligned}\sum_{m\leq x}\mu(m)\ln\frac{x}{m}\sum_{k\leq\frac{x}{m}}(\psi(\frac{x}{mk})-(\frac{x}{mk}-1-\gamma)) &\overset{Lemma\ 5.2}{\geq} -\frac{1}{2}\sum_{m\leq x}\ln^2\frac{x}{m}-0.74\sum_{m\leq x}\ln\frac{x}{m}\\ &> -2.91x+1.08\sqrt{x}-0.37\ln x+0.62\end{aligned}$$

Lemma 5.4

$$\sum_{m|n}\mu(m)\ln m = -\Lambda(n)$$

Proof.
On account of Theorem 2(a) and Lemma 3.1, we have

$$\Lambda(n) = \sum_{m|n}\mu(m)\ln\frac{n}{m} = \ln n\sum_{m|n}\mu(m) - \sum_{m|n}\mu(m)\ln m \overset{Lemma\ 2.1}{=\!=\!=\!=} -\sum_{m|n}\mu(m)\ln m$$

Lemma 5.5

$$F(y)\ln y + \sum_{d\le y}\Lambda(d)F(\frac{y}{d}) = \sum_{d\le y}\mu(d)G(\frac{y}{d})\ln\frac{y}{d}\ (y\ge 1)$$

$$\Longleftrightarrow\ G(x) = \sum_{m\le x}F(\frac{x}{m})\ (x\ge 1)$$

Proof.
If $G(x) = \sum_{m\le x}F(\frac{x}{m})$, then

$$\begin{aligned}\sum_{d\le y}\mu(d)G(\frac{y}{d})\ln\frac{y}{d} &= \sum_{d\le y}\mu(d)\ln\frac{y}{d}\sum_{m\le\frac{y}{d}}F(\frac{y}{md})\\ &\overset{r=md}{=\!=\!=} \sum_{d\le y}\mu(d)\ln\frac{y}{d}\sum_{d|r,r\le y}F(\frac{y}{r})\\ &\overset{order-exchange}{=\!=\!=\!=\!=} \sum_{r\le y}F(\frac{y}{r})\sum_{d|r}\mu(d)\ln\frac{y}{d}\\ &= \ln y\sum_{r\le y}F(\frac{y}{r})\sum_{d|r}\mu(d) - \sum_{r\le y}F(\frac{y}{r})\sum_{d|r}\mu(d)\ln d\\ &\overset{Lemma\ 2.1,\ Lemma\ 5.4}{=\!=\!=\!=\!=\!=} F(y)\ln y + \sum_{r\le y}\Lambda(r)F(\frac{y}{r})\end{aligned}$$

If $F(y)\ln y + \sum_{d\le y}\Lambda(d)F(\frac{y}{d}) = \sum_{d\le y}\mu(d)G(\frac{y}{d})\ln\frac{y}{d}$, then

$$\begin{aligned}G(x)\ln x &\overset{Theorem\ 2(b)}{=\!=\!=\!=\!=} \sum_{m\le x}(F(\frac{x}{m})\ln\frac{x}{m} + \sum_{d\le\frac{x}{m}}\Lambda(d)F(\frac{x}{dm}))\\ &= \ln x\sum_{m\le x}F(\frac{x}{m}) - \sum_{m\le x}F(\frac{x}{m})\ln m + \sum_{m\le x}\sum_{d\le\frac{x}{m}}\Lambda(d)F(\frac{x}{dm})\end{aligned}$$

Combine it with

$$\sum_{m\le x}\sum_{d\le \frac{x}{m}}\Lambda(d)F(\frac{x}{dm}) \xlongequal{order-exchange} \sum_{d\le x}\Lambda(d)\sum_{m\le \frac{x}{d}}F(\frac{x}{dm})$$

$$\xlongequal{r=dm} \sum_{d\le x}\Lambda(d)\sum_{d|r,r\le x}F(\frac{x}{r})$$

$$\xlongequal{order-exchange} \sum_{r\le x}F(\frac{x}{r})\sum_{d|r}\Lambda(d)$$

$$\xlongequal{Lemma\ 3.1} \sum_{r\le x}F(\frac{x}{r})\ln r$$

we get

$$G(x)\ln x = \ln x\sum_{m\le x}F(\frac{x}{m}) \Rightarrow G(x) = \sum_{m\le x}F(\frac{x}{m})$$

Theorem 5

$$2x\ln x - 6.61x < \psi(x)\ln x + \sum_{m\le x}\Lambda(m)\psi(\frac{x}{m}) < 2x\ln x + 5.71x\ (x\ge 1)$$

Proof.

It is easy to prove

$$\psi(n)\ln n + \sum_{m\le n}\Lambda(m)\psi(\frac{n}{m}) > 2n\ln n - 6.61n$$

and

$$\psi(n)\ln(n+1) + \sum_{m\le n}\Lambda(m)\psi(\frac{n}{m}) < 2n\ln n + 5.71n$$

for $1\le n\le 7$. Thus,

$$2x\ln x - 6.61x < \psi(x)\ln x + \sum_{m\le x}\Lambda(m)\psi(\frac{x}{m}) < 2x\ln x + 5.71x\ (1\le x<8)$$

When $x\ge 8$, set

$$F(x) = \psi(x) - (x-1-\gamma),\ G(x) = \sum_{m\le x}(\psi(\frac{x}{m}) - (\frac{x}{m}-1-\gamma))$$

in the result of Lemma 5.5, combined with Lemma 5.3, we get

$$(\psi(x) - (x - 1 - \gamma)) \ln x + \sum_{m \leq x} \Lambda(m)(\psi(\frac{x}{m}) - (\frac{x}{m} - 1 - \gamma))$$

$$\in (-2.91x + 1.08\sqrt{x} - 0.37 \ln x + 0.62, 5.34x - 2.16\sqrt{x} + \frac{1}{2} \ln x - 0.84)$$

Because of $1.57721566 < \gamma < 1.57721567$, Theorem 3 and Theorem 4, we have

$$1.57 \ln x < (1 + \gamma) \ln x < 1.58 \ln x,\ x \ln x - 1.04x < x \sum_{m \leq x} \frac{\Lambda(m)}{m} < x \ln x + 1.04x$$

and

$$0.67x < (1 + \gamma)\psi(x) < 2.61x$$

$$\Rightarrow \psi(x) \ln x + \sum_{m \leq x} \Lambda(m)\psi(\frac{x}{m}) - 2x \ln x < 5.71x - 2.16\sqrt{x} - 1.07 \ln x - 0.84$$

$$\psi(x) \ln x + \sum_{m \leq x} \Lambda(m)\psi(\frac{x}{m}) - 2x \ln x > -6.56x + 1.08\sqrt{x} - 1.95 \ln x + 0.62$$

$$\Rightarrow 2x \ln x - 6.61x < \psi(x) \ln x + \sum_{m \leq x} \Lambda(m)\psi(\frac{x}{m}) < 2x \ln x + 5.71x$$

6 Four properties of $W(x)$

Lemma 6.1

$$|S(x)| < 0.65x \ (x \geq 1)$$

Proof.

$$|S(x)| = \left| \int_2^x \frac{R(s)}{s} ds \right| \leq \int_2^x \left| \frac{R(s)}{s} \right| ds$$

It is easy to prove

$$|R(s)| < 0.77s \ (2 \leq s < 3), \ |R(s)| < 0.65s \ (3 \leq s \leq 8)$$

Hence,

$$|S(x)| < 0.65x \ (1 \leq x \leq 8)$$

When $x > 8$, we have

$$|S(x)| \leq |S(8)| + |S(x) - S(8)| < 5.2 + \int_8^x \left| \frac{R(s)}{s} \right| ds \overset{Theorem\ 3}{<} 5.2 + 0.65(x-8) = 0.65x$$

Then the proof is done.

Lemma 6.2

$$|S(x_2) - S(x_1)| < 0.39(x_2 - x_1) \ (x_2 \geq x_1 \geq e^8)$$

Proof.

$$|S(x_2) - S(x_1)| = \left| \int_{x_1}^{x_2} \frac{R(s)}{s} ds \right| \leq \int_{x_1}^{x_2} \left| \frac{R(s)}{s} \right| ds \overset{Theorem\ 3}{<} 0.39(x_2 - x_1)$$

Lemma 6.3

$$\ln 2 - 3.05 < \int_2^x \frac{S(s)}{s^2} ds < \ln 2 + 0.36 \ (x \geq e^8)$$

Proof.
Set $a(m) = \Lambda(m)$, $A(s) = \psi(s)$, $f(s) = \frac{1}{s}$ and $y = 2$ in the result of Theorem 1(a), we get

$$\sum_{2 < m \leq x} \frac{\Lambda(m)}{m} = \frac{\psi(x)}{x} - \frac{\psi(2)}{2} + \int_2^x \frac{\psi(s)}{s^2} ds$$

$$\Rightarrow \sum_{m \leq x} \frac{\Lambda(m)}{m} = \frac{\psi(x)}{x} + \int_2^x \frac{\psi(s)}{s^2} ds$$

Since
$$\frac{S(s)}{s^2} = \frac{S'(s)}{s} - (\frac{S(s)}{s})' = \frac{R(s)}{s^2} - (\frac{S(s)}{s})'$$
we have
$$\int_2^x \frac{S(s)}{s^2} ds = \int_2^x \frac{R(s)}{s^2} ds - \frac{S(x)}{x} = \int_2^x \frac{\psi(s)}{s^2} ds - \frac{S(x)}{x} - \ln x + \ln 2$$
$$\Rightarrow \int_2^x \frac{S(s)}{s^2} ds = \sum_{m \le x} \frac{\Lambda(m)}{m} - \frac{\psi(x)}{x} - \frac{S(x)}{x} - \ln x + \ln 2$$
Because of Theorem 3, Theorem 4 and Lemma 6.1,
$$\ln x - 1.01 < \sum_{m \le x} \frac{\Lambda(m)}{m} < \ln x + 0.40,\ 0.69 < \frac{\psi(x)}{x} < 1.39,\ -0.65 < \frac{S(x)}{x} < 0.65$$
$$\Rightarrow \ln 2 - 3.05 < \int_2^x \frac{S(s)}{s^2} ds < \ln 2 + 0.36$$

Lemma 6.4

$$-6.96x < R(x) \ln x + \sum_{m \le x} \Lambda(m) R(\frac{x}{m}) < 7.45x\ (x \ge 2)$$

Proof.

$$R(x) \ln x + \sum_{m \le x} \Lambda(m) R(\frac{x}{m}) = \psi(x) \ln x + \sum_{m \le x} \Lambda(m) \psi(\frac{x}{m}) - x \ln x - x \sum_{m \le \frac{x}{2}} \frac{\Lambda(m)}{m}$$

According to Theorem 4,
$$\ln \frac{x}{2} - 1.04 < \sum_{m \le \frac{x}{2}} \frac{\Lambda(m)}{m} < \ln \frac{x}{2} + 1.04 \Rightarrow x \ln x - 1.74x < x \sum_{m \le \frac{x}{2}} \frac{\Lambda(m)}{m} < x \ln x + 0.35x$$

Then, by using Theorem 5, we complete the proof.

Lemma 6.5

$$-7.61x < S(x) \ln x + \sum_{m \le x} \Lambda(m) S(\frac{x}{m}) < 8.10x\ (x \ge 1)$$

Proof.

The proposition is obvious when $1 \le x < 2$.

When $x \geq 2$, on the basis of Lemma 6.4, we have

$$-6.96 < \frac{R(s)}{s} \ln s + \sum_{m \leq s} \Lambda(m) \frac{R(\frac{s}{m})}{s} < 7.45 \; (2 \leq s \leq x)$$

Notice $R(y) = 0 \; (y < 2)$, then we get

$$-6.96x < \int_2^x \frac{R(s)}{s} \ln s \, ds + \sum_{m \leq x} \Lambda(m) \int_2^x \frac{R(\frac{s}{m})}{s} ds < 7.45x$$

Since

$$(S(s) \ln s)' = S'(s) \ln s + \frac{S(s)}{s} = \frac{R(s)}{s} \ln s + \frac{S(s)}{s}$$

we have

$$S(x) \ln x = \int_2^x \frac{R(s)}{s} \ln s \, ds + \int_2^x \frac{S(s)}{s} ds$$

Combine these with

$$\int_2^x \frac{R(\frac{s}{m})}{s} ds \xlongequal{r=\frac{s}{m}} \int_{\frac{2}{m}}^{\frac{x}{m}} \frac{R(r)}{r} dr = \int_2^{\frac{x}{m}} \frac{R(r)}{r} dr = S(\frac{x}{m})$$

we get

$$-6.96x < S(x) \ln x + \sum_{m \leq x} \Lambda(m) S(\frac{x}{m}) - \int_2^x \frac{S(s)}{s} ds < 7.45x$$

Finally, because of Lemma 6.1, we know

$$\left| \int_2^x \frac{S(s)}{s} ds \right| \leq \int_2^x \left| \frac{S(s)}{s} \right| ds < 0.65x$$

$$\Rightarrow -7.61x < S(x) \ln x + \sum_{m \leq x} \Lambda(m) S(\frac{x}{m}) < 8.10x$$

Lemma 6.6

$$|S(x)| \ln^2 x < \sum_{m \leq x} \Lambda_2(m) \left| S(\frac{x}{m}) \right| + 16.12 \, x \ln x \; (x \geq e^8)$$

Proof.

$$(S(x)\ln x + \sum_{m\le x}\Lambda(m)S(\frac{x}{m}))\ln x - \sum_{m\le x}\Lambda(m)(S(\frac{x}{m})\ln\frac{x}{m} + \sum_{k\le\frac{x}{m}}\Lambda(k)S(\frac{x}{mk}))$$
$$= S(x)\ln^2 x + \sum_{m\le x}\Lambda(m)S(\frac{x}{m})\ln m - \sum_{m\le x, k\le\frac{x}{m}}\Lambda(m)\Lambda(k)S(\frac{x}{mk})$$
$$\overset{r=mk}{=\!=\!=} S(x)\ln^2 x + \sum_{m\le x}\Lambda(m)\ln m\, S(\frac{x}{m}) - \sum_{r\le x}\sum_{mk=r}\Lambda(m)\Lambda(k)S(\frac{x}{r})$$
$$= S(x)\ln^2 x + \sum_{r\le x}(\Lambda(r)\ln r - \sum_{mk=r}\Lambda(m)\Lambda(k))S(\frac{x}{r})$$

According to Theorem 4 and Lemma 6.5, the value of the first row is between

$$-7.61\,x\ln x - 8.10x(\ln x + 0.40)$$

and

$$8.10\,x\ln x + 7.61x(\ln x + 0.40)$$

Thus, its absolute value is less than $16.12\,x\ln x$.

$$\Rightarrow |S(x)|\ln^2 x < \left|\sum_{r\le x}(\Lambda(r)\ln r - \sum_{mk=r}\Lambda(m)\Lambda(k))S(\frac{x}{r})\right| + 16.12\,x\ln x$$
$$\Rightarrow |S(x)|\ln^2 x < \sum_{r\le x}\Lambda_2(r)\left|S(\frac{x}{r})\right| + 16.12\,x\ln x$$

Lemma 6.7

$$2x\ln x - 8.26x + 2 < \sum_{m\le x}\Lambda_2(m) < 2x\ln x + 5.36x + 2 \ (x\ge 2)$$

Proof.

Set $a(m)=\Lambda(m)$, $A(s)=\psi(s)$, $f(s)=\ln s$ and $y=1$ in the result of Theorem 1(a), we get

$$\sum_{1<m\le x}\Lambda(m)\ln m = \psi(x)\ln x - \int_1^x \frac{\psi(s)}{s}ds$$
$$\Rightarrow \sum_{m\le x}\Lambda(m)\ln m = \psi(x)\ln x - S(x) - x + 2$$

In addition,

$$\sum_{m\le x}\sum_{kd=m}\Lambda(k)\Lambda(d) = \sum_{kd\le x}\Lambda(k)\Lambda(d) = \sum_{k\le x}\Lambda(k)\sum_{d\le\frac{x}{k}}\Lambda(d) = \sum_{k\le x}\Lambda(k)\psi(\frac{x}{k})$$

Combine these with Theorem 5 and Lemma 6.1, then we get

$$2x\ln x - 8.26x + 2 < \sum_{m\le x}\Lambda_2(m) < 2x\ln x + 5.36x + 2$$

Lemma 6.8

$$\left|\sum_{m\le x}(\Lambda_2(m) - 2\ln m)\right| < 7.89x \ (x\ge 2)$$

Proof.
Based on Lemma 4.1 and Lemma 6.7, we have

$$-6.26x - 2\ln x < \sum_{m\le x}\Lambda_2(m) - 2\ln([x]!) < 7.36x + \ln x + 0.32$$

Then the proof is done.

Lemma 6.9

$$|S(x_2) - S(x_1)| < 0.77(x_2 - x_1) \ (x_2\ge x_1\ge 1)$$

Proof.
According the proof of Lemma 6.1, we have $|R(s)| < 0.77s \ (s\ge 1)$. Then,

$$\Rightarrow |S(x_2) - S(x_1)| = \left|\int_{x_1}^{x_2}\frac{R(s)}{s}ds\right| \le \int_{x_1}^{x_2}\left|\frac{R(s)}{s}\right|ds < 0.77(x_2 - x_1)$$

Lemma 6.10

$$|S(x)|\ln^2 x < 2\sum_{m\le x}\ln m\left|S(\frac{x}{m})\right| + 22.20\,x\ln x \ (x\ge e^8)$$

Proof.

$$\sum_{m\le x}(\Lambda_2(m) - 2\ln m)\left|S(\frac{x}{m})\right|$$

$$= S(\frac{x}{[x]})\sum_{m\le [x]}(\Lambda_2(m) - 2\ln m) + \sum_{1\le m\le [x]-1}(\left|S(\frac{x}{m})\right| - \left|S(\frac{x}{m+1})\right|)\sum_{k\le m}(\Lambda_2(k) - 2\ln k)$$

$$= \sum_{2\le m\le [x]-1}(\left|S(\frac{x}{m})\right| - \left|S(\frac{x}{m+1})\right|)\sum_{k\le m}(\Lambda_2(k) - 2\ln k)$$

Combine it with Lemma 6.8 and Lemma 6.9, we get

$$\left|\sum_{m\le x}\Lambda_2(m)\left|S(\frac{x}{m})\right| - 2\sum_{m\le x}\ln m\left|S(\frac{x}{m})\right|\right| < 6.08x\sum_{2\le m\le[x]-1}\frac{1}{m+1} < 6.08x\int_2^{[x]}\frac{1}{s}ds$$
$$< 6.08\,x\ln x$$

Finally, noticing the result of Lemma 6.6, we complete the proof.

Lemma 6.11

$$|S(x)|\ln^2 x < 2\int_1^x\left|S(\frac{x}{s})\right|\ln s\,ds + 26.34\,x\ln x\ (x\ge e^8)$$

Proof.

Since $S(y)=0\ (s<2)$, we have

$$\left|\sum_{m\le x}\ln m\left|S(\frac{x}{m})\right| - \int_1^x\left|S(\frac{x}{s})\right|\ln s\,ds\right| = \left|\sum_{m\le[x]-1}\int_m^{m+1}(\left|S(\frac{x}{m})\right|\ln m - \left|S(\frac{x}{s})\right|\ln s)ds\right|$$

Notice

$$\left|\left|S(\frac{x}{m})\right|\ln m - \left|S(\frac{x}{s})\right|\ln s\right|$$
$$\le \left|S(\frac{x}{m})\ln m - S(\frac{x}{s})\ln s\right|$$
$$\le \left|S(\frac{x}{m})\right|\ln\frac{s}{m} + \left|S(\frac{x}{m}) - S(\frac{x}{s})\right|\ln s$$
$$\overset{Lemma\ 6.1,\ Lemma\ 6.9}{<} 0.65\frac{x}{m}\ln\frac{m+1}{m} + 0.77\frac{x}{m(m+1)}\ln(m+1)$$
$$< 2.07\frac{x}{m+1}$$

we get

$$\left|\sum_{m\le x}\ln m\left|S(\frac{x}{m})\right| - \int_1^x\left|S(\frac{x}{s})\right|\ln s\,ds\right| < 2.07x\sum_{m\le[x]-1}\frac{1}{m+1} < 2.07x\int_1^{[x]}\frac{1}{s}ds$$
$$\le 2.07\,x\ln x$$

Combine it with Lemma 6.10, then the proof is done.

Theorem 6

(a)

$$W(x) < 0.65 \; (x \geq 0)$$

(b)

$$|W(x_2) - W(x_1)| < 1.04(x_2 - x_1) \; (x_2 \geq x_1 \geq 8)$$

(c) Given $8 \leq a < b$ and $W(s) \neq 0 \; (a < s < b)$. Then,

$$\int_a^b |W(s)| \, ds < 3.41$$

(d)

$$|W(x)| < \frac{2}{x^2} \int_0^x \int_0^y |W(s)| \, ds dy + \frac{26.34}{x} \; (x \geq 8)$$

Proof.

When $x \geq 0$,

$$W(x) = \frac{S(e^x)}{e^x} \overset{Lemma\ 6.1}{<} 0.65$$

When $x_2 \geq x_1 \geq 8$,

$$\begin{aligned} |W(x_2) - W(x_1)| = \left| \frac{S(e^{x_1})}{e^{x_1}} - \frac{S(e^{x_2})}{e^{x_2}} \right| &\leq (\frac{1}{e^{x_1}} - \frac{1}{e^{x_2}}) \, |S(e^{x_1})| + \frac{1}{e^{x_2}} \, |S(e^{x_1}) - S(e^{x_2})| \\ &\overset{Lemma\ 6.1, Lemma\ 6.2}{<} 1.04(1 - e^{x_1 - x_2}) \\ &< 1.04(x_2 - x_1) \end{aligned}$$

For a, b described in (c),

$$\begin{aligned} \int_a^b |W(s)| \, ds = \left| \int_a^b \frac{S(e^s)}{e^s} ds \right| \overset{r=e^s}{=\!=\!=} \left| \int_{e^a}^{e^b} \frac{S(r)}{r^2} dr \right| &= \left| \int_2^{e^b} \frac{S(r)}{r^2} dr - \int_2^{e^a} \frac{S(r)}{r^2} dr \right| \\ &\overset{Lemma\ 6.3}{<} 3.41 \end{aligned}$$

Finally, (d) can be proved by combining Lemma 6.11 and the following equation:

$$\begin{aligned} \int_0^x \int_0^y |W(s)| \, ds dy = \int_0^x (\int_0^y \frac{|S(e^s)|}{e^s} ds) dy &\overset{order-exchange}{=\!=\!=\!=\!=} \int_0^x \frac{|S(e^s)|}{e^s} (\int_0^s dy) ds \\ &= -\int_x^0 \frac{|S(e^s)|}{e^s} (x - s) ds \\ &\overset{r=e^{x-s}}{=\!=\!=} \frac{1}{e^x} \int_1^{e^x} \left| S(\frac{e^x}{r}) \right| \ln r \, dr \end{aligned}$$

7 The estimation of $W(x)$

Lemma 7.1

For $n \geq 2$, $W(x)$ has a finite number of roots in $[\ln n, \ln(n+1)]$.

Proof.

If not, choose three of them: $a < b < c$. Then,

$$\int_2^{e^a} \frac{\psi(s)-s}{s} ds = \int_2^{e^b} \frac{\psi(s)-s}{s} ds = \int_2^{e^c} \frac{\psi(s)-s}{s} ds$$

$$\Rightarrow \int_{e^a}^{e^b} \frac{\psi(s)-s}{s} ds = \int_{e^b}^{e^c} \frac{\psi(s)-s}{s} ds = 0 \;\Rightarrow \frac{e^b-e^a}{b-a} = \psi(n) = \frac{e^c-e^b}{c-b}$$

Noticing $(e^x)' = e^x$ is increasing, the upper equation is impossible.
Then the proof is done.

Lemma 7.2

Given $8 \leq a < b$ and $0 < |W(s)| < y$ $(a < s < b, y \leq 0.65)$. Then,

$$\int_a^b |W(s)|\, ds < (y - \frac{y^3}{3.97})(b-a)$$

Proof.

When $b - a \leq \frac{2y}{1.04}$,

$$\int_a^b |W(s)|\, ds \overset{Theorem\ 6(b)}{<} 1.04 \int_a^{\frac{a+b}{2}} (s-a) ds + 1.04 \int_{\frac{a+b}{2}}^b (b-s) ds < (y - \frac{y^3}{3.97})(b-a)$$

When $b - a \geq \frac{3.41}{y - \frac{y^3}{3.97}}$,

$$\int_a^b |W(s)|\, ds \overset{Theorem\ 6(c)}{<} 3.41 < (y - \frac{y^3}{3.97})(b-a)$$

When $\frac{2y}{1.04} < b - a < \frac{3.41}{y - \frac{y^3}{3.97}}$,

$$\begin{aligned}
\int_a^b |W(s)|\, ds &= \int_a^{a+\frac{y}{1.04}} |W(s)|\, ds + \int_{a+\frac{y}{1.04}}^{b-\frac{y}{1.04}} |W(s)|\, ds + \int_{b-1.04}^{b} |W(s)|\, ds \\
&\overset{Theorem 6(b)}{<} 1.04 \int_a^{a+\frac{y}{1.04}} (s-a) ds + (b - a - \frac{2y}{1.04}) y + \int_{b-1.04}^{b} (b-s) ds \\
&= (b-a)y - \frac{y^2}{1.04} < (y - \frac{y^3}{3.97})(b-a)
\end{aligned}$$

Lemma 7.3

Let $a_n = (\frac{1}{2}n)^{\frac{3}{2}n}$ and $b_n = \frac{2}{\sqrt{n}}$. Then, when $x \geq a_n$, we have $|W(x)| < b_n$.

Proof.

Use induction.

For $n \leq 9$, $|W(x)| \overset{Theorem\ 6(a)}{<} 0.65 < b_n$.

Now, suppose the proposition is true for $n = k$.

When $y \geq a_k$, because of Lemma 7.1, there are finite number of roots in $[a_k, y]$.

Assume all of them are $z_1 < z_2 < \cdots < z_d$. Then,

$$\int_0^y |W(s)|\, ds = \int_0^{a_k} |W(s)|\, ds + \int_{a_k}^{z_1} |W(s)|\, ds + \sum_{i=1}^{d-1} \int_{z_i}^{z_{i+1}} |W(s)|\, ds + \int_{z_d}^{y} |W(s)|\, ds$$

$$\overset{Theorem\ 6(a)(b),\, Lemma 7.2}{<} \quad 0.65a_k + 3.41 + (b_k - \frac{b_k^3}{3.97}) \sum_{i=1}^{d-1} (z_{i+1} - z_i) + 3.41$$

$$< 0.65a_k + 6.82 + (b_k - \frac{b_k^3}{3.97})y$$

Considering this inequality is also true for $0 \leq y < a_k$, we have

$$\frac{2}{x^2} \int_0^x \int_0^y |W(s)|\, dsdy < \frac{2}{x^2} \int_0^x (0.65a_k + 6.82 + (b_k - \frac{b_k^3}{3.97})y)dy$$

$$= \frac{1.3a_k + 13.64}{x} + b_k - \frac{b_k^3}{3.97} \ (x \geq a_{k+1})$$

Combine it with Theorem 6(d), then we get

$$|W(x)| < \frac{1.3a_k + 40}{x} + b_k - \frac{b_k^3}{3.97} < b_k - \frac{b_k^3}{8} < b_{k+1}$$

Theorem 7

$$|W(x)| < \frac{2}{\sqrt[3]{\ln x}} \ (x \geq 1)$$

Proof.

Suppose $(\frac{1}{2}n)^{\frac{3}{2}n} \leq x < (\frac{1}{2}(n+1))^{\frac{3}{2}(n+1)}$, then

$$\ln x < \frac{3}{2}(n+1)\ln(\frac{1}{2}(n+1)) < n^{\frac{3}{2}} \Rightarrow |W(x)| \overset{Lemma\ 7.3}{<} \frac{2}{\sqrt{n}} < \frac{2}{\sqrt[3]{\ln x}}$$

8 The estimation of p_n

Lemma 8.1

$$x - 2 - \frac{2x}{\sqrt[3]{\ln\ln x}} < \int_2^x \frac{\psi(s)}{s} ds < x - 2 + \frac{2x}{\sqrt[3]{\ln\ln x}} \ (x \geq e)$$

Proof.

$$\int_2^x \frac{\psi(s)}{s} ds = x - 2 + \int_2^x \frac{\psi(s) - s}{s} ds = x - 2 + x\, W(\ln x)$$

Combine it with Theorem 7, then we complete the proof.

Lemma 8.2

$$x - \frac{4x}{\sqrt[6]{\ln\ln x}} < \psi(x) < x + \frac{6x}{\sqrt[6]{\ln\ln x}} \ (x \geq e^{e^{4096}})$$

Proof.

$$\begin{aligned}
\psi(x)\ln(1+y) = \int_x^{(1+y)x} \frac{\psi(x)}{s} ds &\leq \int_x^{(1+y)x} \frac{\psi(s)}{s} ds \\
&= \int_2^{(1+y)x} \frac{\psi(s)}{s} ds - \int_2^x \frac{\psi(s)}{s} ds \\
&\overset{Lemma\ 8.1}{<} yx + \frac{2x}{\sqrt[3]{\ln\ln x}} + \frac{2(1+y)x}{\sqrt[3]{\ln\ln((1+y)x)}} \\
&< yx + \frac{(4+2y)x}{\sqrt[3]{\ln\ln x}} \ (y = \frac{2}{\sqrt[6]{\ln\ln x}})
\end{aligned}$$

Since

$$\ln(1+y) > \frac{y}{1+y}$$

we have

$$\psi(x) < (1+y)x + \frac{(1+y)(4+2y)}{y\sqrt[3]{\ln\ln x}} < x + \frac{6x}{\sqrt[6]{\ln\ln x}}$$

Similarly,

$$\begin{aligned}
\psi(x)\ln(1+y) \geq \int_2^x \frac{\psi(s)}{s} ds - \int_2^{\frac{x}{1+y}} \frac{\psi(s)}{s} ds &\overset{Lemma\ 8.1}{>} \frac{y}{1+y}x - \frac{2x}{\sqrt[3]{\ln\ln x}} - \frac{\frac{2x}{1+y}}{\sqrt[3]{\ln\ln\frac{x}{1+y}}} \\
&> \frac{y}{1+y}x - \frac{4x}{\sqrt[3]{\ln\ln x}}
\end{aligned}$$

Since

$$\ln(1+y) < y$$

we have

$$\psi(x) > \frac{1}{1+y}x - \frac{4x}{y\sqrt[3]{\ln\ln x}} > x - \frac{4x}{\sqrt[6]{\ln\ln x}}$$

Lemma 8.3

$$\frac{\psi(x)}{\ln x} - \sqrt{x}\ln x \le \pi(x) \le \frac{\psi(x)}{\lambda \ln x} + x^{\lambda}\ (x \ge 1, 0 < \lambda < 1)$$

Proof.

On the one hand, since

$$\theta(x^{\frac{1}{i}}) \le \sqrt{x}\ln\sqrt{x}\ (i \ge 2)$$

we have

$$\pi(x) \ge \frac{\theta(x)}{\ln x} = \frac{\psi(x) - \sum_{i=2}^{M}\theta(x^{\frac{1}{i}})}{\ln x} \ge \frac{\psi(x)}{\ln x} - \sqrt{x}\ln x\ (M = [\frac{\ln x}{\ln 2}])$$

On the other hand,

$$\psi(x) \ge \theta(x) \ge \sum_{x^{\lambda} < p \le x} \ln p \ge \lambda(\pi(x) - \pi(x^{\lambda}))\ln x \ge \lambda(\pi(x) - x^{\lambda})\ln x$$

$$\Rightarrow \pi(x) \le \frac{\psi(x)}{\lambda \ln x} + x^{\lambda}$$

Theorem 8

(a)

$$\frac{x}{\ln x}(1 - \frac{4.01}{\sqrt[6]{\ln\ln x}}) < \pi(x) < \frac{x}{\ln x}(1 + \frac{6.01}{\sqrt[6]{\ln\ln x}})\ (x \ge e^{e^{4096}})$$

(b) If p_n is the nth prime number ($n \ge e^{e^{4096}}$), then

$$n\ln n < p_n < n\ln n\,(1 + \frac{4.02}{\sqrt[6]{\ln\ln n}})$$

Proof.

On the one hand,

$$\pi(x) \overset{Lemma\ 8.3}{\ge} \frac{\psi(x)}{\ln x} - \sqrt{x}\ln x \overset{Lemma\ 8.2}{>} \frac{x - \frac{4x}{\sqrt[6]{\ln\ln x}}}{\ln x} - \sqrt{x}\ln x > \frac{x}{\ln x}(1 - \frac{4.01}{\sqrt[6]{\ln\ln x}})$$

On the other hand, set $\lambda = 1 - 2\frac{\ln\ln x}{\ln x}$ in the result of Lemma 8.3, then we get

$$\pi(x) \le \frac{\psi(x)}{\ln x - 2\ln\ln x} + \frac{x}{\ln^2 x} \overset{Lemma\ 8.2}{<} \frac{x + \frac{6x}{\sqrt[6]{\ln\ln x}}}{\ln x - 2\ln\ln x} + \frac{x}{\ln^2 x} < \frac{x}{\ln x}(1 + \frac{6.01}{\sqrt[6]{\ln\ln x}})$$

Now (a) is proven. Set $x = p_n$ in the result of (a), then we know

$$n > \frac{p_n}{\ln p_n}(1 - \frac{4.01}{\sqrt[6]{\ln \ln p_n}}) \Rightarrow p_n < n \ln n \, (1 + \frac{4.02}{\sqrt[6]{\ln \ln n}})$$

$$n < \frac{p_n}{\ln p_n}(1 + \frac{6.01}{\sqrt[6]{\ln \ln p_n}}) \Rightarrow p_n > n \ln n$$

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