Miyaoka-Yau type inequalities of complete intersection threefolds in products of projective spaces

*Mengxuan Zhang
affiliation: Chongqing Depu Foreign Language School
address: Building 90-4, Yangguang 100 AErLe, Yunan Road 163, Banan District, Chongqing, China

*Mengyao Zhang
affiliation: Chongqing Depu Foreign Language School
address: Building 90-4, Yangguang 100 AErLe, Yunan Road 163, Banan District, Chongqing, China

e-mail address: 1872479691@qq.com

All authors contributed equally.

Abstract. Geography of projective varieties is one of the fundamental problems in algebraic geometry. There are many researches toward the characteristics of Chern number of some projective spaces, for example Noether’s inequalities, the theorem of Chang-Lopez, and the Miyaoka-Yau inequality. In this paper, we compute the Chern numbers of any smooth complete intersection threefold in the product of projective spaces via the standard exact sequences of cotangent bundles. Then we obtain linear Chern number inequalities for $c_1(X)c_2(X)$ and $c_3(X)\frac{c_1^3(X)}{c_1^2(X)}$ on such threefolds under conditions of $d_{ij} \geq 4$ and $d_{ij} \geq 6$ respectively. They can be considered as a generalization of the Miyaoka-Yau inequality and an improvement of Yau’s inequality for such threefolds.

Keywords: Chen class, Miyaoka-Yau inequality, threefold, complete intersection.

1. Introduction
One of the fundamental problems in algebraic geometry is to study the geography of projective varieties, i.e., determining which Chern numbers occur for a complex smooth projective variety $M$. When $M$ is a minimal surface of general type, we have Noether’s inequalities [1]:

$$p_g(M) = h^0(M, \omega_M)$$

$$K_M^2 \geq 2p_g(M) - 4.$$
5c_f^2(M) \geq c_2(M) - 36.

While on the other hand, we have the Miyaoka-Yau inequality:
\[
c_f^2(M) \leq 3c_2(M).
\]

Hence $\frac{c_2(M)}{c_f^2(M)}$ is bounded. When $M$ is a threefold of general type with ample canonical divisor, Yau’s famous inequality in [2] says
\[
8c_3(M)c_2(M) \leq 3c_3^2(M).
\]

Hunt studied the geography of threefolds in [3]. Later, Chang and Lopez proved in [4] that the region described by the Chern ratios $(\frac{c_3^2(M)}{c_1(M)c_2(M)}, \frac{c_2(M)}{c_1(M)c_2(M)})$ of threefolds with ample canonical divisor is bounded. Sheng, Xu and Zhang gave the inequalities of Chern numbers of complete intersection threefolds with ample canonical divisor in [5]:
\[
86c_1^2(M) \leq c_3(M) \leq \frac{c_f^2(M)}{6}.
\]

The theorem of Chang-Lopez has been generalized to higher dimensional case by Du and Sun in [6].

**Theorem 1.1.** Let $X$ be a nonsingular projective variety of dimension $n$ over an algebraic closed field $\kappa$ with any characteristic. Suppose $K_X$ or $-K_X$ is ample. If the characteristic of $\kappa$ is $0$ or the characteristic of $\kappa$ is positive and $\kappa$ with any characteristic. Suppose $X$ is a smooth hypersurface for $i = 1, 2, \cdots, n$, and $X = H_1 \cap H_2 \cap \cdots \cap H_n$ is a smooth threefold.

Our main result is

**Theorem 1.2.** If $d_{ij} \geq 4$ for any $1 \leq i \leq n, 1 \leq j \leq n + 3$, then we have $\frac{1}{2} \leq \frac{c_3(X)c_2(X)}{c_f^2(X)} < \frac{2}{(n+2)^2} + \frac{2}{4n-2} + 1$. If $d_{ij} \geq 6$ for any $1 \leq i \leq n, 1 \leq j \leq n + 3$, then $\frac{c_3(X)c_2(X)}{c_f^2(X)} - \frac{1}{2} \leq \frac{c_f^2(X)}{c_f^2(X)} < \frac{7}{12}$.

In Section 2, we recall the basic definitions and properties of Chern classes. In section 3, we will computes the Chern numbers of $X$. In section 4, we study the upper and lower bounds of $\frac{c_3(X)}{c_f^2(X)}$ and $\frac{c_3(X)c_2(X)}{c_f^2(X)}$.

2. Chern classes
In this section, we introduce the definition of Chern classes.
Let $M$ be a smooth projective variety of dimension $n$. Let $A(M) = \bigoplus_{i=1}^n A^i(M)$ be the Chow ring of $M$. $E$ is a vector bundle on $M$ of rank $r$. The Chern class $c_i(E)$ is a cycle in $A^i(M)$, here $c_0(E) = 1$. We let $c_t(E) = 1 + c_1(E)t + \cdots + c_r(E)t^r$ be the Chern polynomial of $E$.

Chern class $c_i(E)$ satisfies the properties below:

1. If $D$ is a divisor on $M$ and $E \cong O_M(D)$ is a line bundle, then $c_i(E) = D$.
2. If $f : M' \to M$ is a morphism of projective varieties, then $c_i(f^* E) = f^* c_i(E)$.
3. If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of a vector bundle, then
   $$c_t(E) = c_t(E') \cdot c_t(E'') = (1 + c_1(E')t + \cdots + c_r(E')t^r)(1 + c_1(E'')t + \cdots + c_r(E'')t^r).$$
   Assume that rank $E' = r'$, rank $E'' = r''$, so that rank $E = r' + r''$. As a result, we have $c_{r'+r''}(E) = c_{r'}(E)c_{r''}(E'')$.
4. Let $s$ be a global section of $E$. Assume that the zero set $Z(s)$ of $s$ satisfies that $\dim Z(s) = \dim M - r$, then $c_r(E) = Z(s) \in A^r(M)$.

We call $c_i(M) = c_i(T_M)$ the $i$-th Chern class of $M$.

### 3. Chern numbers of complete intersection three-folds in products of projective spaces

In this section, we compute the Chern numbers of $X$.

$$M = \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

then one sees

$$c_t(M) = c_t(T_M) = c_t(\pi_1^* T_{\mathbb{P}^1} \oplus \cdots \oplus \pi_{n+3}^* T_{\mathbb{P}^1}) = (1 + 2Q_1t)(1 + 2Q_2t)\cdots (1 + 2Q_{n+3}t).$$

From the standard exact sequence

$$0 \to O_{H_1}(-H_2) \to \Omega_M|_{H_1} \to \Omega_{H_1} \to 0,$$

after taking duality, we have

$$0 \to T_{H_1} \to T_M|_{H_1} \to O_{H_1}(H_1) \to 0.$$

Hence we have

$$c_t(H_1) = \frac{c_t(T_M|_{H_1})}{c_t(O_{H_1}(H_1))} = \frac{(1+2Q_1t)(1+2Q_2t)\cdots (1+2Q_{n+3}t)|_{H_1}}{(1+H_1t)|_{H_1}}.$$  

From the exact sequence

$$0 \to T_{H_1 \cap H_2} \to T_{H_1}|_{H_1 \cap H_2} \to O_{H_1 \cap H_2}(H_2) \to 0,$$

We obtain
\[ c_t(H_1 \cap H_2) = \frac{(1+2Q_1t)\cdots(1+2Q_{n+3}t)|H_1 \cap H_2}{(1+H_1t)(1+H_2t)|H_1 \cap H_2} \]

By repeating the procedure above, it can be obtained that

\[ c_t(X) = c_t(H_1 \cap \cdots \cap H_n) = \frac{(1+2Q_1t)(1+2Q_2t)\cdots(1+2Q_{n+3}t)|X}{(1+H_1t)(1+H_2t)\cdots(1+H_n)|X} \]

It follows that

\[
(1 + c_1(X)t + c_2(X)t^2 + c_3(X)t^3)(1 + H_1t)(1 + H_2t) \cdots (1 + H_n)X \\
= (1 + 2Q_1t)(1 + 2Q_2t) \cdots (1 + 2Q_{n+3}t)|X.
\]

By considering the coefficient of \( t \), we can get

\[
c_1(X) + H_1|X + H_2|X + \cdots + H_n|X = 2Q_1|X + 2Q_2|X + \cdots + 2Q_{n+3}|X.
\]

Thus,

\[
c_1(X) = (2Q_1 + 2Q_2 + \cdots + 2Q_{n+3} - H_1 - H_2 - \cdots - H_n)|X
= \sum_{i=1}^{n+3} (2 - d_{1i} - d_{2i} - \cdots - d_{ni})Q_i|X.
\]

(1)

As for the coefficient of \( t^2 \), we see that

\[
c_2(X) + c_1(X)(H_1 + H_2 + \cdots + H_n)|X + \sum_{1 \leq i < j \leq n} H_iH_j|X
= 4\sum_{1 \leq i < j \leq n} Q_iQ_j|X.
\]

Since

\[
H_1 + H_2 + \cdots + H_n
= \sum_{i=1}^n d_{1i}Q_1 + \sum_{i=1}^n d_{2i}Q_2 + \cdots + \sum_{i=1}^n d_{i,n+3}Q_{n+3},
\]

We obtain

\[
c_1(X)(H_1 + H_2 + \cdots + H_n)
= \sum_{1 \leq i,j \leq n+3} (2 - d_{1i} - d_{2i} - \cdots - d_{ni}) \sum_{k=1}^n d_{kj}Q_iQ_j.
\]

Simple computations show that

\[
H_iH_j = (d_{i1}Q_1 + d_{i2}Q_2 + \cdots + d_{i,n+3}Q_{n+3})(d_{j1}Q_1 + d_{j2}Q_2 + \cdots + d_{j,n+3}Q_{n+3})
= \sum_{1 \leq k,l \leq n+3} d_{ik}d_{jl}Q_kQ_l
\]

Hence we have

\[
c_2(X) = 4\sum_{1 \leq i < j \leq n+3} Q_iQ_j|X - \sum_{1 \leq k,l \leq n+3} d_{ik}d_{jl}Q_kQ_l|X
= \sum_{1 \leq i,j \leq n+3} (2 - d_{1i} - d_{2i} - \cdots - d_{ni}) \sum_{k=1}^n d_{kj}Q_iQ_j|X.
\]

(2)

Now considering the coefficient of \( t^3 \), we get
This implies

\[
c_3(X) = \sum_{i_1, \cdots, i_n, i, j, k} d_{i_1} \cdots d_{i_{n-1}} \left( 2 - \frac{1}{2} \sum_{t=1}^{n} d_{ti} \right) \left( 2 - \frac{1}{2} \sum_{t=1}^{n} d_{tj} \right) \left( 2 - \frac{1}{2} \sum_{t=1}^{n} d_{tk} \right)
\]

\[ -(2 - \sum_{t=1}^{n} d_{ti}) \sum_{t=1}^{n} d_{tj} \sum_{t=1}^{n} d_{tk} \]

where \(i_1, \cdots, i_n, i, j, k\) take all the arrangements of \(1, 2, \cdots, n + 3\).

By (1), (2), (3), we can have

\[
c_i^2(X) = \sum_{i_1, \cdots, i_n, i, j, k} d_{i_1} \cdots d_{i_{n}} (2 - \sum_{t=1}^{n} d_{ti}) (2 - \sum_{t=1}^{n} d_{tj}) (2 - \sum_{t=1}^{n} d_{tk})
\]

\[ -(2 - \sum_{t=1}^{n} d_{ti}) \sum_{t=1}^{n} d_{tj} \sum_{t=1}^{n} d_{tk} \]

4. Inequalities of Chern numbers

In this section, we estimate the upper and lower bounds for \(\frac{c_1(X)c_2(X)}{c_1^2(X)}\) and \(\frac{c_4(X)}{c_1^4(X)}\) respectively. Let

\[
A_i = (\sum_{t=1}^{n} d_{ti}) - 2,
\]

\[
B_{ij} = \sum_{1 \leq s, t < n, s \neq t} d_{ts} d_{sj},
\]

\[
C_{ijk} = \sum_{1 \leq r, s, t < n, r \neq t} d_{rt} d_{sj} d_{tk},
\]

We have

\[
-c_i^2(X) = \sum_{i_1, \cdots, i_n, i, j, k} d_{i_1} \cdots d_{i_{n-1}} A_i A_j A_k,
\]

\[
-c_1(X)c_2(X) = \sum_{i_1, \cdots, i_n, i, j, k} d_{i_1} \cdots d_{i_{n-1}} \left( 2 - \frac{1}{2} B_{ij} + A_i (A_j + 2) \right) A_k,
\]

\[
-c_3(X) = \sum_{i_1, \cdots, i_n, i, j, k} d_{i_1} \cdots d_{i_{n-1}} \left( 2 - \frac{1}{2} B_{ij} + A_i (A_j + 2) \right) (A_k + 2)
\]

\[ -\frac{1}{2} A_i B_{jk} + \frac{1}{6} C_{ijk} - \frac{4}{3} \]
Lemma 1. If $d_{ij} \geq 4$ for $1 \leq i \leq n, 1 \leq j \leq n + 3, B_{ij} < A_i A_j$.

Proof. If $d_{ij} \geq 4$, we have
\[
\sum_{t=1}^{n} d_{ti} \sum_{t=1}^{n} d_{ij} - 2 \sum_{t=1}^{n} d_{ti} - 2 \sum_{t=1}^{n} d_{ij} + 4
\]
\[
= \frac{1}{2} \sum_{t=1}^{n} d_{ti} d_{tj} - 2 \sum_{t=1}^{n} d_{ti} + \frac{1}{2} \sum_{t=1}^{n} d_{ti} d_{tj} - 2 \sum_{t=1}^{n} d_{ij} + 4
\]
\[
= \sum_{t=1}^{n} \left( \frac{1}{2} d_{ij} - 2 \right) d_{tj} + \sum_{t=1}^{n} \left( \frac{1}{2} d_{tj} - 2 \right) d_{tj} + 4 \geq 4.
\]
Since
\[
A_i A_j = (\sum_{t=1}^{n} d_{ti} - 2)(\sum_{t=1}^{n} d_{ij} - 2)
\]
\[
= \sum_{t=1}^{n} d_{ti} \sum_{t=1}^{n} d_{ij} - 2 \sum_{t=1}^{n} d_{ti} - 2 \sum_{t=1}^{n} d_{ij} + 4
\]
\[
= B_{ij} + \sum_{t=1}^{n} d_{ti} d_{tj} - 2(A_i + 2) - 2(A_j + 2) + 4
\]
\[
= B_{ij} - 2A_i - 2A_j - 4 + \sum_{t=1}^{n} d_{ti} d_{tj}
\]

One sees that
\[
A_i A_j \geq B_{ij} + 4 > B_{ij}.
\]

Lemma 2. When $d_{ij} \geq 4$ for $1 \leq i \leq n, 1 \leq j \leq n + 3$, then we have
\[
\frac{1}{2} < \frac{2 - \frac{1}{2} B_{ij} + A_i(A_j + 2)}{A_i A_j} < \frac{2}{(4n - 2)^2} + \frac{2}{4n - 2} + 1.
\]

Proof. As $d_{ij} \geq 4$, one sees $A_j \geq 4n - 2$, which means $\frac{1}{A_j} \leq \frac{1}{4n - 2}$. We can also have $\frac{1}{A_i} > 0$. By Lemma 1, we have
\[
\frac{2 - \frac{1}{2} B_{ij} + A_i(A_j + 2)}{A_i A_j} = \frac{2}{A_i A_j} - \frac{1}{2} \frac{B_{ij}}{A_j A_j} + \frac{2}{A_j} + 1 - \frac{1}{2} = \frac{1}{2}
\]

On the other hand, we have
\[
\frac{2}{A_i A_j} - \frac{1}{2} \frac{B_{ij}}{A_j A_j} + \frac{2}{A_j} + 1 < \frac{2}{A_i A_j} + \frac{2}{A_j} + 1 < \frac{2}{(4n - 2)^2} + \frac{2}{4n - 2} + 1
\]

(10)

Theorem 4.1. If $d_{ij} \geq 4$ for any $1 \leq i, j \leq n + 3$, then we have $\frac{1}{2} < \frac{c_{ij}(X)c_{ij}(Y)}{c_{ij}^2(X)} < \frac{2}{(4n - 2)^2} + \frac{2}{4n - 2} + 1$.

Proof. The desired conclusion follows from Lemma 2.

4.2. Inequalities of $\frac{c_{ij}(X)}{c_{ij}^2(X)}$

In order to estimate the range of $\frac{c_{ij}(X)}{c_{ij}^2(X)}$, we need to estimate the range of
\[
\frac{2 - \frac{1}{2} B_{ij} + A_i(A_j + 2)}{A_i A_j} \frac{1}{2} A_i B_{jk} + \frac{1}{6} c_{ij k} - \frac{4}{3}
\]

Lemma 3. If $d_{ij} \geq 6$ for any $i, j$, then we have $A_i A_j A_k > c_{ijk}$.
Proof. One sees that

\[ A_i A_j A_k = (\sum_{t=1}^n d_{ti} - 2)(\sum_{t=1}^n d_{tj} - 2)(\sum_{t=1}^n d_{tk} - 2) \]
\[ = \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tj} \sum_{t=1}^n d_{tk} - 2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tj} - 2 \sum_{t=1}^n d_{tj} \sum_{t=1}^n d_{tk} \]
\[ -2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tk} + 4 \sum_{t=1}^n (d_{ti} + d_{tj} + d_{tk}) - 8, \]

and

\[ \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tj} \sum_{t=1}^n d_{tk} \]
\[ = \sum_{1 \leq r, s \leq n} d_{ri} d_{sj} d_{tk} \]
\[ = c_{ijk} + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{rj} d_{tk} + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{tj} d_{tk} \]
\[ + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{sj} d_{rk} + \sum_{t=1}^n d_{ti} d_{tj} d_{tk}. \]

We can further have that

\[ A_i A_j A_k \]
\[ = c_{ijk} + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{rj} d_{tk} - 2 \sum_{t=1}^n d_{tj} \sum_{t=1}^n d_{tk} \]
\[ + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{tj} d_{tk} - 2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tk} \]
\[ + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{sj} d_{rk} - 2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tj} + \sum_{t=1}^n d_{ti} d_{tj} d_{tk}. \]

In order to see the relationship between \( A_i A_j A_k \) and \( c_{ijk} \), we need to calculate the value of

\[ \sum_{1 \leq r \neq s \leq n} d_{ri} d_{rj} d_{tk} - 2 \sum_{t=1}^n d_{tj} \sum_{t=1}^n d_{tk} + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{tj} d_{tk} \]
\[ -2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tk} + \sum_{1 \leq r \neq s \leq n} d_{ri} d_{sj} d_{rk} - 2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tj}. \]

One sees that

\[ \sum_{1 \leq r \neq s \leq n} d_{ri} d_{rj} d_{tk} - 2 \sum_{t=1}^n d_{tj} \sum_{t=1}^n d_{tk} \]
\[ = \sum_{1 \leq r \neq s \leq n} d_{ri} d_{rj} d_{tk} - 2 \sum_{1 \leq r \neq s \leq n} d_{rj} d_{tk} \]
\[ = \sum_{1 \leq r \neq s \leq n} d_{ri} d_{rj} d_{tk} - 2 \sum_{1 \leq r \neq s \leq n} d_{rj} d_{tk} - 2 \sum_{t=1}^n d_{tj} d_{tk} \]
\[ = \sum_{1 \leq r \neq s \leq n} (d_{ri} - 2)d_{rj} d_{tk} - 2 \sum_{t=1}^n d_{tj} d_{tk} > -2 \sum_{t=1}^n d_{tj} d_{tk}. \]

Similarly, we can obtain that

\[ \sum_{1 \leq r \neq s \leq n} d_{ri} d_{tj} d_{tk} - 2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tk} > -2 \sum_{t=1}^n d_{ti} d_{tk} \]

and

\[ \sum_{1 \leq r \neq s \leq n} d_{ri} d_{sj} d_{rk} - 2 \sum_{t=1}^n d_{ti} \sum_{t=1}^n d_{tj} > -2 \sum_{t=1}^n d_{ti} d_{tj}. \]

By (20), (21) and (22), we can have that

\[ A_i A_j A_k > c_{ijk} - 2 \sum_{t=1}^n d_{ti} d_{tj} - 2 \sum_{t=1}^n d_{ti} d_{tk} - 2 \sum_{t=1}^n d_{tj} d_{tk} + \sum_{t=1}^n d_{ti} d_{tj} d_{tk} + 4 \sum_{t=1}^n (d_{ti} + d_{tj} + d_{tk}) - 8. \]

One sees that
\[ \sum_{t=1}^{n} d_{tij} d_{tk} - 2 \sum_{t=1}^{n} d_{tj} d_{tk} - 2 \sum_{t=1}^{n} d_{tj} d_{tk} - 2 \sum_{t=1}^{n} d_{tj} d_{tk} \\
= \left( \frac{1}{3} \sum_{t=1}^{n} d_{tij} d_{tk} - 2 \sum_{t=1}^{n} d_{tj} d_{tk} \right) + \left( \frac{1}{3} \sum_{t=1}^{n} d_{tj} d_{tk} - 2 \sum_{t=1}^{n} d_{tj} d_{tk} \right) \\
+ \left( \frac{1}{3} \sum_{t=1}^{n} d_{tj} d_{tk} - 2 \sum_{t=1}^{n} d_{tj} d_{tk} \right) \\
= \sum_{t=1}^{n} \left( \frac{1}{3} d_{ti} - 2 \right) d_{tj} d_{tk} + \sum_{t=1}^{n} \left( \frac{1}{3} d_{tj} - 2 \right) d_{tj} d_{tk} + \sum_{t=1}^{n} \left( \frac{1}{3} d_{tk} - 2 \right) d_{tj} d_{tk}. \]

If \( d_{ij} \geq 6 \), then we can have that
\[ \sum_{t=1}^{n} \left( \frac{1}{3} d_{ti} - 2 \right) d_{tj} d_{tk} + \sum_{t=1}^{n} \left( \frac{1}{3} d_{tj} - 2 \right) d_{tj} d_{tk} + \sum_{t=1}^{n} \left( \frac{1}{3} d_{tk} - 2 \right) d_{tj} d_{tk} \geq 0. \]

This implies that
\[ A_i A_j A_k > C_{ijk}. \]

As a result, we have
\[ 0 < \frac{C_{ijk}}{A_i A_j A_k} < 1. \]

**Lemma 4.** If \( d_{ij} \geq 6 \) for any \( i, j \), then we have
\[ \frac{\frac{8}{3} - \frac{1}{2} B_{ij} + A_i A_j + 2 A_i - \frac{1}{2} A_j B_{jk} + \frac{1}{6} C_{ijk}}{A_i A_j A_k} > \frac{\frac{1}{2} B_{jk}}{A_j A_k} - \frac{1}{2}. \]

**Proof.** One sees that
\[ \frac{\frac{8}{3} - \frac{1}{2} B_{ij} + A_i A_j + 2 A_i - \frac{1}{2} A_j B_{jk} + \frac{1}{6} C_{ijk}}{A_i A_j A_k} = \frac{\frac{8}{3} - \frac{1}{2} B_{ij} + A_i A_j + 2 A_i - \frac{1}{2} A_j B_{jk} + \frac{1}{6} C_{ijk}}{A_i A_j A_k} - \frac{\frac{1}{2} B_{jk}}{A_j A_k}. \]

By Lemma 1, we have
\[ B_{ij} < A_i A_j, B_{jk} < A_j A_k. \]

Hence we have
\[ \frac{\frac{8}{3} - \frac{1}{2} B_{ij} + A_i A_j + 2 A_i - \frac{1}{2} A_j B_{jk} + \frac{1}{6} C_{ijk}}{A_i A_j A_k} > \]
\[ \frac{\frac{8}{3} - \frac{1}{2} B_{ij} + 2 A_i - \frac{1}{2} A_j B_{jk} + \frac{1}{6} C_{ijk}}{A_i A_j A_k} > 0. \]

This implies
\[ \frac{\frac{8}{3} - \frac{1}{2} B_{ij} + A_i A_j + 2 A_i - \frac{1}{2} A_j B_{jk} + \frac{1}{6} C_{ijk}}{A_i A_j A_k} > \frac{\frac{1}{2} B_{jk}}{A_j A_k} - \frac{1}{2}. \] \hfill (11)

**Theorem 4.2.** If \( d_{ij} \geq 6 \) for any \( i, j \), then we have
\[ \frac{c_3(X) c_2(X)}{c_1(X)} = \frac{1}{2} < \frac{c_3(X)}{c_1(X)} < \frac{7}{12}. \]
Proof. According to Lemma 4, we have that \(-c_3(X) > -c_1(X)c_2(X) - \frac{1}{2}c_1^3(X)\), i.e., \(\frac{c_3(X)c_2(X)}{c_1^2(X)} > \frac{1}{2}\).

Now, we consider the upper bound of \(\frac{c_3(X)}{c_1^2(X)}\).

Because \(A_i = \sum_{t=1}^{n-3} d_{ti} - 2 \geq 6n - 2\), we have

\[
\frac{8}{3} \frac{1}{A_i A_j A_k} + \frac{1}{A_i A_k} + \frac{2}{A_j A_k} + \frac{1}{2} \frac{c_{ijk}}{A_i A_j A_k} < \frac{8}{3} \frac{1}{(6n-2)^3} + \frac{1}{6n-2} + \frac{2}{(6n-2)^2} + \frac{1}{6} \\
\leq \frac{8}{3} + \frac{1}{4} + \frac{2}{16} + \frac{1}{6} \\
= \frac{1}{24} + \frac{1}{4} + \frac{1}{8} + \frac{1}{6} \\
= \frac{7}{12}.
\]

\[(12)\]

5. Conclusions

In this paper, we take \(M = \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1\) as an example to calculate the Chern numbers of complete intersection three-folds in products of projective spaces. Thus, in our conclusion, we get its Chern number and the inequalities that it will satisfy:

If \(d_{ij} \geq 4\) for any \(1 \leq i \leq n, 1 \leq j \leq n + 3\), then we have \(\frac{1}{2} < \frac{c_1(X)c_2(X)}{c_1^2(X)} < \frac{2}{(4n-2)^2} + \frac{2}{4n-2} + 1\). If \(d_{ij} \geq 6\) for any \(1 \leq i \leq n, 1 \leq j \leq n + 3\), then \(\frac{c_1(X)c_2(X)}{c_1^2(X)} - \frac{1}{2} < \frac{c_3(X)}{c_1^2(X)} < \frac{7}{12}\).

However, those conclusions build up on an important assumption, which is the value of \(d_{ij}\). This means that there is still room for exploration and explanation of those results when applying other values of \(d_{ij}\).

As for the future meaning of research into this field, it may help in the field of physics. For instance, Miyaoka-Yau type inequalities are widely applied to the quantum mechanics and field theory, so we believe researches like this can be applied to more different conditions.

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