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MODULAR RELATIONS FOR HURWITZ ZETA FUNCTIONS AND DIRICHLET L-SERIES

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Abstract

We undertake a comprehensive study of Ramanujan's famous identity for odd zeta values from the perspective of Hurwitz zeta functions. We obtain several new transformation formulae whose specializations produce new and interesting identities, as well as recover several other well-known identities from the literature, such as the transformation formula for the logarithm of the Dedekind eta function and the convolution of the Riemann zeta function at odd and even integer arguments by Dixit et al. Our method of deriving transformation formulae can also be applied to other Dirichlet series. In particular, we illustrate this phenomenon by applying our method to Dirichlet L-series and thereby producing some new transformation formulae involving the convolution Dirichlet L-series at positive integer arguments as the residual term.

Keywords: Hurwitz zeta functions, Mellin transform, Dirichlet L-series, Lambert series.

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1. Introduction and Motivation

The Riemann zeta function $\zeta(s)$, defined by the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$, is undoubtedly one of the most paramount functions in analytic number theory as it plays a crucial role in studying the distribution of primes and has applications in various branches of physics, probability theory, applied mathematics, and statistics. While the critical strip $0 < \Re(s) < 1$ is indeed the most important region in the complex plane on account of the unsolved problem regarding the location of non-trivial zeros of $\zeta(s)$, namely, the Riemann Hypothesis, the right-half plane $\Re(s) > 1$ also has its own share of interesting unsolved problems to contribute to; such as the algebraic nature of $\zeta(s)$ at integer arguments.

Over the past few centuries, the zeta values and their algebraic behaviour have been studied extensively by many mathematicians. In 1734, Euler established an explicit formula for $\zeta(2n)$ in terms of powers of π and Bernoulli number B_{2n}. In particular, he proved that that, for all $n \in \mathbb{N}$,

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2\,(2n)!},$$

which immediately implies that all even zeta values are transcendental due to the well-known fact that π is transcendental and Bernoulli numbers are rational. However, no such explicit formula is known to exist for $\zeta(2k+1)$, as a result of which, its arithmetic nature still remains largely mysterious and open. Only in 1978 did Apéry [2] famously prove the irrationality of $\zeta(3)$.

In this direction, Lerch [20], in 1901, proved that, for all $m \in \mathbb{N}$,

$$\zeta(4n+3) = \pi^{4n+3} 2^{4n+2} \sum_{k=0}^{2n+2} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{4n+4-2k}}{(4n+4-2k)!} - 2 \sum_{k=1}^{\infty} \frac{1}{k^{4n+3}(e^{2\pi k} - 1)}. \tag{1.1}$$

On account of the rapid convergence of the infinite series in (1.1), we can deduce that $\zeta(4n+3)$ is "almost a rational multiple π", as recently stated by Berndt and Straub [6].

Ramanujan [21, p. 173, Ch. 14, Entry 21(i)], who made many intriguing discoveries in his short life of 32 years, obtained the following elegant generalization of Lerch's identity (1.1).

Theorem 1.1 (Ramanujan's formula for $\zeta(2n+1)$)**.** *Let $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha\beta = \pi^2$. Then, for all $n \in \mathbb{Z} \setminus \{0\}$, the following identity holds.*

$$\alpha^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\alpha m} - 1} \right\} - (-\beta)^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\beta m} - 1} \right\}$$

$$(1.2) \qquad = 2^{2n} \sum_{k=0}^{n+1} \frac{(-1)^{k-1} B_{2k} B_{2n-2k+2}}{(2k)!\,(2n-2k+2)!} \alpha^{n-k+1} \beta^{k},$$

where, as usual, B_n denotes the n-th Bernoulli number.

Over the decades, this identity took the attention of many mathematicians, and several generalizations of Theorem 1.1 of different kinds were studied. To know more about the history of this formula and its further generalizations, we refer the reader to [6, 13, 18].

A. Dixit et. al. [14] recently asked whether a Ramanujan-type identity exists for the Hurwitz zeta functions. We answer this question positively in this work via Theorem 3.2 and also provide Hurwitz zeta generalization and analogs of several other identities from the literature. Moreover, our method of proving and obtaining these results can also be applied to other exotic Dirichlet series, say, the Dirichlet L-series for instance, which we talk about later in the manucript.

Notice that Ramanujan's identity (1.2) can also be interpreted as a quasimodular transformation, a transformation of the form $F(z) = F\left(-\frac{1}{z}\right) + R(z)$, where $R(z)$ is the residual term given by the convolution of Bernoulli numbers. Toward this end, the following notable extension of Ramanujan's formula was provided by Grosswald [16].

Theorem 1.2 (Grosswald [16])**.** *Let $z \in \mathfrak{h}$ and let*

$$F_k(z) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^k} e^{2i\pi nz}.$$

Then, we have

$$F_{2k+1}(z) - z^{2k} F_{2k+1}\left(\frac{-1}{z}\right) = \frac{\zeta(2k+1)}{2}(z^{2k}-1) + \frac{(2i\pi)^{2k+1}}{2z} R_{2k+1}(z).$$

This formula is important due to the fact that it relates Ramanujan's formula with Eisenstein series $E_{2k}(z)$ over the full modular group $\mathrm{SL}_2(\mathbb{Z})$ since the Fourier expansion of $E_{2k}(z)$ is given by

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} F_{1-2k}(z).$$

1.1. **Organisation of the manuscript.** We devote Section 2 to compiling basic properties of Hurwitz zeta functions and Mellin transforms. In Section 3, we state some of our main results and obtain several transformation formulae involving convolution of Hurwitz zeta functions. The primary method of proving these results is straightforward and involves standard techniques in complex analysis. In Section 4, we apply this methodology to Dirichlet L-series and thereby generate similar transformation formulae. Finally, in section 5, we provide detailed proofs of all our results.

2. Prelimnaries

The *Mellin transform* of a function $\varphi(t)$ is defined by

$$\widetilde{\varphi}(s) = \int_0^\infty t^{s-1}\varphi(t)\,dt.$$

For some $c \in \mathbb{R}$, denote by

$$\int_{\Re(s)=c} F(s)\,ds = \int_{(c)} F(s)\,ds.$$

We can recover the original function $\varphi(t)$ from the following result.

Theorem 2.1. *Let $\widetilde{\varphi}(s)$ be a function of the complex variable $s = \sigma + it$ such that it is holomorphic in the strip $S = \{s \mid a < \sigma < b\}$ and $|\widetilde{\varphi}(s)| \to 0$ as $\Im(s) \to \infty$ uniformly in the strip $a - \eta \leqslant \sigma \leqslant b - \eta$ for any arbitrary small $0 < \eta$. Then if*

$$\int_{-\infty}^{\infty} \left|\widetilde{\varphi}(\sigma + it)\right| dt$$

is finite for each $\sigma \in (a, b)$, and if a function $\varphi(t)$ is defined by

$$\varphi(t) = \frac{1}{2i\pi}\int_{(c)} t^{-s}\widetilde{\varphi}(s)\,ds,$$

for $t > 0$ and some fixed $c \in (a, b)$, then

$$\widetilde{\varphi}(s) = \int_0^\infty t^{s-1}\varphi(t)\,dt.$$

The *Bernoulli polynomials* $B_n(x)$ are defined through their generating function as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{2.1}$$

where $|t| < 2\pi$. The Bernoulli numbers B_n are defined by $B_n = B_n(0)$.

The *Hurwitz zeta function* for complex variable s with $\Re(s) > 1$ and $a \in \mathbb{C}/\mathbb{Z}_{\leqslant 0}$ is defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}. \tag{2.2}$$

The Hurwitz zeta function has the integral representation [3, p. 251]

$$\zeta(s, a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-ax}}{1 - e^{-x}}ds, \tag{2.3}$$

valid for $\Re(s) > 1$. Moreover, it can be analytically continued to a meromorphic function via the following contour integral [23]:

$$\zeta(s, a) = -\frac{\Gamma(1 - s)}{2i\pi}\int_C \frac{(-z)^{s-1}e^{-az}}{1 - e^{-z}}dz, \tag{2.4}$$

where C is the Hankel contour counterclockwise around the positive real axis and principal branch of logarithm is used. The integral (2.4) defines $\zeta(s,a)$ for all $s \in \mathbb{C}$, with a single pole at $s = 1$ and corresponding residue 1. The value of Hurwitz zeta function at negative integers is given by

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}.$$

The Laurent series expansion of Hurwitz zeta centered at $s = 1$ is given by

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a)\,(s-1)^n\,, \tag{2.5}$$

where $\gamma_n(a)$ are the generalized Stieltjes constants. Note that $\gamma_0(a) = -\psi(a)$.

3. Hurwitz zeta functions

In this Section, we undertake a comprehensive study of Ramanujan-type identities from the perspective of Hurwitz zeta functions. Moreover, we also rederive several other identities from the literature as special cases of our general formulae. Note that most results from this section (except Theorem 3.12) have been published by the author in [9].

The function $\frac{1}{e^{2\pi x}-1}$, as seen in Theorem 1.1, also appears in several of Ramanujan's identities and has the following integral representation:

$$\frac{1}{e^{2\pi x}-1} = \frac{1}{2i\pi}\int_{(c)} \frac{\zeta(1-s)}{2\cos\left(\frac{\pi s}{2}\right)} x^{-s} ds,$$

where (c) denotes the vertical line $\Re(s) = c$ with c an arbitrary real number such that $1 < c < 2$. This representation can be deduced by simply realizing that

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x-1} dx,$$

and using Mellin inversion. We call this function *Ramanujan's kernel* throughout the rest of the manuscript. We now introduce a two-parameter generalization of Ramanujan's kernel. One reason to study this new kernel is to obtain more information on the arithmetic nature of the odd zeta values. By including a free parameter a, one can derive other results involving the zeta function by differentiating or integrating with respect to a.

Ramanujan's kernel has simple poles at $x = 0$ and $x = \pm in, n \in \mathbb{N}$ with residue $\frac{1}{2\pi}$ at $0, \pm in$, and thus has the partial fraction expansion

$$\frac{1}{e^{2\pi x}-1} = -\frac{1}{2} + \frac{1}{2\pi x} + \frac{x}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}.$$

In this manuscript, we generalize this result to a meromorphic function with simple poles at

$$x=0,\quad x=e^{\frac{(2j+1)i\pi}{2k}}\left(n+a\right),\quad \left(n\in\mathbb{N},\, a\in\mathbb{C}\right),$$

where $j\in\{0,1,\ldots,2k-1\}$, with residue $\frac{1}{2k\pi}$ at $e^{\frac{(2j+1)i\pi}{2k}}(n+a)$ defined as follows.

Definition 3.1. *Let $x\in\mathbb{R}^+, a\in\mathbb{C}$ and $k\in\mathbb{N}$. Define the Hurwitz kernel by*

$$\begin{aligned}\Psi\left(x,a;k\right) &:= \frac{1}{2i\pi}\int_{(c)}\frac{\zeta\left(1-s,a\right)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}x^{-s}\mathrm{d}s\\ &= \frac{2a-1}{2\pi x}-\frac{1}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{x^{2k-1}}{x^{2k}+(n+a)^{2k}},\end{aligned}\tag{3.1}$$

where $c\in(1,2)$.

The second equality in equation (3.1) can be derived by shifting the line of integration to negative infinity and collecting the residues at corresponding poles. Note that $\Psi(x,1;1)$ is Ramanujan's kernel. A. Dixit et al. ask in [14] whether a Ramanujan type identity for Hurwitz zeta function exists and we answer this question positively in this work.

Denote by $\Psi_\alpha(x,a;k)=\Psi\left(\frac{\alpha}{\pi}x,a;k\right)$. Observing the analogy between Ramanujan's kernel and Hurwitz kernel $\Psi(x,a;k)$ we derive a Ramanujan-type formula involving $\Psi(x,a;k)$ as follows.

Theorem 3.2. *Let $\alpha,\beta\in\mathbb{R}^+$ such that $\alpha\beta=\pi^2$ and let $k,N\in\mathbb{N}$. Then, we have*

$$\begin{aligned}&\beta^{k(N+1)-1}\left(\sum_{n=0}^{\infty}\frac{\Psi_\alpha(n+b,a;k)}{(n+b)^{2k(N+1)-1}}+\frac{\zeta(2k\left(N+1\right)-1,b)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\right)\\ &=(-1)^N\alpha^{k(N+1)-1}\left(\sum_{n=0}^{\infty}\frac{\Psi_\beta(n+a,b;k)}{(n+a)^{2k(N+1)-1}}+\frac{\zeta(2k\left(N+1\right)-1,a)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\right)\\ &+\sum_{p=0}^{N+1}(-1)^{p+1}\zeta(2kp,a)\,\zeta(2k\left(N-p+1\right),b)\,\alpha^{kp-1}\beta^{k(N+1-p)-1}.\end{aligned}\tag{3.2}$$

Ramanujan's identity gives an expression for the convolution of Riemann zeta at even arguments. In the same spirit it can be asked whether such an expression exists for the convolution of Riemann zeta at odd arguments. This was positively answered in [12]:

Theorem 3.3. *Let α and β be two complex numbers such that $\Re(\alpha)>0, \Re(\beta)>0$ and $\alpha\beta=4\pi^2$. Let ψ denote the digamma function. Then for $m\in\mathbb{N}$, we have*

$$(-\beta)^{-m}\left\{2\gamma\zeta(2m+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\left(\psi\left(\frac{\imath n\beta}{2\pi}\right)+\psi\left(-\frac{\imath n\beta}{2\pi}\right)\right)\right\}$$

$$+\alpha^{-m}\left\{2\gamma\zeta(2m+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\left(\psi\left(\frac{in\alpha}{2\pi}\right)+\psi\left(-\frac{in\alpha}{2\pi}\right)\right)\right\}$$
$$=-2\sum_{k=1}^{m-1}(-1)^k\,\zeta(2k+1)\,\zeta(2m-2k+1)\,\alpha^{k-m}\beta^{-k}.$$

We claim that Theorem 3.3 is related to the integral kernel

$$\frac{1}{2i\pi}\int_{(c)}\frac{\zeta\left(1-s\right)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\,\mathrm{d}s, \tag{3.3}$$

where $1<c<2$. This will be proved as a special case of Theorem 3.5. We now proceed to provide a series expansion of the integral in equation (3.3) by shifting the line of integration to negative infinity and collecting the residues at corresponding poles.

Consider rectangular the contour determined by the line segments $[c-iT,c+iT],[c+iT,c'+iT],[c'+iT,c'-iT],[c'-iT,c-iT]$ where $c'=1-c$. Inside this contour,

$$\frac{\zeta\left(1-s\right)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)},$$

has a pole of order 2 at $s=0$. It can be calculated

$$\mathrm{Res}\left(\frac{\zeta\left(1-s\right)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\right)_{s=0}=\frac{\log(x)+\gamma}{\pi}.$$

Thus by Cauchy's residue Theorem, we have

$$\frac{1}{2i\pi}\left[\int_{c-iT}^{c+iT}+\int_{c+iT}^{c'+iT}+\int_{c'+iT}^{c'-iT}+\int_{c'-iT}^{c-iT}\right]\frac{\zeta\left(1-s,a\right)\zeta(2kN+2k-1+s,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s$$
$$=\frac{\log(x)+\gamma}{\pi}.$$

From Lemma 5.1 and equation (5.2), it can be seen that as $T\to\infty$, the integrals along horizontal segments tend to zero. Thus we have,

$$\frac{1}{2i\pi}\int_{(c)}\frac{\zeta\left(1-s\right)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\,\mathrm{d}s=\frac{\log(x)+\gamma}{\pi}+\frac{1}{2i\pi}\int_{(c')}\frac{\zeta\left(1-s\right)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\,\mathrm{d}s.$$

On account of absolute convergence of the series representing $\zeta(s)$ for $\Re(s)>1$, we have,

$$\int_{(c')}\frac{\zeta\left(1-s\right)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\,\mathrm{d}s=\sum_{n=1}^{\infty}\int_{(c')}\frac{n^{s-1}x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\,\mathrm{d}s.$$

On shifting the line of integration to negative infinity and using Cauchy's Residue Theorem, we get the following equality for $|x|<1$

(3.4) $$\frac{1}{2i\pi}\int_{(c')}\frac{n^{s-1}x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\mathrm{d}s=\sum_{i=1}^{\infty}\mathrm{Res}\left(\frac{n^{s-1}x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\right)_{s=-2i}=\frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^i\frac{x^{2i}}{n^{2i+1}}=-\frac{1}{\pi}\frac{x^2}{n\,(x^2+n^2)}.$$

Noticing that the integral on the left-hand side of equation (3.4) is analytic in the plane $\Re(x)>0$, we conclude that equation (3.4) holds for all $x\in\mathbb{C}$ satisfying $\Re(x)>0$. Thus, the following equality holds for all $x\in\mathbb{R}^+$

$$\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\mathrm{d}s=\frac{\log(x)+\gamma}{\pi}-\frac{x^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n(x^2+n^2)}.$$

We now give an alternate expression of the above series in terms of the Digamma function $\psi(x)$. The Digamma function satisfies the following identities [1]:

$$\psi(x+1)=\psi(x)+\frac{1}{x},$$

$$\psi(x+1)=-\gamma+\sum_{n=1}^{\infty}\left(\frac{x}{n(n+x)}\right),\qquad x\notin\{-1,-2,-3,\ldots\}.$$

Using the above two well-known formulas for the digamma function $\psi(x)$, we conclude that

$$\begin{aligned}\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s)x^{-s}}{2\sin\left(\frac{\pi s}{2}\right)}\mathrm{d}s&=\frac{\log(x)+\gamma}{\pi}-\frac{x^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n(x^2+n^2)}\\&=\frac{1}{\pi}\left(\log(x)-\frac{\psi(ix)+\psi(-ix)}{2}\right).\end{aligned}$$

This suggests defining a similar analog of the Hurwitz kernel.

Definition 3.4. *Let $x\in\mathbb{R}^+, a\in\mathbb{C}$ and $k\in\mathbb{N}$. Define the odd Hurwitz kernel by*

$$\Phi(x,a;k):=\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s,a)}{2k\sin\left(\frac{\pi s}{2k}\right)}x^{-s}\,\mathrm{d}s$$

(3.5) $$=\frac{\log(x)-\psi(a)}{\pi}-\frac{x^{2k}}{\pi}\sum_{n=0}^{\infty}\frac{1}{(n+a)((n+a)^{2k}+x^{2k})},$$

As before, denote by $\Phi_\alpha(x,a;k)=\Phi\left(\frac{\alpha}{\pi}x,a;k\right)$. Observing the analogy between the analog of Ramanujan's kernel and the odd Hurwitz kernel, we now state a generalization of Theorem 3.3.

Theorem 3.5. *Let $\alpha,\beta>0$ and $\alpha\beta=\pi^2$. Then, the following identity holds*

$$\beta^{km}\left(\sum_{n=0}^{\infty}\frac{\Phi_\alpha(n+b,a;k)}{(n+b)^{2km+1}}-\frac{1}{\pi}\left(\zeta(2km+1,b)\left(\log\left(\frac{\alpha}{\pi}\right)-\psi(a)\right)-\frac{\partial}{\partial s}\zeta(2km+1+s,b)\Big|_{s=0}\right)\right)$$

$$=(-1)^m\frac{\alpha^{km}}{\pi}\left(\zeta(2km+1,a)\left(\log\left(\frac{\beta}{\pi}\right)-\psi(b)\right)-\frac{\partial}{\partial s}\zeta(2km+1+s,a)\Big|_{s=0}\right)$$

$$+(-1)^{m+1}\,\alpha^{km}\sum_{n=0}^{\infty}\frac{\Phi_\beta(n+a,b;k)}{(n+a)^{2km+1}}+\frac{1}{\pi}\sum_{i=1}^{m-1}(-1)^i\,\zeta(2ki+1,a)\,\alpha^{ki}\zeta(2k(m-i)+1,b)\,\beta^{k(m-i)}. \tag{3.6}$$

Finally, the following result [11] answers the question of determining a formula relating the convolution of Riemann zeta at odd and even arguments:

Theorem 3.6. *For $\alpha,\beta>0$ with $\alpha\beta=\pi^2$ and $m\in\mathbb{N}$, we have*

$$\beta^{-(m-\frac{1}{2})}\left\{\frac{1}{2}\zeta(2m)+\sum_{n=0}^{\infty}\frac{n^{-2m}}{e^{2n\beta}-1}\right\}-\sum_{k=0}^{m-1}(-1)^{k+1}\frac{\zeta(2k)\zeta(2m-2k+1)}{\pi^{2k}}\beta^{2k-m-\frac{1}{2}}$$
$$=(-1)^{m+1}\,\alpha^{-(m-\frac{1}{2})}\left\{\frac{\gamma}{\pi}\zeta(2m)+\frac{1}{2\pi}\sum_{n=1}^{\infty}n^{-2m}\left(\psi\left(\frac{in\alpha}{\pi}\right)+\psi\left(\frac{-in\alpha}{\pi}\right)\right)\right\}. \tag{3.7}$$

As before, we now state an elegant generalization of this result to the convolution of Hurwitz zeta function at odd and even arguments, which also relates $\Phi(x,a;k)$ with $\Psi(x,a;k)$.

Theorem 3.7. *Let $\alpha,\beta>0$ and $\alpha\beta=\pi^2$. Then, we have*

$$\pi\beta^{km}\left(\sum_{n=0}^{\infty}\frac{\Phi_\alpha(n+b,a;k)}{(n+b)^{2km}}-\frac{1}{\pi}\left(\zeta(2km,b)\left(\log\left(\frac{\alpha}{\pi}\right)-\psi(a)\right)-\frac{\partial}{\partial s}\zeta(2km+s,b)\Big|_{s=0}\right)\right)$$
$$=(-1)^m\,\pi^2\alpha^{km-1}\left(\sum_{n=0}^{\infty}\frac{\Psi_\beta(n+a,b;k)}{(n+a)^{2km}}+\frac{\zeta(2km,a)}{2k\sin\left(\frac{\pi}{2k}\right)}\right)+\sum_{i=1}^{m}(-1)^i\,\alpha^{ki}\zeta(2ki+1,a)\beta^{k(m-i)}\zeta(2k(m-i),b).$$

Ramanujan's identity (1.2) has the following specialization, namely, for $\alpha,\beta>0$ with $\alpha\beta=\pi^2$,

$$\alpha^{m+1}\sum_{n=1}^{\infty}\frac{n^{2m+1}}{e^{2\alpha n}-1}-(-\beta)^{m+1}\sum_{n=1}^{\infty}\frac{n^{2m+1}}{e^{2\beta n}-1}=\left(\alpha^{m+1}-(-\beta)^{m+1}\right)\frac{B_{2m+1}}{4m+1}, \tag{3.8}$$

with $m>1$, which can be found in [21]. The following result provides a generalization of the above identity, that is, equation (3.8) involving the Hurwitz kernel $\Psi(x,a;k)$.

Theorem 3.8. *Let $\alpha,\beta\in\mathbb{R}^+$ such that $\alpha\beta=\pi^2$ and $B_j(x)$ denote Bernoulli polynomials. Then, we have*

$$\alpha^{km+1}\sum_{n=0}^{\infty}(n+b)^{2km+1}\left[\Psi_\alpha(n+b,a;k)-\sum_{p=1}^{m}\frac{B_{2kp+1}(a)}{2kp+1}\left(\frac{\pi}{\alpha}\right)^{2kp+1}\right].$$
$$=(-1)^{m+1}\,\beta^{km+1}\sum_{n=0}^{\infty}(n+a)^{2km+1}\left[\Psi_\beta(n+a,b;k)-\sum_{p=1}^{m}\frac{B_{2kp+1}(b)}{2kp+1}\left(\frac{\pi}{\beta}\right)^{2kp+1}\right]$$
$$+\frac{\alpha^{km+1}B_{2km+2}(b)}{4k(km+1)\cos\left(\frac{\pi(k-1)}{2k}\right)}+\frac{(-1)^m\,\beta^{km+1}B_{2km+2}(a)}{4k\,(km+1)\cos\left(\frac{\pi(k-1)}{2k}\right)}+\sum_{p=0}^{m}(-1)^p\,\frac{B_{2kp+1}(a)\,\beta^{kp}B_{2k(m-p)+1}(b)\,\alpha^{k(m-p)}}{(2kp+1)\,(2k\,(m-p)+1)}.$$

As a special case, plugging in $a=b=1$ in Theorem 3.8 yields the following interesting identity.

Proposition 3.9. *Let $\alpha,\beta>0$ and $\alpha\beta=\pi^2$. Then, the following identity holds*

$$\alpha^{km+1}\left(\sum_{n=1}^{\infty}\frac{\Psi_\alpha(n,1;k)}{n^{-2km-1}}-\frac{B_{2km+2}}{4k(km+1)\cos\left(\frac{\pi(k-1)}{2k}\right)}\right)$$
$$=(-1)^{m+1}\beta^{km+1}\left(\sum_{n=1}^{\infty}\frac{\Psi_\beta(n,1;k)}{n^{-2km-1}}-\frac{\beta^{km+1}B_{2km+2}}{4k(km+1)\cos\left(\frac{\pi(k-1)}{2k}\right)}\right). \tag{3.9}$$

Moreover, substituting in $\alpha=\beta=\pi$ with $m=2p$ in Proposition 3.9 produces

$$\sum_{n=1}^{\infty}\frac{\Psi_\pi(n,1;k)}{n^{-4kp-1}}=\frac{B_{4kp+2}}{4k\,(2kp+1)\cos\left(\frac{\pi(k-1)}{2k}\right)}, \tag{3.10}$$

which is a generalization of Glaisher's famous identity [15]

$$\sum_{n=1}^{\infty}\frac{n^{4m+1}}{e^{2\pi n}-1}=\frac{B_{4m+2}}{2(4m+2)}. \tag{3.11}$$

Notice that we recover equation (3.8) as a special case $k=1$ of Proposition 3.9.

As a companion of Theorem 4.3, we have the following intriguing identity for $\Phi(x,a;k)$.

Theorem 3.10. *Let $\alpha,\beta>0$ and $\alpha\beta=\pi^2$. Then, the following identity holds*

$$\alpha^{km}\sum_{n=0}^{\infty}(n+b)^{2km-1}\left[\Phi_\alpha(n+b,a;k)-\sum_{p=1}^{m}\frac{B_{2kp}(a)}{2kp}\left(\frac{\pi}{\alpha}\right)^{2kp}\right]$$
$$+\frac{\alpha^{km}}{\pi}\left(\frac{B_{2km}(b)}{2km}\left(\log\left(\frac{\alpha}{\pi}\right)-\psi(a)\right)+\frac{\partial}{\partial s}\zeta(s-2km+1,b)|_{s=0}\right)$$
$$=(-1)^{m+1}\beta^{km}\sum_{n=0}^{\infty}(n+a)^{2km-1}\left[\Phi_\beta(n+a,b;k)-\sum_{p=1}^{m}\frac{B_{2kp}(b)}{2kp}\left(\frac{\pi}{\beta}\right)^{2kp}\right]$$
$$+\frac{(-1)^{m+1}\beta^{km}}{\pi}\left(\frac{B_{2km}(a)}{2km}\left(\log\left(\frac{\beta}{\pi}\right)-\psi(b)\right)+\frac{\partial}{\partial s}\zeta(s-2km+1,a)\Big|_{s=2km}\right)$$
$$+\frac{1}{\pi}\sum_{p=1}^{m-1}(-1)^p\,\frac{B_{2kp}(a)\,\beta^{kp}B_{2k(m-p)}(b)\,\alpha^{k(m-p)}}{(2kp)\,(2k\,(m-p))} \tag{3.12}$$

Ramanujan also gave the transformation formula for the logarithm of the Dedekind eta function in the following form.

Theorem 3.11. *If α,β are positive reals such that $\alpha\beta=\pi^2$, then*

$$\sum_{m=1}^{\infty}\frac{1}{m(e^{2\alpha m}-1)}-\sum_{m=1}^{\infty}\frac{1}{m(e^{2\beta m}-1)}=\frac{1}{4}\log\left(\frac{\alpha}{\beta}\right)-\frac{\alpha-\beta}{12}.$$

We now provide a generalization of this formula in our context.

Theorem 3.12. *If α, β are positive reals such that $\alpha\beta = \pi^2$, then*

$$\sum_{m=0}^{\infty} \frac{\Psi_\alpha(m+b, a; 1)}{m+b} - \sum_{m=0}^{\infty} \frac{\Psi_\beta(m+a, b; 1)}{m+a} = \frac{1}{4}\left(\log\left(\frac{\alpha}{\beta}\right) + 2(\psi(b) - \psi(a))\right) + \frac{(2a-1)\zeta(2,b)}{2\alpha} + \frac{(1-2b)\zeta(2,a)}{2\beta}.$$

Plugging in $a = b$ produces

$$\sum_{m=0}^{\infty} \frac{\Psi_\alpha(m+a, a; 1)}{m+a} - \sum_{m=0}^{\infty} \frac{\Psi_\beta(m+a, a; 1)}{m+a} = \frac{1}{4}\log\left(\frac{\alpha}{\beta}\right) + \frac{(2a-1)\zeta(2,a)}{2}\left(\frac{\beta-\alpha}{\pi^2}\right).$$

3.1. **Sketch of the proofs.** In this Subsection, we briefly render a rough outline of the proof of our results. In particular, we highlight the common element when proving and obtaining such results. For the sake of argument, we ignore the constants and convergence conditions. However, further details of all the proofs in rigor can be found in Section 5.

Let

$$\zeta_{x,a}(s) = \sum_{n=1}^{\infty} \frac{a_n}{x_n^s}, \quad \zeta(y, b) = \sum_{n=1}^{\infty} \frac{b_n}{y_n^s},$$

and let $P(\Gamma(s), k)$ denote some product of Gamma functions with rational linear combinations of s, k as their arguments. It can be noticed that all of our results involve expressions of the form

$$\sum_{n=1}^{\infty} \frac{a_n \Phi_y(x_n z, k)}{x_n^N}, \tag{3.13}$$

where $N \in \mathbb{N}$ and

$$\Phi_y(z, k) = \frac{1}{2i\pi} \int_{\Re(s)=c} \frac{\zeta_{y,b}(1-s)}{P(\Gamma(s), k)} z^{-s} ds,$$

for some $c \in \mathbb{R}$. By suitably shifting the line of integration beyond the abscissa of convergence of $\zeta(s, a)$ we can interchange summation and integral in equation (3.13), and it can be rewritten as

$$\sum_{n=1}^{\infty} \frac{a_n \Phi_y(x_n z, k)}{x_n^N} + R_1(z) = \frac{1}{2i\pi} \int_{\Re(s)=d} \frac{\zeta_{y,b}(1-s)\zeta_{x,a}(N+s)}{P(\Gamma(s), k)} z^{-s} ds. \tag{3.14}$$

Now assuming that

$$\frac{\zeta_{y,b}(1-s)\zeta_{x,a}(N+s)}{P(\Gamma(s), k)} z^{-s},$$

decays exponentially as $\Im(s) \to \infty$, we can shift the line of integration to $\Re(s) = -c - N + 1$ and use the residue theorem from complex analysis to write

$$\frac{1}{2i\pi} \int_{(d)} \frac{\zeta_{y,b}(1-s)\zeta_{x,a}(N+s)}{P(\Gamma(s), k)} z^{-s} ds = R_2(z) + \frac{1}{2i\pi} \int_{(-d-N+1)} \frac{\zeta_{y,b}(1-s)\zeta_{x,a}(N+s)}{P(\Gamma(s), k)} z^{-s} ds, \tag{3.15}$$

where $R_2(z)$ is the sum of residues of the integrand inside the strip $S = \{s \in \mathbb{C} \mid -N-d+1 < \Re(s) < d\}$. Now if $P(\Gamma(s), k)$ is invariant up to a constant under the change of variable $s \to -s - N + 1$, we can apply the same change of variable to write

$$\int_{(-d-N+1)} \frac{\zeta_{y,b}(1-s)\zeta_{x,a}(N+s)}{P(\Gamma(s),k)} z^{-s} ds = z^N c_1 \int_{(d)} \frac{\zeta_{x,a}(1-s)\zeta_{y,b}(N+s)}{P(\Gamma(s),k)} z^s ds, \tag{3.16}$$

for some constant c_1. Noticing the similarity between equations (3.14) and (3.16), upon shifting the line of integration beyond the abscissa of convergence of $\zeta_{y,b}(N+s)$, we can write

$$\int_{(d)} \frac{\zeta_{x,a}(1-s)\zeta_{y,b}(N+s)}{P(\Gamma(s),k)} z^{-s} ds = R_3(z) + \sum_{n=1}^{\infty} \frac{b_n \Phi_x(y_n z, k)}{y_n^N}. \tag{3.17}$$

Upon combining all these equations we obtain

$$\sum_{n=1}^{\infty} \frac{a_n \Phi_y(x_n z, k)}{x_n^N} + R_1(z) - z^N c_1 \left(R_3(z) + \sum_{n=1}^{\infty} \frac{b_n \Phi_x\left(\frac{y_n}{z}, k\right)}{y_n^N} \right) = R_2(z). \tag{3.18}$$

Notice that this method can be specialized to obtain reciprocity relations for any Dirichlet series. An astute reader would notice that a similar method was obtained by Bochner in his seminal work [7] which enables one to obtain reciprocity relations. Note that in general, we do not need $P(\Gamma(s), k)$ to be invariant under some change of variable. This situation will be illustrated via proof of Theorem 3.7.

4. Dirichlet-L series

In this Section, we state several new reciprocity formulae for Dirichlet L-series. These formulae can be considered analogs of formulae from the previous section.

We first provide a primer on Dirichlet L-series.

4.1. **Background.** Given a fixed positive integer m, a *Dirichlet character* χ of modulus m is a complex-valued arithmetic function $\chi : \mathbb{Z} \to \mathbb{C}$ that satisfies the following conditions

(1) $\chi(1) = 1$.
(2) $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{N}$.
(3) $\chi(a + m) = \chi(a)$.
(4) $\chi(a) = 0$ for $\gcd(a, m) > 1$.

The value of the minimal period of χ is called as the *conductor* of χ and is denoted by f_χ. The simplest Dirichlet character of modulus m is called the *principal character* and is defined by

$$\chi(n) = \begin{cases} 1 & \gcd(m,n) = 1 \\ 0 & \gcd(m,n) > 1. \end{cases}$$

Given a Dirichlet character χ and a complex number s satisfying $\Re(s) > 1$, the corresponding *Dirichlet L-series* is defined as

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Dirichlet L-series can be analytically continued on the whole complex plane to a meromorphic function and are also called *Dirichlet L-function*. Let $\varphi(n)$ denote Euler's totient function. For the principal character χ_1 of modulus m, the analytic continuation of $L(s,\chi_1)$ has a simple pole at $s = 1$ with corresponding residue $\frac{\varphi(m)}{m}$ whereas $L(s,\chi)$ can be analytically continued to an entire function for all non-principal characters χ.

Let $\delta \in \{0,1\}$ be defined by $\chi(-1) = (-1)^\delta$. A Dirichlet character χ is even if $\delta = 0$ and odd if $\delta = 1$. A Dirichlet character of modulus m is said to be *primitive* if $f_\chi = m$ and *imprimitive* otherwise. In what follows, we assume χ to be primitive. Given a character of modulus m, the Laurent series expansion of $L(s,\chi)$ is known to be

$$L(s,\chi) = \frac{\delta_\chi}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^n \gamma_n(\chi)}{n!}(s-1)^n$$

where

$$\delta_\chi = \begin{cases} \frac{\varphi(m)}{m} & \chi \text{ is principal} \\ 0 & \text{otherwise.} \end{cases}$$

The *generalized Bernoulli numbers* attached to a character χ are defined by

$$F_\chi(t) := \frac{1}{e^{ft}-1} \sum_{a=1}^{f_\chi} \chi(a)\, te^{at} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n,$$

which converges for $|t| < \frac{2\pi}{f_\chi}$. By means of analytic continuation, it can be deduced that

$$L(-n,\chi) = -\frac{B_{n+1,\chi}}{n+1}.$$

The Gaussian sum $g(\chi)$ associated to a character χ is defined as

$$g(\chi) = \sum_{a=1}^{f_\chi} \chi(a) \exp\left(\frac{2i\pi a}{f_\chi}\right).$$

The following result can be used to calculate special values of $L(s,\chi)$ at positive integers [4]

Theorem 4.1. *Let k be a natural number. If $\chi(-1) = (-1)^k$, then the special value of $L(k,\chi)$ is given by*

$$L(k,\chi) = \frac{(-1)^{k-1}(2i\pi)^k}{2k!\,f_\chi^k} g(\chi) B_{k,\chi}.$$

The above theorem implies

(1) If χ is even, then for all integers $n \geqslant 1$ we have

$$L(2n,\chi) = -\frac{(2i\pi)^{2n}}{2(2n)!\,f^{2n}} g(\chi) B_{2n,\chi}.$$

(2) If χ is odd, then for all integers $n \geqslant 1$ we have

$$L(2n+1,\chi) = \frac{(2i\pi)^{2n+1}}{2(2n+1)!\,f^{2n+1}} g(\chi) B_{2n+1,\chi}.$$

Example. Let χ_4 be the alternating character of modulus 4. The associated L-series is

$$L(s,\chi_4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

This series is also called *Dirichlet Beta function*. The corresponding Gauss sum is

$$g(\chi_4) = \sum_{a=1}^{4} \chi(a) \exp\left(\frac{i\pi}{2}\right) = 2i,$$

and the corresponding generalization of Bernoulli numbers is

$$\frac{1}{e^{4t}-1} \sum_{a=1}^{4} \chi(a) t e^{at} = \frac{t(e^t - e^{3t})}{e^{4t}-1} = \frac{-t}{e^t + e^{-t}}.$$

It is well known that

$$\frac{2}{e^t + e^{-t}} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!},$$

where E_k are Euler numbers. Thus

$$B_{k,\chi_4} = \begin{cases} 0 & k = 0 \\ -\frac{kE_{k-1}}{2} & k \geq 1. \end{cases}$$

Since χ_4 is odd, invoking Theorem 4.1 we obtain

$$L(2n+1,\chi_4) = (-1)^n \frac{E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}. \tag{4.1}$$

We now introduce analogs of Hurwitz and Odd Hurwitz kernels in the context of the Dirichlet L-series. Given a complex number $z \in \mathbb{C}$, an integer $k > 0$ and a Dirichlet character χ, define

$$\Psi(z,\chi;k) := \frac{1}{2i\pi} \int_{(c)} \frac{L(\chi, 1-s)}{2k \cos\left(\frac{\pi(s+k-1)}{2k}\right)} z^{-s} ds,$$

and

$$\Phi(z,\chi;k) := \frac{1}{2i\pi}\int_{(c)} \frac{L(\chi,1-s)}{2k\sin\left(\frac{\pi s}{2k}\right)} z^{-s} ds.$$

Next, we evaluate $\Psi(z,\chi;k)$. Let

$$F(s,\chi;k) = \frac{L(\chi,1-s)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)} z^{-s}.$$

From equation (5.3) and lemma 5.1 we get that $|F(s,\chi;k)|$ decays exponentially as $\Im(s)\to\infty$. This allows us to shift the line of integration to the left. We now shift the line of integration to negative infinity and collect residues at corresponding poles. The integrand has simple poles at $-2km+1$ where $m\in\mathbb{N}_{\geq 0}$. It also has a simple pole at $s=0$ if χ is principal. The residues are

$$R_0 = \frac{-\varphi(m)}{2km\cos\left(\frac{\pi(k-1)}{2k}\right)}$$

$$R_{-2km+1} = (-1)^{m+1}\frac{L(\chi,2km)}{\pi} z^{2km-1}.$$

Let $h(\chi)=R_0$ if χ is principal or zero otherwise. Using residue theorem we have

$$\Psi(z,\chi;k) = h(\chi) - \frac{1}{\pi z}\sum_{m=0}^{\infty} (-1)^m L(2km,\chi) z^{2km}.$$

Next, we use the fact that $B_{2j+1,\chi}=0$ for $j\geq 1$ when χ is even to write

$$\begin{aligned}\Psi(z,\chi;1) &= h(\chi) - \frac{L(0,\chi)}{2\pi z} - \frac{1}{\pi z}\sum_{m=1}^{\infty}(-1)^m L(2m,\chi) z^{2m}\\ &= h(\chi) - \frac{L(0,\chi)}{2\pi z} + \frac{g(\chi)}{2\pi z}\sum_{m=1}^{\infty}\frac{B_{2m,\chi}}{(2m)!}\left(\frac{2\pi z}{f}\right)^{2m}\\ &= h(\chi) - \frac{L(0,\chi)}{\pi z} + \frac{g(\chi)}{2\pi z}\left(F_\chi\left(\frac{2\pi z}{f}\right) - B_{0,\chi} - 2\pi z B_{1,\chi}\right).\end{aligned}$$

For instance, when χ is the principal character of modulus one, we recover

$$\Psi(z,\chi;1) = \frac{1}{e^{2\pi z}-1}.$$

A similar method can be employed to evaluate $\Phi(z,\chi;k)$. The funtion $\frac{L(1-s,\chi)}{2k\sin\left(\frac{\pi s}{2k}\right)}z^{-s}$ has simple poles at $s=-2km$ where $m\in\mathbb{N}$. It also has a simple or double pole at $s=0$ depending upon χ. The corresponding residues are

$$R_0 = \begin{cases} \frac{1}{\pi}\left(\frac{\varphi(m)\log(z)}{m} + \gamma_0(\chi)\right) & \chi \text{ is principal}\\ \frac{L(1,\chi)}{\pi} & \text{otherwise.}\end{cases}$$

$$R_{-2km} = (-1)^m\frac{L(\chi,2km+1)}{\pi} z^{2km}.$$

Let

$$h_2(\chi) = \begin{cases} \frac{1}{\pi}\left(\frac{\varphi(m)\log(z)}{m} + \gamma_0(\chi)\right) & \chi \text{ is principal,} \\ \frac{L(1,\chi)}{\pi} & \text{otherwise.} \end{cases}$$

Using a similar argument as before we deduce

$$\Phi(z,\chi;k) = h_2(\chi) + \sum_{m=1}^{\infty}(-1)^m \frac{L(\chi, 2km+1)}{\pi} z^{2km}.$$

Upon substituting $\chi = \chi_4$ and $k = 1$ we obtain

$$\begin{aligned} \Phi(z,\chi;k) &= \frac{L(1,\chi_4)}{\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^m L(\chi_4, 2m+1) z^{2m} \\ &= \frac{1}{4} + \frac{1}{4}\sum_{m=1}^{\infty} \frac{E_{2m}}{(2m)!}\left(\frac{\pi z}{2}\right)^{2m} \\ &= \frac{1}{4\cosh\left(\frac{\pi z}{2}\right)}. \end{aligned} \tag{4.2}$$

Finally, we are now ready to state our main results. We begin by providing the following elegant generalization of Ramanujan's identity.

Theorem 4.2. *Let χ_1, χ_2 be Dirichlet characters of moduli m_1, m_2 respectively and let $\mu \in \mathfrak{h}$. Then, we have*

$$\sum_{n=0}^{\infty} \frac{\chi_2(n)\Psi(\mu n, \chi_2; k)}{n^{2km-1}} + (-1)^m \mu^{2km-2} \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^{2km-1}} \Psi\left(\frac{n}{\mu}, \chi_2; k\right)$$
$$= h(\chi_1, k) + h(\chi_2, k) + \frac{1}{\pi}\sum_{p=0}^{m} (-1)^{p+1} L(\chi_1, 2k(m-p)) L(\chi_2, 2kp) \mu^{2kp-1},$$

where

$$h(\chi_1, k) = -\frac{\delta_{\chi_1} L(2km-1, \chi_2)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}, \quad h(\chi_2, k) = (-1)^{m-1}\frac{\delta_{\chi_2} L(2km-1, \chi_1)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\mu^{2km-2}.$$

We now provide an analog of Theorem 3.7 in this context.

Theorem 4.3. *Let χ_1, χ_2 be Dirichlet characters of moduli m_1, m_2 respectively and $\mu \in \mathfrak{h}$. Then,*

$$\sum_{n=1}^{\infty} \frac{\chi_1(n)\Phi(\mu n, \chi_2; k)}{n^{2km}} - (-1)^m \mu^{2km-1} \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^{2km}} \Psi\left(\frac{n}{\mu}, \chi_1; k\right)$$
$$= h(\chi_1, k) + h(\chi_2, k) + \frac{1}{\pi}\sum_{p=1}^{m} (-1)^p L(\chi_1, 2k(m-p)) L(\chi_2, 2kp+1) \mu^{2kp},$$

where

$$h(\chi_1) = \begin{cases} \frac{\delta_{\chi_2}}{\pi}\left(L(2km,\chi_1)\log(\mu) - \frac{\partial}{\partial s}L(2km+s,\chi_1)\mid_{s=0}\right) + \frac{\gamma_0(\chi_2)}{\pi}L(2km,\chi_1) & \chi_2 \text{ is principal} \\ \frac{L(1,\chi_2,)L(2km,\chi_1)}{\pi} & \text{otherwise,} \end{cases}$$

$$h(\chi_2) = (-1)^m \frac{\delta_{\chi_1}L(2km,\chi_2)\mu^{2km-1}}{2k\sin\left(\frac{\pi}{2k}\right)}.$$

Substituting $k = 1$ yields the following.

Corollary 4.4. *Let $N \in \mathbb{N}$ and χ_1, χ_2 be Dirichlet characters of moduli m_1, m_2 respectively. Then, we have*

$$\sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^{2N}} \Psi(\mu n, \chi_2; 1) - \left(-\mu^2\right)^N \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^{2N}} \Phi\left(\frac{n}{\mu}, \chi_1; 1\right)$$
$$= h(\chi_1) + h(\chi_2) + \frac{1}{\pi}\sum_{k=1}^{N} (-1)^k L(2N-2k, \chi_1) L(2k+1\chi_2)\mu^{2k}.$$

where where

$$h(\chi_1) = \begin{cases} \frac{\delta_{\chi_2}}{\pi}\left(L(2m,\chi_1)\log(\mu) - \frac{\partial}{\partial s}L(2m+s,\chi_1)\mid_{s=0}\right) + \frac{\gamma_0(\chi_2)}{\pi}L(2m,\chi_1) & \chi_2 \text{ is principal} \\ \frac{L(1,\chi_2,)L(2m,\chi_1)}{\pi} & \text{otherwise,} \end{cases}$$

$$h(\chi_2) = (-1)^m \frac{\delta_{\chi_1}L(2m,\chi_2)\mu^{2km-1}}{2}.$$

On substituting χ_1 to be the principal character of modulus one, $\chi_2(n) = \chi_4$ and $\mu = \alpha/\pi$ in corollary 4.4 we recover the following result by Bradley [8, Corollary 4].

Corollary 4.5. *Let $N \in \mathbb{N}$ and let $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha\beta = \pi^2$. Then, the following identity holds*

$$\alpha^{\frac{1}{2}-N}\left(\frac{\beta(2N)}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^{-2N}}{e^{(4n+2)\alpha}-1}\right) - \frac{(-1)^N}{4}\beta^{\frac{1}{2}-N}\sum_{n=1}^{\infty} \frac{n^{-2N}}{\cosh\left(\frac{n\beta}{2}\right)}$$
$$= 2^{2N-3}\sum_{k=0}^{N} \frac{(-1)^k E_{2k}B_{2N-2k}}{2^{4k}(2k)!\,(2N-2k)!}\alpha^{N-k}\beta^{k+\frac{1}{2}}.$$

Substituting χ_1, χ_2 to be the principal characters of modulus one and $\mu = \alpha/\pi$ in corollary 4.4 we recover the following identity by A. Dixit et. al. [11].

Corollary 4.6. *Let $N \in \mathbb{N}$ and let $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha\beta = \pi^2$. Then, the following identity holds*

$$\beta^{-\left(m-\frac{1}{2}\right)}\left\{\frac{1}{2}\zeta(2m) + \sum_{n=0}^{\infty} \frac{n^{-2m}}{e^{2n\beta}-1}\right\} - \sum_{k=0}^{m-1} \frac{(-1)^{k+1}}{\pi^{2k}}\zeta(2k)\zeta(2m-2k+1)\beta^{2k-m-\frac{1}{2}}$$
$$= (-1)^{m+1}\alpha^{-\left(m-\frac{1}{2}\right)}\left\{\frac{\gamma}{\pi}\zeta(2m) + \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{1}{n^{2m}}\left(\psi\left(\frac{in\alpha}{\pi}\right) + \psi\left(\frac{-in\alpha}{\pi}\right)\right)\right\}.$$

Lastly, we provide an analog of Theorem 3.5

Theorem 4.7. *Let χ_1, χ_2 be dirichlet characters of moduli m_1, m_2 respectively. Then, we have*

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{2km+1}} \Phi(\mu n, \chi_2, k) - (-1)^m \mu^{2km} \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^{2km+1}} \Phi\left(\frac{n}{\mu}, \chi_1; k\right)$$
$$= h(\chi_1, k) + h(\chi_2, k) + \frac{1}{\pi} \sum_{p=1}^{m-1} (-1)^p L(\chi_1, 2k(m-p)+1) L(\chi_2, 2kp+1) \mu^{2kp},$$

where

$$h_1(\chi_1) = \begin{cases} \frac{\delta_{\chi_2}}{\pi}\left(\log(\mu) L(2km+1, \chi_1) - \frac{\partial}{\partial s} L(2km+s+1, \chi_1)\mid_{s=0}\right) + \frac{\gamma_0(\chi_2)}{\pi} L(2km+1, \chi_1) & \chi_2 \text{ is principal} \\ \frac{L(1,\chi_2) L(2km+1, \chi_1)}{\pi} & \text{otherwise,} \end{cases}$$

and, when χ_2 is principal

$$h_2(\chi_2) = (-1)^{m-1} \mu^{2km} \left(\frac{\delta_{\chi_2}}{\pi}\left(\log(\mu) L(2km+1, \chi_1) - \frac{\partial}{\partial s} L(2km+s+1, \chi_1)\mid_{s=0}\right) + \frac{\gamma_0(\chi_2)}{\pi} L(2km+1, \chi_1)\right),$$

otherwise we have

$$h_2(\chi_2) = (-1)^m \mu^{2km} \frac{L(1, \chi_1) L(2km+1, \chi_2)}{\pi}.$$

Substituting $k = 1, \chi_1 = \chi_2 = \chi_4$, and $\mu = \frac{4\alpha}{\pi}$ we recover the following result by Berndt [5].

Corollary 4.8. *Let $N > 0$ and $\alpha\beta = \pi^2/16$. Then*

$$\alpha^{-N} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2N+1} \cosh(2(2n+1)\alpha)} + (-\beta)^{-N} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2N+1} \cosh(2(2n+1)\beta)}$$
$$= 2^{2N-2} \pi \sum_{j=0}^{N} (-1)^j \frac{E_{2j}}{(2j)!} \frac{E_{2N-2j}}{(2N-2j)!} \alpha^{N-j} \beta^j.$$

5. Proofs

Before proving these results we state an important inequality that we will use throughout the proofs. Stirling's formula for the gamma function on a vertical strip states that for $a \leqslant \sigma \leqslant b$ and $|t| \geqslant 1$,

$$|\Gamma(\sigma+it)| = (2\pi)^{\frac{1}{2}} |t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} \left(1 + O\left(\frac{1}{|t|}\right)\right). \tag{5.1}$$

The reflection formula for the gamma function is $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, z \notin \mathbb{Z}$ Thus as $\Im(s) \to \infty$ using the reflection formula and equation (5.1) we have the inequality

$$\frac{1}{\left|\sin\left(\frac{\pi s}{2k}\right)\right|} = 2\exp\left(\frac{-\pi}{2}\left|\frac{\Im(s)}{k}\right|\right)\left(1 + O\left(\frac{1}{|\Im(s)|}\right)\right). \tag{5.2}$$

A variant of the reflection formula for the gamma function is $\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{\cos(\pi z)}, z\notin\mathbb{Z}-\frac{1}{2}$. Thus as $\Im(s)\to\infty$ using the variant of reflection formula and equation (5.1) we have the inequality

$$\frac{1}{\left|\cos\left(\frac{\pi(s+k-1)}{2k}\right)\right|}=2\exp\left(\frac{-\pi}{2}\left|\frac{\Im(s)}{k}\right|\right)\left(1+O\left(\frac{1}{|\Im(s)|}\right)\right). \tag{5.3}$$

The next result gives an upper bound on growth rate of the Riemann zeta function.

Lemma 5.1. *For $\sigma>\sigma_0$, there exists a constant $C(\sigma_0)$ such that*

$$|\zeta(\sigma+iT)|\ll|T|^{C(\sigma)}$$

as $|T|\to\infty$.

Proof. The proof can be found in [22, p. 95]. ∎

Most of the proofs follow a similar structure and thus some of them can be skipped. However, we provide complete proofs of all our results for the convinience of the reader.

5.1. **Proof of Theorem 3.2.** On account of the absolute convergence of the Hurwitz zeta function for $\Re(s)>1$ we have

$$\sum_{n=0}^{\infty}\frac{\Psi_\alpha(n+b,a)}{(n+b)^{2kN+2k-1}}=\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s,a)\,\zeta(2kN+2k-1+s,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s.$$

We now evaluate this integral by shifting the line of integration. Consider rectangular the contour determined by the line segments $[c-iT,c+iT],[c+iT,d+iT],[d+iT,d-iT],[d-iT,c-iT]$ where $d=-c-2kN-2k+2$. Inside this contour, the integrand has simple poles at $0,-2kN-2k+2$ and at the integers $-2kp+1$ where $p\in\{0,1,\ldots,N+1\}$. The residues at these poles are

$$R_0=\frac{-\zeta(2kN+2k-1,b)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)},$$

$$R_{-2kN-2k+2}=\left(\frac{\alpha}{\pi}\right)^{2kN+2k-2}\frac{\zeta(2kN+2k-1,a)}{2k\cos\left(\frac{\pi(-2kN-k+1)}{2k}\right)},$$

$$R_{-2kp+1}=(-1)^{p+1}\left(\frac{\alpha}{\pi}\right)^{2kp-1}\zeta(2kp,a)\zeta(2k(N+1-p),b).$$

Thus by Cauchy's Residue Theorem, we have

$$\frac{1}{2i\pi}\left[\int_{c-iT}^{c+iT}+\int_{c+iT}^{d+iT}+\int_{d+iT}^{d-iT}+\int_{d-iT}^{c-iT}\right]\frac{\zeta(1-s,a)\,\zeta(2kN+2k-1+s,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s$$

$$=R_0+R_{-2kN-2k+2}+\sum_{p=0}^{N+1}R_{-2kp+1}.$$

From Lemma 5.1 and equation (5.3), it can be seen that as $T \to \infty$, the integrals along horizontal segments tend to zero. Under the change of variables $s \to -s - 2kN - 2k + 2$ we have

$$\int_{(d)} \frac{\zeta(1-s,a)\,\zeta(2kN+2k-1+s,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}$$
$$= (-1)^N \left(\frac{\alpha}{\pi}\right)^{2kN+2k-2} \int_{(c)} \frac{\zeta(1-s,b)\zeta(2kN+2k-1+s,a)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\beta}{\pi}\right)^{-s},$$

which completes the proof of Theorem 3.2. ∎

5.2. **Proof of Theorem 3.5.** On account of the absolute convergence of the series representing $\zeta(s,b)$ for $\Re(s) > 1$ we have

$$\sum_{n=0}^{\infty} \frac{\Phi_\alpha(n+b,a;k)}{(n+b)^{2km+1}} = \frac{1}{2i\pi}\int_{(c)} \frac{\zeta(1-s,a)\,\zeta(2km+1+s,b)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s} \mathrm{d}s \tag{5.4}$$

where $1 < c < 2$. We now evaluate this integral by shifting the line of integration and using the residue Theorem. Consider the rectangular contour determined by the line segments $[c-iT, c+iT]$, $[c+iT, d+iT]$, $[d+iT, d-iT]$, $[d-iT, c-iT]$ where $d = -2km - c$. Inside this contour, the integrand has poles of order two at $0, -2km$ and simple poles at the integers $-2ki$ where $i \in \{1, \ldots, m-1\}$. The residues at these poles are

$$R_0 = \frac{1}{\pi}\left(\zeta(2km+1,b)\left(\log\left(\frac{\alpha}{\pi}\right) - \psi(a)\right) - \frac{\partial}{\partial s}\zeta(2km+1+s,b)\Big|_{s=0}\right)$$
$$R_{-2km} = \frac{(-1)^m}{\pi}\left(\frac{\alpha}{\pi}\right)^{2km}\left(-\zeta(2km+1,a)\left(\psi(b) + \log\left(\frac{\alpha}{\pi}\right)\right) + \frac{\partial}{\partial s}\zeta(2km+1+s,a)\Big|_{s=0}\right)$$
$$R_{-2i} = (-1)^i\left(\frac{\alpha}{\pi}\right)^{2ki}\frac{\zeta(2ki+1,a)\zeta(2k(m-i)+1,b)}{\pi}$$

Thus we have

$$\frac{1}{2i\pi}\left[\int_{c-iT}^{c+iT} + \int_{c+iT}^{d+iT} + \int_{d+iT}^{d-iT} + \int_{d-iT}^{c-iT}\right]\frac{\zeta(1-s,a)\,\zeta(s+2km+1,b)}{2\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s$$
$$= R_0 + R_{-2km} + \sum_{i=1}^{m-1} R_{-2ki}$$

From Lemma 5.1 and equation (5.3), it can be seen that as $T \to \infty$, the integrals along horizontal segments tend to zero. Under the change of variables $s \to -2km - s$ we have

$$\int_{(d)} \frac{\zeta(1-s,a)\,\zeta(s+2km+1,b)}{2k\sin\left(\frac{\pi s}{2}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s$$
$$= \left(\frac{\alpha}{\pi}\right)^{2km}(-1)^m\int_{(c)} \frac{\zeta(1-s,b)\zeta(s-2km-1,a)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\beta}{\pi}\right)^{-s}\mathrm{d}s$$

which gives us the desired result. ∎

5.3. **Proof of Theorem 3.7.** On account of the absolute convergence of $\zeta(s,a)$ for $\Re(s) > 1$, $k > 1$ we have

$$\sum_{n=0}^{\infty} \frac{\Phi_\alpha(n+b,a;1)}{(n+b)^{2km}} = \frac{1}{2i\pi}\int_{(c)} \frac{\zeta(1-s,a)\,\zeta(2km+s,b)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s} \mathrm{d}s$$

We now evaluate this integral by shifting the line of integration and using Cauchy's Residue Theorem. Consider the contour determined by the line segments $[c-iT, c+iT]$, $[c+iT, d+iT]$, $[d+iT, d-iT]$, $[d-iT, c-iT]$ where $d = -c-2km+1$. Inside this domain, the integrand has a simple pole at $-2km+1$ due to $\zeta(2km+s,b)$ and a pole of order two at $s=0$. It also has simple poles at the integers $-2ki$ where $i \in \{1,\ldots,m\}$ due to sine term in the denominator. The residues at these poles are

$$R_0 = \frac{1}{\pi}\left(\zeta(2km,b)\left(\log\left(\frac{\alpha}{\pi}\right)-\psi(a)\right) - \frac{\partial}{\partial s}\zeta(2km+s,b)\Big|_{s=0}\right),$$

$$R_{-2km+1} = \left(\frac{\alpha}{\pi}\right)^{2km-1} \frac{(-1)^m\zeta(2km,a)}{2\sin\left(\frac{\pi}{2k}\right)},$$

$$R_{-2ki} = (-1)^i\left(\frac{\alpha}{\pi}\right)^{2ki} \frac{\zeta(2ki+1,a)\zeta(2k(m-i),b)}{\pi}.$$

Thus by Cauchy's Residue Theorem, we have

$$\frac{1}{2i\pi}\left[\int_{c-iT}^{c+iT} + \int_{c+iT}^{d+iT} + \int_{d+iT}^{d-iT} + \int_{d-iT}^{c-iT}\right] \frac{\zeta(1-s,a)\,\zeta(s-2km-1,b)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s} \mathrm{d}s$$
$$= R_0 + R_{-2km+1} + \sum_{i=1}^{m} R_{-2ki}.$$

From Lemma 5.1 and equation (5.2), it can be seen that as $T \to \infty$, the integrals along horizontal segments tend to zero. Under the change of variables $s \to -s-2km+1$ we have

$$\int_{(d)} \frac{\zeta(1-s,a)\,\zeta(2km+s,b)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s} \mathrm{d}s$$
$$= \left(\frac{\alpha}{\pi}\right)^{2km-1}(-1)^m \int_{(c)} \frac{\zeta(2km+s,a)\zeta(1-s,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\beta}{\pi}\right)^{-s} \mathrm{d}s.$$

However,

$$\frac{1}{2i\pi}\int_{(c)} \frac{\zeta(2km+s,a)\zeta(1-s,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\beta}{\pi}\right)^{-s} \mathrm{d}s = \sum_{n=0}^{\infty}\frac{\Psi_\beta(n+a,b;k)}{(n+a)^{2m}},$$

which completes the proof of Theorem 3.7. ∎

5.4. **Proof of Theorem 3.8.** On account of the absolute convergence of $\zeta(s,a)$ for $\Re(s) > 1$ we have

$$\sum_{n=0}^{\infty} (n+b)^{2km+1}\left[\Psi_{\alpha}(n+b,a;k) - \sum_{p=1}^{m} \frac{B_{2kp+1}(a)}{2kp+1}\left(\frac{\pi}{\alpha}\right)^{2kp+1}\right]$$
$$= \frac{1}{2i\pi}\int_{(d)} \frac{\zeta(1-s,a)\,\zeta(s-2km-1,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s} \mathrm{d}s,$$

where $2km+2 < d < 2km+3$, since

$$\int_{(c)} \frac{\zeta(1-s,a)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}x^{-s}\mathrm{d}s = \int_{(d)} \frac{\zeta(1-s,a)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}x^{-s}\mathrm{d}s - \sum_{p=1}^{m} \frac{B_{2kp+1}(a)}{2kp+1}x^{-2kp-1}.$$

We now evaluate this integral by shifting the line of integration and using the Cauchy's Residue Theorem. Consider the rectangular contour determined by the line segments $[d-iT, d+iT], [d+iT, e+iT], [e+iT, e-iT], [e-iT, d-iT]$ where $e = 2km+2-d$. Inside this contour, the integrand has simple poles at $0, 2km+2$ and at integers $2kp+1$ where $p \in \{0, 1, \ldots, m\}$. The residues at these poles are

$$R_0 = \frac{B_{2km+2}(b)}{2k(2km+2)\cos\left(\frac{\pi(k-1)}{2k}\right)},$$
$$R_{2km+2} = (-1)^m\left(\frac{\pi}{\alpha}\right)^{2km+2}\frac{B_{2km+2}(a)}{2k(2km+2)\cos\left(\frac{\pi(k-1)}{2k}\right)},$$
$$R_{2kp+1} = (-1)^p\left(\frac{\pi}{\alpha}\right)^{2kp+1}\frac{B_{2kp+1}(a)B_{2k(m-p)+1}(b)}{\pi\,(2kp+1)\,(2k\,(m-p)+1)}.$$

Thus by Cauchy's Residue Theorem, we have

$$\frac{1}{2i\pi}\left[\int_{d-iT}^{d+iT} + \int_{d+iT}^{e+iT} + \int_{e+iT}^{e-iT} + \int_{e-iT}^{d-iT}\right]\frac{\zeta(1-s,a)\,\zeta(s-2km-1,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s} \mathrm{d}s$$
$$= R_0 + R_{2km+2} + \sum_{p=0}^{m} R_{2kp+1}.$$

From Lemma 5.1 and equation 5.3, it can be seen that as $T \to \infty$, the integrals along horizontal segments tend to zero. Under the change of variables $s \to 2km+2-s$ we have

$$\int_{(e)} \frac{\zeta(1-s,a)\,\zeta(s-2km-1,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s} \mathrm{d}s$$
$$= \left(\frac{\alpha}{\pi}\right)^{-2km-2}(-1)^{m+1}\int_{(d)} \frac{\zeta(1-s,b)\zeta(s-2km-1,a)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\left(\frac{\beta}{\pi}\right)^{-s} \mathrm{d}s$$

which proves Theorem 4.3. ∎

5.5. **Proof of Theorem 3.10.** On account of absolute convergence of $\zeta(s,a)$ for $\Re(s) > 1$ we have

$$\sum_{n=0}^{\infty}(n+b)^{2km-1}\left[\Phi_\alpha(n+b,a;k)-\sum_{p=1}^{m}\frac{B_{2kp}(a)}{2kp}\left(\frac{\pi}{\alpha}\right)^{2kp}\right]$$
$$=\frac{1}{2i\pi}\int_{(d)}\frac{\zeta(1-s,a)\,\zeta(s-2km+1,b)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s,$$

where $2km < d < 2km+1$ since

$$\int_{(c)}\frac{\zeta(1-s,a)}{2k\sin\left(\frac{\pi s}{2k}\right)}x^{-s}\mathrm{d}s=\int_{(d)}\frac{\zeta(1-s,a)}{2k\sin\left(\frac{\pi s}{2k}\right)}x^{-s}\mathrm{d}s-\sum_{p=1}^{m}\frac{B_{2kp}(a)}{2kp}x^{-2kp}.$$

We now evaluate this integral by shifting the line of integration and using the Residue Theorem. Consider the contour determined by the line segments $[d-iT, d+iT], [d+iT, e+iT], [e+iT, e-iT], [e-iT, d-iT]$ where $e = 2km-d$. Inside this countour, the integrand has poles of order two at $0, 2km$ and simple poles at the integers $2kp$ where $p \in \{1,\ldots,m-1\}$. The residues at these poles are

$$R_0=\frac{-1}{\pi}\left(\frac{B_{2km}(b)}{2km}\left(\log\left(\frac{\alpha}{\pi}\right)-\psi(a)\right)+\frac{\partial}{\partial s}\zeta(s-2km+1,b)|_{s=0}\right)$$
$$R_{2km}=\frac{(-1)^m}{\pi}\left(\frac{\pi}{\alpha}\right)^{2km}\left(\frac{B_{2km}(a)}{2km}\left(\log\left(\frac{\alpha}{\pi}\right)+\psi(b)\right)-\frac{\partial}{\partial s}\zeta(s-2km+1,a)\Big|_{s=2km}\right)$$
$$R_{2kp}=(-1)^p\left(\frac{\pi}{\alpha}\right)^{2kp}\frac{B_{2kp}(a)B_{2k(m-p)}(b)}{\pi\,(2kp)\,(2k\,(m-p))}.$$

Thus we have

$$\frac{1}{2i\pi}\left[\int_{d-iT}^{d+iT}+\int_{d+iT}^{e+iT}+\int_{e+iT}^{e-iT}+\int_{e-iT}^{d-iT}\right]\frac{\zeta(1-s,a)\,\zeta(s-2km+1,b)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s$$
$$=R_0+R_{2km}+\sum_{p=1}^{m-1}R_{2kp}.$$

From Lemma 5.1 and equation 5.3, it can be seen that as $T\to\infty$, the integrals along horizontal segments tend to zero. Under the change of variables $s\to 2km-s$ we have

$$\int_{(e)}\frac{\zeta(1-s,a)\,\zeta(s-2km-1,b)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}\mathrm{d}s$$
$$=\left(\frac{\alpha}{\pi}\right)^{-2km}(-1)^{m-1}\int_{(d)}\frac{\zeta(1-s,b)\zeta(s-2km-1,a)}{2k\sin\left(\frac{\pi s}{2k}\right)}\left(\frac{\beta}{\pi}\right)^{-s}\mathrm{d}s$$

which proves Theorem 3.10. ∎

5.6. **Proof of Theorem 3.12.** Using the line integral representation of Ψ and on account of convergence of $\zeta(s,b)$ for $\Re(s)>1$, we obtain

$$\sum_{m=0}^{\infty}\frac{\Psi_\alpha(m+b,a;1)}{m+b}=\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s,a)\zeta(1+s,b)}{2\cos\left(\frac{\pi s}{2}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}ds.$$

We now shift the line of integration to $\Re(s)=-c$. From Lemma 5.1 and equation 5.3, it can be seen that as $\Im(s)\to\infty$, the integrand decays exponentially and thus vanishes over horizontal line segments. The poles inside the contour are at $1,0,-1$ with corresponding residues

$$\begin{aligned}R_1&=-\frac{\zeta(0,a)\zeta(2,b)}{\alpha},\\ R_0&=\frac{1}{4}\log\left(\frac{\alpha}{\beta}\right)+\frac{\psi(b)-\psi(a)}{2}\\ R_{-1}&=\frac{\zeta(0,b)\zeta(2,a)}{\beta},\end{aligned}$$

Since the integral over horizontal line segments vanishes, using the residue theorem we obtain

$$\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s,a)\zeta(1+s,b)}{2\cos\left(\frac{\pi s}{2}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}ds=R_1+R_0+R_{-1}+\frac{1}{2i\pi}\int_{(c)}\frac{\zeta(1-s,a)\zeta(1+s,b)}{2\cos\left(\frac{\pi s}{2}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}ds.$$

Upon using the substitution $s\to -s$ we get

$$\int_{(c)}\frac{\zeta(1-s,a)\zeta(1+s,b)}{2\cos\left(\frac{\pi s}{2}\right)}\left(\frac{\alpha}{\pi}\right)^{-s}ds=\int_{(c)}\frac{\zeta(1-s,b)\zeta(1+s,a)}{2\cos\left(\frac{\pi s}{2}\right)}\left(\frac{\beta}{\pi}\right)^{-s}ds.$$

Putting all things together gives us the desired result. ∎

5.7. **Proof of Theorem 4.2.** On account of the absolute convergence of the Dirichlet L-series for $\Re(s)>1$ we have

$$\sum_{n=0}^{\infty}\frac{\chi_2(n)\Psi\left(\mu n,\chi_2;k\right)}{n^{2km-1}}=\frac{1}{2i\pi}\int_{(c)}\frac{L\left(1-s,\chi_1\right)L(2km+s-1,\chi_2)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\mu^{-s}\,\mathrm{ds}.$$

We now evaluate this integral by shifting the line of integration. Consider rectangular the contour determined by the line segments $[c-iT,c+iT],[c+iT,d+iT],[d+iT,d-iT],[d-iT,c-iT]$ where $d=-c-2km+2$. Inside this contour, the integrand has simple poles at the integers $-2kp+1$ where $p\in\{0,1,\dots,m\}$. It also has simple poles at $0,-2km+2$ is χ_1,χ_2 are principal. The residues at these poles are

$$\begin{aligned}R_0&=\frac{-\delta_{\chi_1}L(2km-1,\chi_2)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)},\\ R_{-2km+2}&=(-1)^{m-1}\frac{\delta_{\chi_2}L(2km-1,\chi_1)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\mu^{2km-2},\end{aligned}$$

$$R_{-2kp+1} = \frac{(-1)^{p+1}}{\pi} L(2kp, \chi_1) L(2k(m-p), \chi_2) \mu^{2kp-1}.$$

Thus by Cauchy's Residue Theorem, we have

$$\frac{1}{2i\pi}\left[\int_{c-iT}^{c+iT} + \int_{c+iT}^{d+iT} + \int_{d+iT}^{d-iT} + \int_{d-iT}^{c-iT}\right] \frac{\zeta\left(1-s,a\right)\zeta(2kN+2k-1+s,b)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\mu^{-s}\mathrm{d}s$$

$$= R_0 + R_{-2kN-2k+2} + \sum_{p=0}^{m} R_{-2kp+1}.$$

From Lemma 5.1 and equation (5.3), it can be seen that as $T \to \infty$, the integrals along horizontal segments tend to zero. Under the change of variables $s \to -s - 2km + 2$ we have

$$\int_{(d)} \frac{L\left(1-s,\chi_1\right)L(2km+s-1,\chi_2)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\mu^{-s}$$

$$= (-1)^{m-1}\mu^{2km-2}\int_{(c)} \frac{L(1-s,\chi_2)L(2km+s-1,\chi_1)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\mu^{s}ds.$$

Since $\Re(2km + s - 1) > 0$, we obtain

$$\frac{1}{2i\pi}\int_{(c)} \frac{L(1-s,\chi_2)L(2km+s-1,\chi_1)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\mu^{s}ds = \sum_{n=1}^{\infty}\frac{\chi_1(n)\Psi\left(\frac{n}{\mu},\chi_2;k\right)}{n^{2km-1}}.$$

Substituting this, we get the desired result. ∎

5.8. **Proof of Theorem 4.3.** On account of absolute convergence of $L(\chi, s)$ for $\Re(s) > 1$ we have

$$\sum_{n=0}^{\infty}\frac{\chi_1(n)\Phi(\mu n,\chi_2;k)}{n^{2km}} = \int_{(c)} \frac{L(2km+s,\chi_1)L(1-s,\chi_2)}{2k\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds.$$

We now shift the line of integration to $-c - 2km + 1$. As before, due to the exponential decay of the denominator, the integrals over horizontal segments vanish. Inside this strip, the integrand has poles at $-2kp$ where $p \in \{1, 2, \ldots, m\}$. It also has poles at $0, -2km + 1$ if χ_1, χ_2 are principal. The corresponding residues are

$$R_{-2kp} = \frac{(-1)^p}{\pi} L(2k(m-p), \chi_1) L(2kp+1, \chi_2) \mu^{2kp}$$

$$R_0 = \begin{cases} \frac{\delta_{\chi_2}}{\pi}\left(L(2km,\chi_1)\log(\mu) - \frac{\partial}{\partial s}L(2km+s,\chi_1)\mid_{s=0}\right) + \frac{\gamma_0(\chi_2)}{\pi}L(2km,\chi_1) & \chi_2 \text{ is principal,} \\ \frac{L(1,\chi_2)L(2km,\chi_1)}{\pi} & \text{otherwise,} \end{cases}$$

$$R_{-2km+1} = (-1)^m \frac{\delta_{\chi_1} L(2km,\chi_2)\mu^{2km-1}}{2k\sin\left(\frac{\pi}{2k}\right)}$$

Next, using the residue theorem we obtain

$$\int_{(c)} \frac{L(\chi_1, 2N+s)L(\chi_2, 1-s)}{2\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds$$
$$= R_0 + R_{-2km+1} + \sum_{k=1}^{m} R_{-2k} + \int_{(-c-2km+1)} \frac{L(\chi_1, 2N+s)L(\chi_2, 1-s)}{2\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds,$$

Plugging in $s \to -s-2km+1$ produces

$$\int_{(-c-2km+1)} \frac{L(\chi_1, 2km+s)L(\chi_2, 1-s)}{2\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds = (-1)^m \mu^{2km-1} \int_{(c)} \frac{L(\chi_2, 2km+s)L(\chi_2, 1-s)}{2\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\mu^{s}ds,$$

Due to absolute convergence of the series representation $L(\chi_2, s)$ for $\Re(s) > 0$, this can be rewritten as

$$\int_{(c)} \frac{L(\chi_2, 2km+s)L(\chi_1, 1-s)}{2\cos\left(\frac{\pi(s+k-1)}{2k}\right)}\mu^{s}ds = \sum_{n\geqslant 0} \frac{\chi_2(n)\Psi\left(\frac{n}{\mu}, \chi_1; k\right)}{n^{2km}}.$$

Substituting these produces the desired result ∎

5.9. **Proof of Theorem 4.7.** On account of the absolute convergence of $L(s,\chi)$ for $\Re(s) > 1$ we obtain

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{2km+1}}\Phi(\mu n, \chi_2, k) = \frac{1}{2i\pi}\int_{(c)} \frac{L(1-s,\chi_2)L(2km+1+s,\chi_1)}{2k\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds.$$

We now shift the line of integration to $-c-2km+1$. As before, due to the exponential decay of the denominator, the integrals over horizontal segments vanish. Inside this strip, the integrand has simple poles at $-2kp$ where $p \in \{1, 2, \ldots, m-1\}$. It also has poles of order two at $0, -2kp$ if χ_1, χ_2 are principal and simple poles otherwise. The corresponding residues are

$$R_0 = \begin{cases} \frac{\delta_{\chi_2}}{\pi}\left(\log(\mu)L(2km+1,\chi_1) - \frac{\partial}{\partial s}L(2km+s+1,\chi_1)\mid_{s=0}\right) + \frac{\gamma_0(\chi_2)}{\pi}L(2km+1,\chi_1) & \chi_2 \text{ is principal,} \\ \frac{L(1,\chi_2)L(2km+1,\chi_1)}{\pi} & \text{otherwise.} \end{cases}$$

$$R_{-2kp} = \frac{(-1)^i}{\pi}L(2kp+1,\chi_2)L(2k(m-p)+1,\chi_1),$$

and when χ_1 is principal,

$$R_{-2km} = (-1)^{m-1}\mu^{2km}\left(\frac{\delta_{\chi_2}}{\pi}\left(\log(\mu)L(2km+1,\chi_1) - \frac{\partial}{\partial s}L(2km+s+1,\chi_1)\mid_{s=0}\right) + \frac{\gamma_0(\chi_2)}{\pi}L(2km+1,\chi_1)\right),$$

otherwise, we have

$$R_{-2km} = (-1)^m \mu^{2km}\frac{L(1,\chi_1)L(2km+1,\chi_2)}{\pi}.$$

Next, using the residue theorem we obtain

$$\int_{(c)} \frac{L(2km+1+s,\chi_1)L(1-s,\chi_2)}{2k\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds = R_0 + R_{-2km} + \sum_{p=1}^{m-1} R_{-2kp} + \int_{(-c-2km+1)} \frac{L(2km+1+s,\chi_1)L(1-s,\chi_2)}{2k\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds.$$

On substituting $s \to -s-2km$ we have

$$\int_{(-c-2km+1)} \frac{L(1-s,\chi_2)L(2km+s,\chi_1)}{2\sin\left(\frac{\pi s}{2k}\right)}\mu^{-s}ds = (-1)^m \mu^{2km} \int_{(c)} \frac{L(2km+s,\chi_2)L(1-s,\chi_1)}{2k\sin\left(\frac{\pi s}{2k}\right)}\mu^{s}ds.$$

Due to absolute convergence of the series form of $L(\chi_2, s)$ for $\Re(s) > 0$, this can be rewritten as

$$\int_{(c)} \frac{L(2km+s,\chi_2)L(1-s,\chi_1)}{2k\sin\left(\frac{\pi s}{2k}\right)}\mu^{s}ds = \sum_{n\geqslant 0} \frac{\chi_2(n)}{n^{2km+1}}\Phi\left(\frac{n}{\mu},\chi_1;k\right).$$

Substituting these we get the desired result ■

6. Concluding Remarks and Further Research

In this work, we design a procedure geared towards obtaining reciprocity relations involving convolution of specified Dirichlet series as the residual term and use it to obtain several new reciprocity relations for Hurwitz zeta functions and Dirichlet L-functions. An elementary method was recently obtained by the author and collaborators [10] to prove such identities. However, unlike the other method, the procedure employed in this work comes with rigorous convergence conditions. The current method can also be used to generate results like Theorem 3.8, Theorem 3.10, and Theorem 3.12 which cannot be proved otherwise by borrowing tools from [10].

In a recent paper A. Dixit and R. Gupta [13] found a Ramanujan-type identity for squares of the Riemann zeta function at even arguments by considering the kernel

$$\Omega(x) = \frac{1}{2i\pi}\int_{(c)} \frac{\zeta(1-s)^2}{2\cos\left(\frac{\pi s}{2}\right)}x^{-s}\,ds,$$

which they call Koshliakov kernel first introduced by N.S. Koshliakov in [19]. The transformation is stated as follows:

$$\left(-\beta^2\right)^{-N}\left\{\zeta^2(2N+1)\left(\gamma+\log\left(\frac{\beta}{\pi}\right)-\frac{\zeta'(2N+1)}{\zeta(2N+1)}\right)+\sum_{n=1}^{+\infty}\frac{\tau_0(n)}{n^{2N+1}}\,\Omega\left(\frac{\beta^2 n}{\pi^2}\right)\right\}$$
$$-\left(\alpha^2\right)^{-N}\left\{\zeta^2(2N+1)\left(\gamma+\log\left(\frac{\alpha}{\pi}\right)-\frac{\zeta'(2N+1)}{\zeta(2N+1)}\right)+\sum_{n=1}^{+\infty}\frac{\tau_0(n)}{n^{2N+1}}\,\Omega\left(\frac{\alpha^2 n}{\pi^2}\right)\right\}$$

(6.1) $$= 2^{4N}\pi \sum_{j=0}^{N+1} \frac{(-1)^j \mathcal{B}_{2j}^2 \mathcal{B}_{2N+2-2j}^2}{((2j)!)^2 ((2N+2-2j)!)^2} \left(\alpha^2\right)^j \left(\beta^2\right)^{N+1-j}.$$

We have restated their transformation in terms of the Koshliakov kernel $\Omega(x)$, which has equivalent expressions given in [13].

By considering the generalized Koshliakov kernel

$$\Omega(x,a;k) = \frac{1}{2i\pi} \int_{(c)} \frac{\zeta(1-s,a)^2}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)} x^{-s}\, \mathrm{d}s,$$

one can obtain a two-parameter generalization of equation (6.1). However, $\zeta(1-s,a)^2$ does not have simple coefficients so we consider the special case $a = \frac{1}{n}$ where $n \in \mathbb{N}$. Similar to the odd zeta kernel one can define a new kernel

$$\Theta(x,a;k) = \frac{1}{2i\pi} \int_{(c)} \frac{\zeta(1-s,a)^2}{2k\sin\left(\frac{\pi s}{2k}\right)} x^{-s}\, \mathrm{d}s,$$

to find an identity analogous to Theorem 3.5 for Hurwitz zeta squares at odd arguments. Finally we can provide a relation between the kernels $\Omega(x,a;k)$ and $\Theta(x,a;k)$ analogous to Theorem 3.7.

A comprehensive study of reciprocity relations was undertaken by Bochner in his seminal work [7]. His results imply the equivalence between modular relations and the functional equation satisfied by the Dirichlet series. We briefly state an equivalent formulation of Bochner's results adopted from [17]: Let

$$\varphi(s) = \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^s} \quad \psi(s) = \sum_{m=1}^{\infty} \frac{b_m}{\mu_m^s},$$

be Dirichlet series admitting finite abscissae of convergence. Suppose these Dirichlet series satisfy a functional equation, subject to certain additional constraints, with multiple Gamma factor $\Delta(s)$ of the form

(6.2) $$A^{-s}\Delta(s)\varphi(s) = A^{-(\delta-s)}\Delta(\delta-s)\psi(\delta-s).$$

Now let

(6.3) $$E(x) = \int_{(\kappa)} \Delta(s) x^{-s} ds,$$

and let

$$\Phi(x) = \sum_{m=1}^{\infty} a_n E(\lambda_n x) \quad \Psi(x) = \sum_{m=1}^{\infty} b_n E(\mu_n x).$$

Note that we need to impose certain conditions on $\Delta(s)$ so that the inverse Mellin integral and the Lambert series are convergent. Bochner's result states that the functional equation 6.2 implies the modular relation

$$\Phi(Ax) - cx^{-\delta}\Psi\left(\frac{A}{x}\right) = P(x), \tag{6.4}$$

where $P(x)$ is a certain residual function.

Many authors later generalized and rediscovered these results. The asserted equivalence between modular relations and functional equations is now famously known as the Riemann-Hecke correspondence. We now elucidate how Ramanujan's identity is a special case of equation 6.4. The Riemann zeta function satisfies the functional equation

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{6.5}$$

Now for an odd integer n we have

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-n) = (-1)^{\frac{n+1}{2}}(2\pi)^{-(n+1-s)}\Gamma(n+1-s)\zeta(n+1-s)\zeta(1-s).$$

It's easy to deduce that

$$\zeta(s)\zeta(s-n) = \sum_{m=1}^{\infty}\frac{\sigma_n(m)}{m^s},$$

where

$$\sigma_n(m) = \sum_{d|m} d^n.$$

In this case, (6.3) reduces to the well-known Inverse Mellin transform of the Gamma function

$$\frac{1}{2i\pi}\int_{(\kappa)}\Gamma(s)x^{-s}ds = e^{-x}.$$

Applying equation (6.4) we obtain

$$\sum_{k=1}^{\infty}\sigma_n(k)e^{-2k\pi x} = \left(\frac{1}{x}\right)^{n+1}(-1)^{\frac{n+1}{2}}\sum_{k=1}^{\infty}\sigma_n(k)e^{-\frac{2k\pi}{x}} + P(x) \tag{6.6}$$

where $P(x)$ is the sum of residues of the function $\Gamma(s)\zeta(s)\zeta(s-n)x^{-s}$. By the formula

$$\sum_{k=1}^{\infty}\sigma_{-2n-1}(k)e^{-2k\pi x} = \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2k\pi x}-1},$$

we deduce Ramanujan's formula as a special case of equation (6.6).

In order to obtain reciprocity relations involving Hurwitz zeta functions, we cannot apply Bochner's results since Hurwitz zeta functions don't satisfy a functional equation of the form (6.5).

Moreover, the method employed in this article can also be applied to more exotic Dirichlet series such as the Barnes zeta function defined by

$$\zeta_N(s, w \mid a_1, \ldots, a_N) = \sum_{n_1,\ldots,n_N \geqslant 0} \frac{1}{(w + n_1 a_1 + \ldots + n_N a_N)^s},$$

where $\Re(w) > 0, \Re(a_i) > 0$ and $\Re(s) > N$. Finding reciprocity relations involving the convolution of Barnes zeta functions at integer arguments will be the subject of future work.

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