

# Access Is Not Use: Device Subsidies and Rehabilitation Capacity in Hearing-Aid Provision

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## Abstract

Device subsidies expand access, but their benefits can depend on services shared with existing users. We study how a fixed rehabilitation resource changes the choice between subsidizing hearing aids and expanding service capacity. In a model with heterogeneous benefits, equal service entitlements, and distortionary public finance, an entrant gains assistance while reducing the support available to incumbents. The resulting local subsidy criterion separates service productivity, the gap between marginal and average adopter benefits, and the fiscal cost of subsidizing existing recipients. Capacity expansion improves services per user even after induced adoption, yet its effect on the relative congestion loss depends on technological curvature and recipient selection. Matched-baseline comparisons isolate this distinction. Distribution sensitivity tests the policy rankings, and interval bounds distinguish robust local subsidy signs from unresolved cases. Numerical budget scenarios yield device-only, capacity-only, mixed, and no-intervention solutions as costs and external benefits vary. Public NHANES data and service-delivery trials provide descriptive and clinical context; they do not identify the shared-capacity mechanism or calibrate policy values. The framework identifies which service responses and benefit measures a program must establish before an access target can guide its allocation of public funds.

**Keywords:** hearing aids; rehabilitation; congestion externality; subsidy design; public capacity.

## 1 Introduction

What should a hearing-care program fund next: another device or the support that helps recipients benefit from devices already provided? Fitting, adjustment, and follow-up can affect outcomes, while effective self-fitting approaches can reduce dependence on some professional services. The World Health Organization’s integrated approach to hearing care places these inputs within the same policy problem [1].

The institutional setting matters. WHO guidance for community-level programs organizes hearing-aid provision around assessment, fitting, and follow-up, including delivery by trained non-specialists [2]. We consider a public program in which a funder subsidizes devices and finances a common pool of divisible post-fitting support, provided without an additional charge at the point of use. Over the policy period, new and existing recipients draw on the same available service time. If each has an equal entitlement, expanding the recipient population dilutes the support per person. This sharing rule is our maintained institutional assumption; applying it to a particular program requires evidence on its actual entitlements, charges, and service allocation.

An entrant anticipates their own benefit but does not internalize the service loss imposed on others. The planner must value both. Because the marginal entrant has a lower benefit type than the average incumbent, simply counting additional devices or applying an average benefit to each new recipient can misstate the gain from expanding access. Public spending also changes on two margins: the number of recipients and the subsidy paid to each existing recipient. These mechanisms motivate our comparison of device subsidies and capacity investment.

## 1.1 Contribution and relation to existing work

Corrective taxation provides the public-finance foundation [3]; club theory relates shared provision to membership [4]; queueing models value the congestion costs an entrant imposes on other users [5], [6]. In our model, the shared resource enters benefit production together with a device. Entry reduces incumbents' realized assistance, and the loss is weighted by their average benefit type.

The policy comparison also builds on close precedents. Qian and Zhuang [7] examine tax/subsidy and capacity choices in a two-tier health system with a redistributive objective. Li et al. [8] compare patient subsidies and hospital capacity investment through a queueing model. Coulombel and Monchambert [9] show that congestion and scale diseconomies can overturn the case for public-transport subsidies. These studies establish the broader demand–capacity problem. We focus on a common service entitlement with heterogeneous money-metric benefits and derive its implications for entry, service dilution, and public expenditure.

Arbel et al. [10] also model hearing-aid adoption as a divergence between private and social incentives, with an evolutionary extension. Our focus is the service resource shared by adopters: entry changes both realized benefits and their distribution across recipients. This mechanism connects the adoption decision to capacity provision and public expenditure.

Two characterizations are central. First, an exact local subsidy threshold separates the service elasticity of benefit, the ratio of average to marginal adopter type, and the financing burden. This decomposition explains why the value of an additional recipient depends on who already uses the service. Second, a curvature decomposition establishes when capacity expansion lowers the congestion loss relative to the marginal entrant's benefit. More service per recipient does not guarantee a lower ratio. Matching technologies on their initial benefit and marginal productivity isolates this effect in the numerical analysis. The same welfare accounting yields consistent capacity values and budget-allocation conditions. We also derive interval bounds for the local policy sign and compare population distributions both at fixed primitives and after matching initial adoption. These exercises show which numerical conclusions survive specified changes in assumptions.

Evidence enters at a different level from these theoretical results. Service-model trials motivate the dependence of benefit on delivery arrangements and show that this dependence varies across settings [11], [12], [13]. The earlier rehabilitation review [14] and the National Academies' outcome framework [15] also distinguish wearing behavior from meaningful hearing benefit. NHANES supplies descriptive context on hearing loss, current use, and recent specialist testing among adults aged at least 70, complementing the broader analysis of Humes [16]. Neither this survey association nor expenditure associations such as those in Arbel et al. [10] identify the model's congestion or external-benefit parameters.

Sections 2–4 develop the model and policy conditions. Section 5 evaluates the scope of the hearing-care evidence. Section 6 illustrates the theoretical mechanisms under explicit parameter choices, and Section 7 sets out the measurements needed for policy application. Proofs, extensions, data definitions, and reconstruction algorithms are included in the appendix.

## 2 Model and equilibrium

### 2.1 Population, technology, and timing

A unit mass of potential recipients has type  $\theta \in [\underline{\theta}, \bar{\theta}]$ , where  $0 \leq \underline{\theta} < \bar{\theta} < \infty$ . The distribution  $F$  is continuous and strictly increasing, with a continuous positive density  $f$  on the interior. Type measures the money-metric benefit from effective assistance; it can reflect hearing need and opportunities to benefit, and is not identified with an audiometric threshold.

The device has competitive price and resource cost  $p = c_D > 0$ . A subsidy  $s \in [0, p]$  reduces the purchaser's payment to  $p - s$ . Initial service capacity is  $K_0 > 0$ . An expansion  $x \geq 0$ , at constant unit resource cost  $c_x > 0$ , gives  $K = K_0 + x$ . Capacity is measured over the same period as benefits and costs. Existing capacity costs are sunk for the policy comparison; expansion costs are incremental. For a multi-year interpretation, prices, benefits, and costs must all be converted to the same present-value basis.

If a mass  $N > 0$  adopts, equal entitlement supplies each adopter with intensity

$$h = \frac{K}{N}, \quad q(h) = a + g(h). \quad (1)$$

We assume  $a \geq 0$ , continuity of  $g$  on  $[0, \infty)$ ,  $g(0) = 0$ ,  $g' > 0$ , and  $g'' \leq 0$  for  $h > 0$ , with  $q$  twice continuously differentiable there. We also impose  $q(h)/h \rightarrow 0$  as  $h \rightarrow \infty$ . This mild sublinear-growth condition, satisfied by bounded and square-root technologies, makes aggregate benefit converge to zero as the adopter mass vanishes. Let  $q_\infty = \lim_{h \rightarrow \infty} q(h)$ , possibly infinite.

An adopter receives utility  $\theta q(h) - (p - s)$ ; non-adoption gives zero. The baseline benefit  $a$  allows a device to have value without the modeled support. The shared input represents divisible post-fitting services, not the first indivisible fitting appointment. There is no additional service charge at the margin. Equal entitlement and unpriced congestion are institutional assumptions defining the application, not assertions that every hearing-care system operates this way.

Competitive device supply isolates service congestion from markups and subsidy incidence. The common money-metric type captures heterogeneity in benefit levels, while every type shares the same service response  $q(h)$ . It does not separately rank recipients by clinical need or allow type-specific service productivity. Welfare below uses equal weights, leaving redistribution and affordability constraints for a different policy objective.

The planner first sets  $(s, x)$ . Individuals then adopt, anticipating the resulting service intensity. Capacity is allocated and benefits are realized. Individuals are atomistic: they treat aggregate adoption as given when deciding, although a change in the mass of adopters changes service intensity for everyone.

### 2.2 Threshold representation and existence

At an interior equilibrium, the lowest-benefit adopter has type

$$\theta_m(N) = F^{-1}(1 - N), \quad \theta_m(N)q(K/N) = p - s. \quad (2)$$

Write

$$M(N) = \int_{\theta_m(N)}^{\bar{\theta}} \theta f(\theta) d\theta, \quad \bar{\theta}_A(N) = \frac{M(N)}{N}, \quad \mathcal{Q}(N, K) = M(N)q(K/N). \quad (3)$$

These are aggregate type, average adopter type, and total private gross benefit. Differentiation gives  $\theta'_m = -1/f(\theta_m)$  and  $M' = \theta_m$ . Consequently  $\bar{\theta}'_A = (\theta_m - \bar{\theta}_A)/N < 0$ : expanding adoption selects lower types into the recipient population.

**Proposition 1** (Unique equilibrium and comparative statics). *For  $K > 0$  and  $s \in [0, p]$ , equilibrium adoption is unique. It is interior exactly when*

$$\underline{\theta}q(K) < p - s < \bar{\theta}q_\infty.$$

*The upper-price region has  $N = 0$ , the lower-price region has  $N = 1$ , and  $s = p$  gives full adoption, up to a measure-zero indifferent type. At an interior equilibrium, define*

$$H = \frac{q(h)}{f(\theta_m)} + \frac{\theta_m h q'(h)}{N} > 0. \quad (4)$$

*Then*

$$N_s = \frac{1}{H} > 0, \quad N_K = \frac{\theta_m q'(h)}{NH} > 0, \quad h_s = -\frac{h}{N} N_s < 0, \quad h_K = \frac{1 - h N_K}{N} > 0. \quad (5)$$

*Average adopter type falls with both a subsidy and capacity expansion.*

*Proof.* On  $(0, 1)$ , inverse willingness to pay  $b(N, K) = \theta_m(N)q(K/N)$  is continuous and has derivative  $b_N = -H < 0$ . Its endpoint limits are  $\bar{\theta}q_\infty$  and  $\underline{\theta}q(K)$ . Crossing the constant  $p - s$  establishes the interior result and uniqueness; the endpoint inequalities give the corners. Implicit differentiation of  $b(N, K) = p - s$  gives  $N_s = 1/H$  and  $N_K = b_K/H = \theta_m q'/(NH)$ . Finally,

$$h N_K = \frac{\theta_m h q'/N}{q/f + \theta_m h q'/N} \in (0, 1),$$

so expanding capacity raises service intensity even after induced adoption. Composing  $\bar{\theta}'_A < 0$  with the positive adoption responses proves the selection result.  $\square$

Uniqueness is a static result. It does not establish convergence of repeated best responses, which would require an adjustment process. Similarly,  $h_s < 0$  means less rehabilitation per adopter, not necessarily less daily device wearing. An incumbent's modeled gross benefit falls because  $\theta q'(h)h_s < 0$ ; aggregate benefit can rise or fall because additional adopters also benefit.

### 2.3 Welfare and public expenditure

A nonnegative parameter  $\eta$  represents external benefits proportional to realized private gross benefit. Only benefits not already counted privately should enter this wedge. With excess financing burden  $\lambda \geq 0$ , public spending and welfare are

$$E(s, x) = sN(s, K) + c_x x, \quad (6)$$

$$W(s, x) = (1 + \eta)Q(N(s, K), K) - pN(s, K) - c_x x - \lambda E(s, x). \quad (7)$$

The subsidy principal is a transfer and is not deducted again as a resource cost. Each unit of capacity expansion costs  $(1 + \lambda)c_x$  including financing. Welfare weights are otherwise equal and utility is quasilinear; liquidity constraints, insurance value, and redistributive weights are not represented.

## Device access, shared services, and welfare

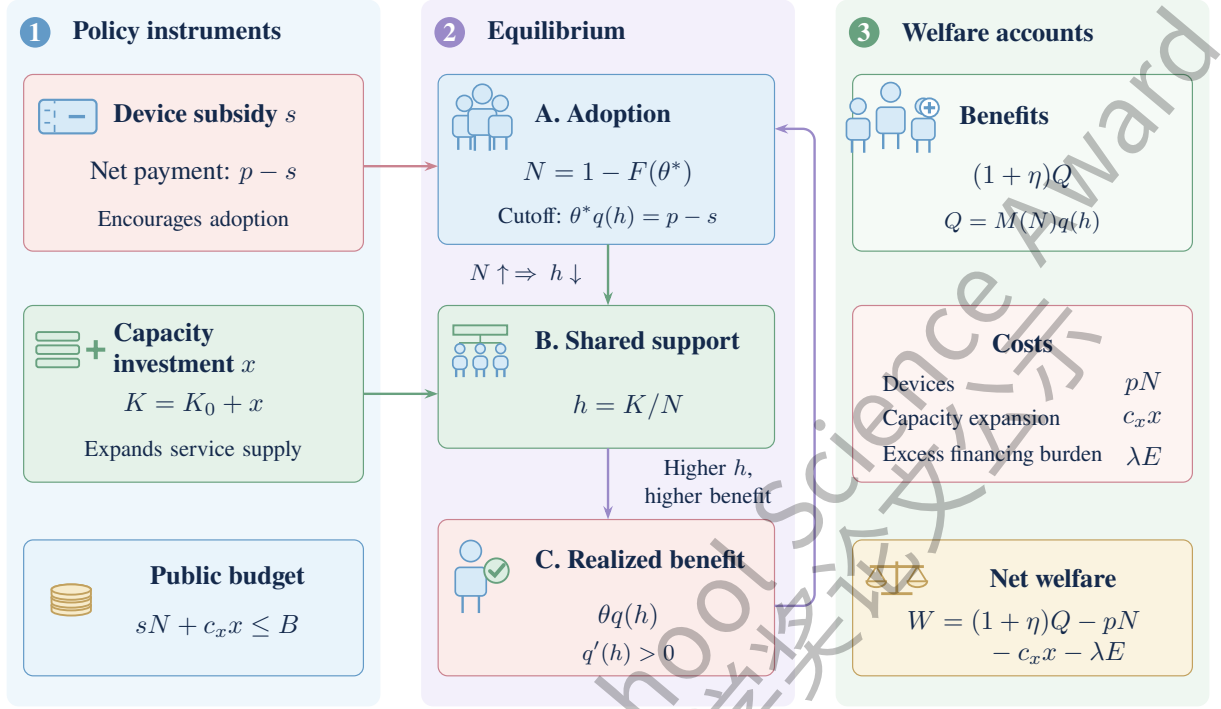


**Figure 1:** Model structure and welfare accounting. A–C are jointly determined at a static interior equilibrium; the return arrow represents benefit-dependent adoption. Here  $\theta^* = \theta_m(N)$ ,  $M(N)$  is aggregate adopter type, and  $Q = Q(N, K)$ . Public expenditure is  $E = sN + c_x x$ . Subsidy payments are transfers; welfare subtracts device and capacity resource costs and the excess financing burden. Initial capacity costs are sunk.

For interior equilibria, the inverse implementation rule is

$$s(N, K) = p - \theta_m(N)q(K/N), \quad s_N = H, \quad s_K = -\frac{\theta_m q'}{N}. \quad (8)$$

It gives an equivalent welfare function in allocation coordinates,

$$V(N, x) = (1 + \eta)Mq + \lambda\theta_m qN - (1 + \lambda)(pN + c_x x). \quad (9)$$

The two expressions agree on implementable interior allocations. Their partial derivatives hold different objects fixed, a distinction essential to capacity valuation. At full adoption with  $\theta > 0$ , several subsidies may implement the same allocation; actual fiscal outlays must be calculated from (7), not inferred from an equality for a marginal type that need not bind.

Figure 1 connects these policy choices to the equilibrium allocation and its welfare accounts.

### 3 The welfare effects of a device subsidy

#### 3.1 The benefit of entry and the loss imposed on incumbents

**Proposition 2** (Congestion decomposition). *At an interior allocation with fixed capacity,*

$$\mathcal{Q}_N = \theta_m q(h) - C(N, K), \quad C(N, K) = hq'(h)\bar{\theta}_A(N) > 0. \quad (10)$$

*Proof.* Since  $M' = \theta_m$  and  $h_N = -h/N$ ,

$$\mathcal{Q}_N = M'q + Mq'h_N = \theta_m q - \frac{M}{N}hq' = \theta_m q - C. \quad \square$$

The entrant's gross benefit is valued at the marginal type  $\theta_m$ . The service reduction experienced by existing adopters is valued at their higher average type  $\bar{\theta}_A$ . Thus counting the entrant's benefit alone overstates the marginal aggregate benefit. The congestion loss is a forgone benefit, not an additional resource cost to be added separately to (7).

To differentiate welfare with respect to policy, define

$$J = (1 + \eta)(\theta_m q - C) - p - \lambda s. \quad (11)$$

This is the marginal welfare of changing the adopter mass while the subsidy is held fixed. Because changing the subsidy also changes payments to existing adopters,

$$W_s = JN_s - \lambda N = [\eta p - (1 + \eta + \lambda)s - (1 + \eta)C]N_s - \lambda N. \quad (12)$$

The second expression uses the equilibrium condition  $\theta_m q = p - s$ . In allocation coordinates the same calculation is

$$V_N = J - \lambda NH, \quad W_s = V_N N_s. \quad (13)$$

The term  $\lambda NH$  is the financing cost of increasing the subsidy received by the incumbent adopter population when the allocation is expanded.

**Corollary 1** (No external-benefit benchmark). *Fix  $K$  and suppose the unsubsidized equilibrium is interior. If  $\eta = 0$ , welfare is strictly decreasing in a nonnegative subsidy while adoption remains interior, for any  $\lambda \geq 0$ . Hence  $s^* = 0$  over  $[0, p]$ . This result does not require a uniform distribution or a square-root technology.*

*Proof.* Equation (12) becomes  $W_s = -[(1 + \lambda)s + C]N_s - \lambda N < 0$ . Once full adoption is reached, benefits and resource costs no longer change with the subsidy and  $W_s = -\lambda \leq 0$ . Continuity completes the argument.  $\square$

When  $\eta = \lambda = 0$ , a small reduction in adoption from the unsubsidized equilibrium increases welfare by avoiding congestion that the marginal adopter ignores. With taxes allowed, the local corrective direction is therefore a device tax. This benchmark isolates the externality under the model's welfare weights. It does not imply that actual hearing-aid access is excessive, because redistribution, affordability constraints, and other subsidy motives have not been measured or included.

### 3.2 An exact local threshold with financing costs

Let  $N^0 = N(0, K)$  be an interior unsubsidized equilibrium. All quantities in the following definitions are evaluated there:

$$\rho = \frac{C}{p}, \quad D = N^0 H = \frac{N^0}{N_s^0}, \quad \delta = \frac{D}{p}. \quad (14)$$

$\rho$  compares the marginal congestion loss with the entrant's private gross benefit, equal to  $p$  at zero subsidy.  $D$  is the public-spending derivative with respect to adoption at that point:  $d(sN)/dN = N^0 s_N = N^0 H$ . The new entrant initially receives a zero subsidy, but raising the subsidy applies to all existing recipients.

**Proposition 3** (Local subsidy criterion). *For an interior unsubsidized equilibrium,*

$$W_s|_{s=0} = pN_s^0[\eta(1 - \rho) - \rho - \lambda\delta]. \quad (15)$$

*If  $\rho < 1$ , the initial marginal welfare effect  $W_s^0$  is strictly positive if and only if*

$$\eta > \eta_c \equiv \frac{\rho + \lambda\delta}{1 - \rho}. \quad (16)$$

*If  $\rho \geq 1$ , its marginal welfare effect is strictly negative for all  $\eta \geq 0$ . At equality in (16), the first derivative is zero and a general global conclusion requires additional curvature conditions.*

*Proof.* At  $s = 0$ , (12) gives  $W_s = [\eta p - (1 + \eta)C]N_s^0 - \lambda N^0$ . Divide by the positive quantity  $pN_s^0$  and use (14). For  $\rho < 1$ , rearranging gives (16); for  $\rho \geq 1$  the bracket is at most  $-\rho < 0$ .  $\square$

Without financing distortions the condition reduces to  $\eta > \rho/(1 - \rho)$ . This is a threshold on the external-benefit wedge required for an initial subsidy; it is not itself an upper bound on a subsidy rate or dollar amount. Away from zero subsidy, the relevant condition is the full derivative (12), with endogenous  $N$ ,  $C$ , and fiscal expenditure. A positive subsidy should be expanded only while that derivative, adjusted for any binding budget, remains positive.

Two decompositions make the threshold interpretable. At  $N^0$ , define the service elasticity of benefit  $\varepsilon_q = hq'/q$ . Then

$$\rho = \varepsilon_q \frac{\bar{\theta}_A}{\theta_m}, \quad \delta = \frac{N^0}{f(\theta_m)\theta_m} + \varepsilon_q. \quad (17)$$

Even a moderate service elasticity can create a large congestion ratio when average adopter benefit substantially exceeds the marginal entrant's benefit. The fiscal term additionally depends on the density of types at the adoption threshold. It can also be recovered from the response  $N_s^0$  if a valid subsidy-response estimate is available.

A consistent change of monetary units leaves  $\rho$  and  $\eta_c$  unchanged. Their values nevertheless depend on technology, population composition, and policy.

Figure 2 visualizes the fixed-capacity mechanism and the local welfare decomposition in (15). The unchanged density and capacity bar distinguish a shift in the adoption cutoff from a change in the population or the service resource.

### 3.3 What the criterion does and does not imply

The threshold concerns net welfare, not a count of devices worn. To see the distinction, suppose a successful user obtains benefit  $b\theta$  with probability  $r(h)$ , where  $r(0) = 0$ ,  $r' > 0$ ,  $r'' \leq 0$ , and  $0 \leq r \leq 1$ . This is a subclass of the model with  $q = br$ . At fixed capacity, the expected mass of successful users is  $L = Nr(K/N)$ , and

$$L_N = r(h) - hr'(h) \geq 0.$$

Concavity makes total successful use nondecreasing even though success per recipient falls. The welfare loss from diluting services and the resource cost of entry can still outweigh the entrant's contribution. Neither this probability interpretation nor the baseline identifies daily wearing hours from service intensity.

The strong result for  $\rho \geq 1$  also depends on the external-benefit specification. If the external benefit is instead  $vN$ , independent of service intensity, the initial-subsidy condition becomes

## A subsidy expands access and dilutes shared support

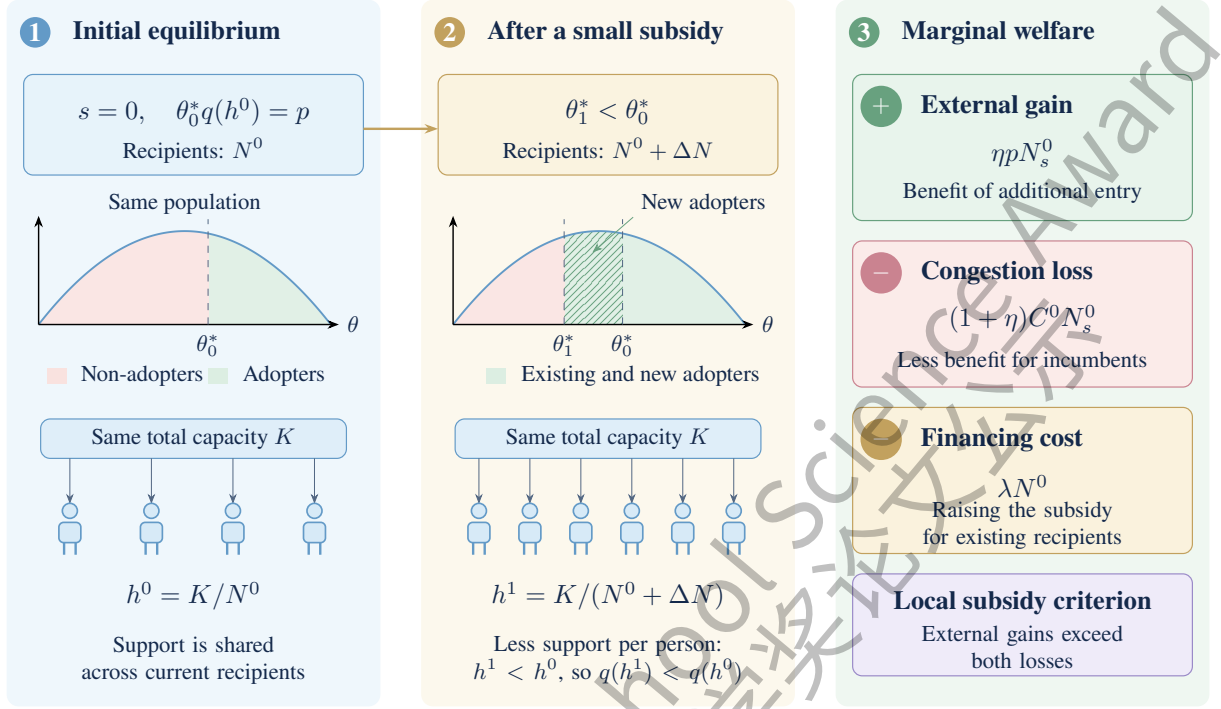


**Figure 2:** Local subsidy effects at fixed capacity, starting from an interior equilibrium with  $s = 0$ . Superscripts 0 and 1 denote the initial and post-change states;  $N_s^0$  is the initial adoption response. The same type distribution and capacity are shown in both states, with schematic person counts. In the welfare decomposition, the entrant's private benefit offsets the device cost; the remaining contributions are external gain, congestion loss, and excess financing cost. Here  $C^0 = h^0 q'(h^0) \bar{\theta}_A(N^0)$ . Their sum gives (15); a binding budget additionally requires the feasible-direction conditions in Section 4.2.

$v > C + \lambda D$ ; there is then no impossibility region based only on  $C/p$ . Appendix B derives the general condition. The proportional case is useful because private and external benefits share the same production process, but it is not a universal restriction on subsidy motives.

### 3.4 When the local result determines the global optimum

**Lemma 1** (Benchmark concavity). *For  $F$  uniform on  $[0, 1]$  and  $q(h) = a + \sqrt{h}$ ,  $V(N, x)$  is strictly concave in  $N \in (0, 1)$  at every fixed  $x$ , for  $a, \eta, \lambda \geq 0$ .*

The second derivatives are given in Appendix A. Because  $N(s, K)$  increases continuously from  $N^0$  to one, concavity implies that the fixed-capacity optimum is  $s^* = 0$  whenever  $\rho \geq 1$  or  $\eta \leq \eta_c$ . If  $\rho < 1$  and  $\eta > \eta_c$ , the optimum is positive and interior: the slope is initially positive, while at  $N = 1$  it is negative. It is uniquely characterized by  $V_N = 0$ . A numerical root or scalar optimization then determines its magnitude. The same global conclusion for arbitrary  $F$  and concave  $q$  does not follow from the local threshold alone.

### 3.5 A congestion-blind evaluation benchmark

Consider an evaluator who uses the actual adoption response to a subsidy but freezes service intensity at the unsubsidized level  $h^0 = K/N^0$ . At  $\lambda = 0$  and a fixed expansion  $x = K - K_0 \geq 0$ , including the same capacity cost in both appraisals, this evaluation is

$$\widetilde{W}(N) = (1 + \eta)M(N)q(h^0) - pN - c_x x. \quad (18)$$

At  $N = N^0$  it agrees with actual welfare. For  $N > N^0$ ,

$$\widetilde{W}(N) - W(N) = (1 + \eta)M(N)[q(h^0) - q(K/N)] > 0. \quad (19)$$

The bias is exactly the benefit credited to expanded access without accounting for reduced service intensity.

**Proposition 4** (Optimal-subsidy ranking in the benchmark). *In the uniform, square-root benchmark at  $\lambda = 0$ , the true optimal subsidy is strictly smaller than the congestion-blind optimum whenever the latter is positive.*

The proof in Appendix A compares slopes at the blind optimum and uses strict concavity in adoption. The evaluator in (18) is an explicit counterfactual appraisal rule. An evaluator who also changes the adoption model by assuming unlimited service capacity would be solving a different problem, and the subsidy ranking should not be transferred to that problem without re-derivation.

### 3.6 A decision under parameter uncertainty

The local criterion can be used without selecting one parameter vector. Let

$$T(\eta, \rho, \lambda, \delta) = \eta(1 - \rho) - \rho - \lambda\delta,$$

so that its sign is the sign of  $W_s^0$ . Suppose the four nonnegative statistics lie in finite intervals, denoted by subscripts  $L$  and  $U$ , and every admissible specification has an interior zero-subsidy equilibrium. These are bounds on the statistics at that equilibrium, not ranges over which adoption is held fixed.

**Corollary 2** (Bounds on the local policy sign). *On the rectangular envelope of the four intervals, the exact extrema of  $T$  are*

$$T_{\min} = \min_{e \in \{\eta_L, \eta_U\}} e(1 - \rho_U) - \rho_U - \lambda_U \delta_U, \quad (20)$$

$$T_{\max} = \max_{e \in \{\eta_L, \eta_U\}} e(1 - \rho_L) - \rho_L - \lambda_L \delta_L. \quad (21)$$

If  $T_{\min} > 0$ , the derivative of welfare with respect to a subsidy at zero is positive throughout the envelope. If  $T_{\max} < 0$ , it is negative throughout. Otherwise the bounds do not establish one strict sign throughout the envelope.

*Proof.*  $T$  decreases in  $\rho$ , since  $T_\rho = -(1 + \eta) < 0$ , and is nonincreasing in  $\lambda$  and  $\delta$ . At the corresponding endpoints it is affine in  $\eta$ , so an endpoint attains each extremum. Equation (15) has the positive prefactor  $pN_s^0$  at every admissible equilibrium.  $\square$

If  $\rho_U < 1$ , the minimum uses  $\eta_L$  and the maximum uses  $\eta_U$ . For example, take the assumed ranges  $\rho \in [0.3, 0.5]$ ,  $\delta \in [2, 3]$ , and  $\lambda \in [0.05, 0.15]$ . Table 1 shows how uncertainty about external

**Table 1:** Local policy signs within an assumed parameter envelope.

| Assumed $\eta$ interval | $T_{\min}$ | $T_{\max}$ | Sign of $W_s^0$ throughout |
|-------------------------|------------|------------|----------------------------|
| $[0.1, 0.4]$            | $-0.9000$  | $-0.1200$  | Negative                   |
| $[0.5, 3]$              | $-0.7000$  | $1.7000$   | Not determined             |
| $[2, 3]$                | $0.0500$   | $1.7000$   | Positive                   |

benefits changes the decision. These ranges are illustrative assumptions, not estimates or confidence intervals.

The statistics can be linked by common model primitives. A rectangular envelope may therefore include combinations that no admissible model produces. A uniform strict sign on that envelope remains sufficient for the feasible subset; an unresolved envelope need not imply ambiguity on that subset. Equality allows a zero derivative and requires higher-order analysis. The bounds concern an initial subsidy at fixed capacity, not the optimal subsidy amount, a uniform finite policy step, or the joint budget allocation. Their empirical use requires defensible bounds on the underlying statistics.

## 4 Capacity investment and the policy mix

### 4.1 Capacity value with a fixed device subsidy

Capacity improves the benefits of existing recipients and makes adoption attractive to additional recipients. Keeping the subsidy fixed while differentiating the primitive welfare expression preserves the interpretation of both effects.

**Proposition 5** (Equilibrium marginal value of capacity). *At an interior adoption equilibrium, the marginal value of effective capacity before its own expansion cost is*

$$\mu_K^{\text{eq}} = \underbrace{(1 + \eta)q'(h)\bar{\theta}_A}_{\mu_K^{\text{fix}}} + \underbrace{JN_K}_{\text{induced-adoption effect}}. \quad (22)$$

The first term holds both adoption and the subsidy fixed. Therefore

$$W_x = \mu_K^{\text{eq}} - (1 + \lambda)c_x. \quad (23)$$

A fixed-adoption appraisal overstates capacity value when  $J < 0$  and understates it when  $J > 0$ .

*Proof.* At fixed  $s$ , differentiation of (7) gives

$$W_x = (1 + \eta)\mathcal{Q}_K + [(1 + \eta)\mathcal{Q}_N - p - \lambda s]N_K - (1 + \lambda)c_x.$$

Since  $\mathcal{Q}_K = Mq'/N = q'\bar{\theta}_A$  and  $\mathcal{Q}_N = \theta_m q - C$ , this is (22)–(23).  $\square$

The induced-adoption correction can be negative even though capacity increases every incumbent's benefit. It measures the additional recipients' net social contribution after their resource costs and their congestion effects are counted. It does not mean that induced adoption fully offsets the service improvement: Proposition 1 establishes  $h_K > 0$ .

There is a second, algebraically equivalent decomposition in allocation coordinates:

$$\mu_K^{\text{eq}} = \underbrace{(1 + \eta)q'\bar{\theta}_A + \lambda\theta_m q'}_{\mu_K^{\text{impl}}} + V_N N_K. \quad (24)$$

Here, holding  $N$  fixed while increasing  $K$  requires lowering the implementing subsidy, because  $s_K = -\theta_m q'/N$ . Thus  $\lambda\theta_m q'$  is a financing saving, not an extra physical benefit of capacity at a fixed subsidy. Equivalence follows from  $V_N = J - \lambda NH$  and  $NHN_K = \theta_m q'$ . At  $\lambda = 0$  the two decompositions coincide. With financing costs they answer different fixed-adoption counterfactuals.

The implementation counterfactual also has a feasibility limit. At  $s = 0$ , increasing capacity while maintaining adoption would require a negative subsidy. It is then a formal derivative outside the nonnegative-subsidy policy set. Starting from  $s > 0$ , a sufficiently small adjustment is feasible while the subsidy remains nonnegative. The fixed-subsidy equilibrium value in (22) applies in either case.

If the subsidy is reoptimized at an interior optimum without a binding expenditure cap,  $W_s = 0$  implies  $V_N = 0$  and  $\mu_K^{\text{eq}} = \mu_K^{\text{impl}}$ . It does *not* imply  $J = 0$  when  $\lambda > 0$ ; instead,  $J = \lambda N/N_s$ . If a budget is binding, even  $V_N = 0$  generally fails, as the next subsection shows.

## 4.2 The constrained government problem

For a finite budget  $B \geq 0$ , the government solves

$$\max_{0 \leq s \leq p, x \geq 0} W(s, x) \quad \text{subject to} \quad E(s, x) \leq B. \quad (25)$$

The feasible set is compact because  $x \leq B/c_x$ . Welfare is continuous, including the zero-adoption limit under the sublinear-growth assumption, so an optimum exists. Joint concavity is not assumed. The following conditions are necessary at a differentiable local optimum satisfying the usual constraint qualification; numerical comparisons must still examine boundaries and alternative local optima.

Let  $\tau \geq 0$  be the multiplier on the spending constraint and let  $\nu_s, \omega_s, \nu_x \geq 0$  correspond to  $s \geq 0$ ,  $s \leq p$ , and  $x \geq 0$ . The Lagrangian is

$$\mathcal{L} = W + \tau(B - E) + \nu_s s + \omega_s(p - s) + \nu_x x.$$

With

$$E_s = N + sN_s > 0, \quad E_x = c_x + sN_K > 0, \quad (26)$$

the stationarity conditions are

$$JN_s - \lambda N - \tau(N + sN_s) + \nu_s - \omega_s = 0, \quad (27)$$

$$\mu_K^{\text{eq}} - (1 + \lambda)c_x - \tau(c_x + sN_K) + \nu_x = 0. \quad (28)$$

Complementary slackness requires

$$\tau(B - E) = 0, \quad \nu_s s = 0, \quad \omega_s(p - s) = 0, \quad \nu_x x = 0.$$

Together with primal and dual feasibility, these equations complete the local characterization. At corners where adoption is not differentiable, feasible one-sided welfare comparisons replace the interior derivatives.

When both instruments are interior,

$$\frac{W_s}{E_s} = \frac{W_x}{E_x} = \tau. \quad (29)$$

At an interior optimum with a slack budget, both marginal net welfare gains are zero. Otherwise, the gains per additional public dollar equal the nonnegative multiplier  $\tau$ . They are strictly positive when  $\tau > 0$ ; budget exhaustion alone does not imply strict positivity. For example, a ceiling equal to expenditure at an unconstrained optimum can bind with  $\tau = 0$ . A capacity expansion at a fixed positive subsidy increases spending on newly induced device recipients as well as on capacity itself; the denominator  $c_x + sN_K$  accounts for this effect. Dividing the capacity derivative only by  $c_x$  would omit it.

At an interior subsidy choice, (27) also gives

$$V_N = \tau \left( s + \frac{N}{N_s} \right). \quad (30)$$

Thus the induced-adoption term in (24) vanishes only when the budget multiplier is zero. A binding budget changes the envelope calculation as well as the ranking of instruments.

### 4.3 How capacity changes the congestion ratio

Additional capacity raises intensity, but this alone does not determine whether the subsidy threshold falls. Let  $\rho(K) = C(N(0, K), K)/p$  follow the unsubsidized equilibrium. Define

$$\zeta = \frac{KN_K}{N} \in (0, 1), \quad \kappa(h) = -\frac{hq''(h)}{q'(h)} \geq 0.$$

Here  $\zeta$  is the capacity elasticity of adoption and  $\kappa$  measures how rapidly marginal service productivity declines.

**Proposition 6** (Capacity–congestion decomposition). *Along an interior unsubsidized equilibrium with fixed  $p$ ,*

$$\frac{d \log \rho}{d \log K} = (1 - \kappa)(1 - \zeta) - \left( 1 - \frac{\theta_m}{\theta_A} \right) \zeta. \quad (31)$$

*In particular, if  $\kappa \geq 1$ , additional capacity strictly lowers  $\rho$ .*

*Proof.* Since  $C = hq'\theta_A$ , logarithmic differentiation gives  $d \log C = (1 - \kappa)d \log h + d \log \bar{\theta}_A$ . Now  $d \log h/d \log K = 1 - \zeta$  and  $d \log \bar{\theta}_A/d \log K = (\theta_m/\theta_A - 1)\zeta$ . Their sum yields (31). The second term is strictly negative; the first is nonpositive when  $\kappa \geq 1$ .  $\square$

This identifies two opposing forces. Higher intensity changes the loss associated with diluting a unit of capacity, through  $hq'$ . Induced entry lowers the average benefit type exposed to that loss. If  $\kappa < 1$ , the first force can outweigh the second. For a square-root technology  $\kappa = 1/2$ ; for  $g(h) = 1 - e^{-h}$  it is  $h$ . A universal instruction to expand capacity until the congestion ratio falls is therefore unsupported. The sign in (31) helps interpret a system, but costs, external benefits, and financing still determine its policy choice. When  $\lambda > 0$ , a fall in  $\rho$  alone need not lower  $\eta_c$ , because the fiscal statistic  $\delta$  also changes along the equilibrium.

## 5 Descriptive evidence and the scope of the application

### 5.1 Population, measurements, and selection

We analyze public-use NHANES 2017–March 2020 demographic, audiometry, and hearing-questionnaire files to describe device use and recent testing in the population relevant to the application. This combined pre-pandemic release requires its own survey weights [17]. We retain this cycle because it supplies the detailed device-use and wearing items used below. The newer 2021–2023 audiometry questionnaire omits AUQ147, AUQ149A, and AUQ153, so it cannot directly replace this analysis [18]. Detailed device questions and audiometry cover ages 6–19 and 70 or older; we study the older domain. The analysis does not represent all US adults or institutionalized populations. It complements the broader multi-wave description in Humes [16] rather than presenting the general severity–use gradient as a new finding.

Of 1,493 examined participants aged at least 70, requiring complete bilateral four-frequency audiometry retains 1,230. Among them, 199 report current hearing-aid use and valid wearing information; excluding two unknown specialist-test recency responses leaves 197. The 263 participants excluded for audiometry availability are older on average and have a higher weighted prevalence of reported hearing-aid use (Table 13). Examination weights do not automatically correct this additional complete-case selection.

Current hearing-aid use combines AUQ147 and AUQ149A. Recent testing means specialist testing within the preceding year (AUQ144). This item records a hearing test, which may be diagnostic; it does not measure post-fitting support, rehabilitation hours, or service capacity. The outcome combines the AUQ153 categories corresponding to at least four hours per day during the preceding two weeks [19]. We use “regular wearing” as an analytic label, not a recommended clinical threshold. AUQ153 may refer to more than one reported device; three primary-sample participants also report another device. Better-ear pure-tone average (PTA) averages 0.5, 1, 2, and 4 kHz thresholds [20]. It measures hearing sensitivity, not the model’s money-metric benefit type. Recorded age is top-coded at 80 [21].

All estimates use examination weights and Taylor variances, retaining the full sampled PSU structure for domain estimation. Appendix C specifies coding, degrees of freedom, and all exploratory sensitivities. The analysis is not prospectively registered.

### 5.2 Severity, current use, and wearing

Figure 3(a) shows higher current hearing-aid use with greater measured hearing loss. The lowest three bands have weighted estimates of 2.6%, 18.5%, and 53.0%. The highest-band estimate, 79.6%, is marked as low precision: it rests on 32 people and three domain design degrees of freedom. Its wide interval warrants more caution than a point estimate alone conveys. We retain the analytic bands and intervals while explicitly flagging this limitation, informed by NCHS reliability guidance [22].

Among the 197 current hearing-aid users, weighted regular wearing is 91.9% after recent testing and 77.1% after earlier or no testing. The unadjusted probability difference is 14.8 percentage points (95% CI 5.3–24.3), using the covariance between the two domain estimates. Adjustment for PTA, recorded age, and sex gives an odds ratio of 2.82 (95% CI 1.18–6.73;  $p = 0.023$ ). This odds ratio is not a ratio of wearing probabilities, and the unadjusted difference is not an adjusted treatment effect.

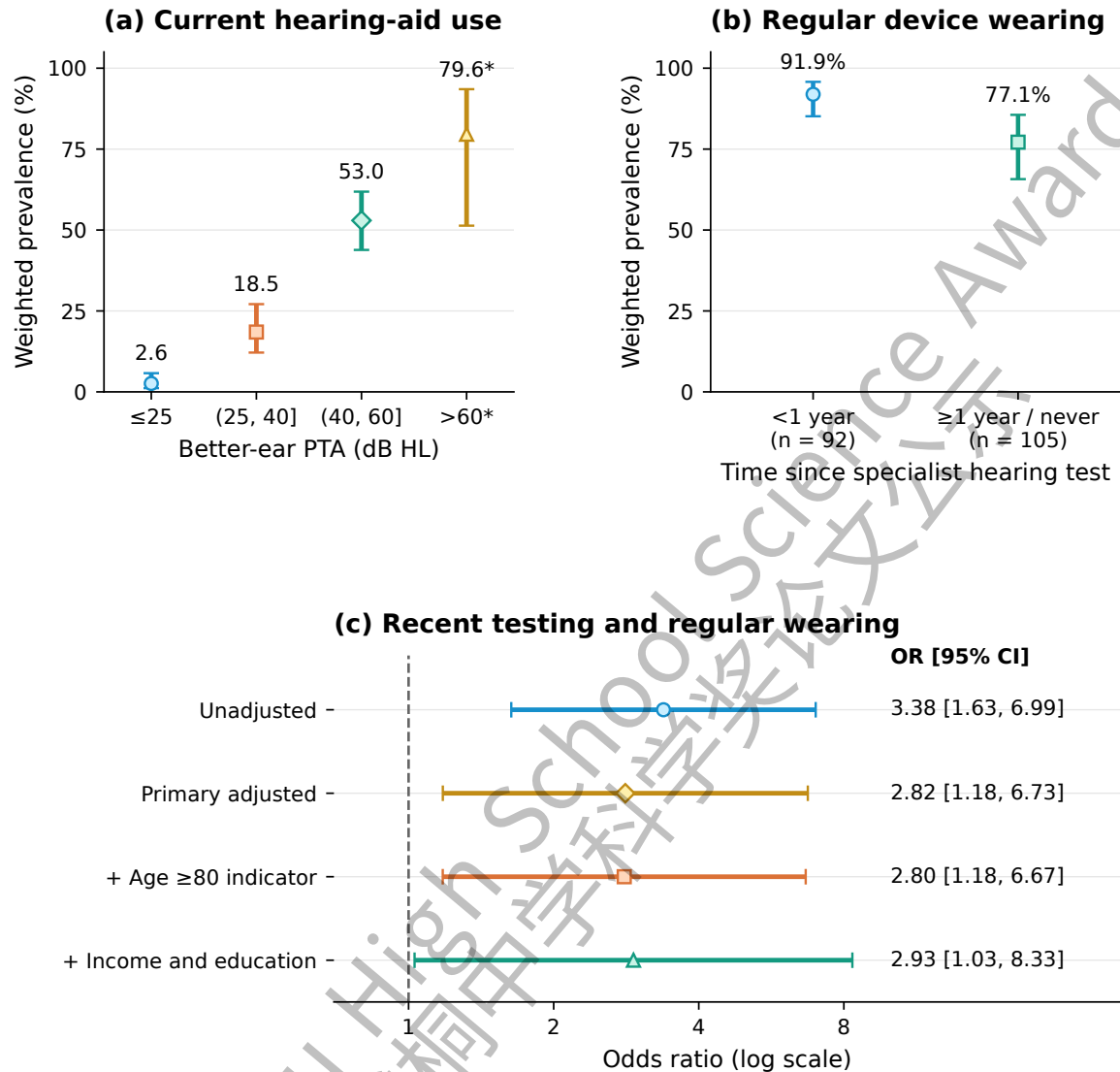


**Figure 3:** Survey-weighted estimates with 95% design-based confidence intervals. (a) Current hearing-aid use by better-ear PTA; band sizes are 520, 423, 255, and 32. \* Low precision: the highest band has three domain design degrees of freedom. (b) Regular device wearing, defined as at least four hours per day in the preceding two weeks, among 197 current hearing-aid users with known test recency. Three also report another device. (c) Odds ratios for recent versus earlier/never testing. Primary adjustment includes PTA, recorded age, and sex. The age-indicator and socioeconomic specifications separately extend that model; the latter retains 177 participants. Other displayed models use 197. These are descriptive associations. Sources: [17], [19], [20], [21]. Tables 9–12 supply estimates and sensitivities.

The association remains positive after excluding the three mixed-device users (OR 2.85; 95% CI 1.19–6.78) and with an alternative outcome of at least eight hours per day (2.42; 1.10–5.32). Preferential use of valid insert-earphone retest thresholds leaves better-ear PTA unchanged in the primary sample. These checks address selected measurement choices. They do not resolve unmeasured confounding or reverse causality: people who wear their devices regularly may seek more testing, while access, motivation, and device age may influence both measures.

The income/education extension retains 177 complete cases and has a wider interval. Adding all released race categories produces separation, so no conventional finite estimate is reported for that extension. Across all 199 current hearing-aid users, 85.1% report regular wearing and 3.4% report no wearing in the previous two weeks. These conditional proportions do not measure abandonment among all device owners. Detailed results appear in Tables 10–12.

### 5.3 Service evidence: heterogeneous effects, different outcomes

Experimental evidence concerns specified delivery arrangements rather than the survey’s testing-recency proxy. Among 245 completers in Wu et al. [11], audiologist best-practice delivery produced better primary patient-reported outcomes than two simulated OTC arrangements. The technology-level contrast was not significant, and the limited-support and independent OTC arms did not differ significantly on the primary outcome. These results support the relevance of the service package, not a universal dose-response relationship for professional hours or equivalence of all hardware.

In contrast, De Sousa et al. [12] randomized 68 adults and analyzed 64 participants, comparing self-fitting with professional fitting of the same device. Remote support for the self-fitting group became available after the initial two weeks. Outcomes were comparable at six weeks. An approximately eight-month extension included 44 participants and found no significant group differences in the main self-reported outcomes at that endpoint [13]. The extension follows the same cohort, with attrition; it is not an independent randomized replication. The earlier service-delivery and purchase-price trial of Humes et al. [23] is also relevant to the distinction between product and service, but is not used here to calibrate a subsidy elasticity.

Daily wearing is a different endpoint. The rehabilitation review of Barker et al. [14] found limited, low- or very-low-quality evidence for changes in wearing time. The National Academies’ proposed core outcome set instead emphasizes speech understanding in complex listening situations and hearing-related psychosocial health [15]. Such clinical outcomes are closer to benefit than acquisition or wearing alone, but they still require an explicit valuation step to enter money-metric welfare.

The evidence supports evaluating service arrangements alongside device access, while leaving the shared-capacity response unresolved. Neither the trials nor NHANES identify the loss imposed on existing recipients by additional entry. The numerical analysis therefore uses explicit hypothetical technologies and costs to examine the theory, with no mapping from the survey odds ratio to  $q'(h)$ ,  $N_s$ , or  $\eta$ .

## 6 Numerical illustrations

### 6.1 Purpose and parameter choices

The calculations examine the consequences of explicit assumptions, not estimated policies for a real hearing-care system. The baseline uses uniform types on  $[0, 1]$ ,  $a = 0.2$ , and  $p = 0.5$ ; Section 6.6 varies the type distribution. Initial capacity is  $K_0 = 1$  unless varied. We compare  $g(h) = \sqrt{h}$  with

$g(h) = 1 - e^{-h}$ . The functions are not calibrated to common clinical outcomes or equal benefit levels; between-family differences cannot be attributed to curvature alone. The analytic decomposition in (31), rather than a comparison of uncalibrated welfare levels, identifies the role of curvature.

We examine  $\eta \in \{0.2, 2, 8\}$  to place the model on both sides of its subsidy threshold. These are stress scenarios. In particular,  $\eta = 8$  assumes external benefits equal to eight times private gross benefit, an intentionally large wedge rather than a plausible clinical estimate. Financing burden is  $\lambda = 0.1$  except in explicitly labeled comparisons with  $\lambda = 0$ . For the budget exercise,  $B = 0.6$  and  $c_x = 0.25$ . No observational expenditure estimate is used to justify these point values.

Equilibria are solved by a bracketed root method. Scalar policy problems include endpoints; the joint problem uses multiple starting values, feasibility checks, and a separate feasible grid. Appendix D.1 specifies the algorithms, tolerances, starting values, and validation grids; Appendix D.3 specifies the numerical figure construction. These numerical checks corroborate calculations, while the propositions rest on their analytical proofs.

## 6.2 Access expansion and the subsidy threshold

At the square-root baseline, unsubsidized adoption is  $N^0 = 0.6523$  and per-user intensity is  $h^0 = 1.5330$ . Raising  $s$  from zero to 0.49 increases adoption by 52.0% and reduces service intensity by 34.2%. The reciprocal relation  $h = K/N$  makes the second change mechanically consistent with the first. This is an access-service divergence; the calculation does not predict a 34.2% fall in daily wearing.

The baseline congestion ratio is  $\rho = 0.8343$  and the fiscal statistic is  $\delta = 2.3067$ . The external-benefit threshold is approximately 5.0350 without financing distortions and 6.4271 with  $\lambda = 0.1$ . As Table 2 shows, the fixed-capacity optimum is zero at  $\eta = 0.2$  and 2. At  $\eta = 8$ , the optimum is 0.0423 when  $\lambda = 0$  but 0.0221 when  $\lambda = 0.1$ . Financing costs reduce the optimal subsidy, but its magnitude and welfare gain should be read together: at  $\lambda = 0.1$ , the gain is only  $0.8184 \times 10^{-3}$  model units, approximately 0.0153% of no-intervention welfare. A positive optimum in this high-wedge scenario therefore delivers a small improvement. The large threshold also shows how strongly the subsidy case depends on external benefits when other access motives are excluded.

| $\eta$ | $\lambda$ | Optimal $s$ | Welfare | Gain over $s = 0$ ( $\times 10^{-3}$ ) |
|--------|-----------|-------------|---------|----------------------------------------|
| 0.2    | 0         | 0.0000      | 0.4324  | 0.0000                                 |
| 2      | 0         | 0.0000      | 1.5703  | 0.0000                                 |
| 8      | 0         | 0.0423      | 5.3661  | 2.9739                                 |
| 0.2    | 0.1       | 0.0000      | 0.4324  | 0.0000                                 |
| 2      | 0.1       | 0.0000      | 1.5703  | 0.0000                                 |
| 8      | 0.1       | 0.0221      | 5.3640  | 0.8184                                 |

**Table 2:** Fixed-capacity optimal subsidies in the square-root benchmark.

Uniform types,  $a = 0.2$ ,  $p = 0.5$ , and  $K = 1$ . The last column reports  $10^3 \Delta W$ . All values are illustrative model units, not estimated monetary benefits or recommended subsidies.

At  $\eta = 8$  and  $\lambda = 0$ , the congestion-blind evaluation of Section 3.5 chooses an implementing subsidy of 0.4529, compared with the true optimum of 0.0423. Both use the same actual adoption response. The gap illustrates the ranking in Proposition 4, without implying that a particular real-world appraisal makes this error.

Freezing service intensity credits the same entrants with higher benefits than the resource permits, producing the widening appraisal gap in Figure 4(d).

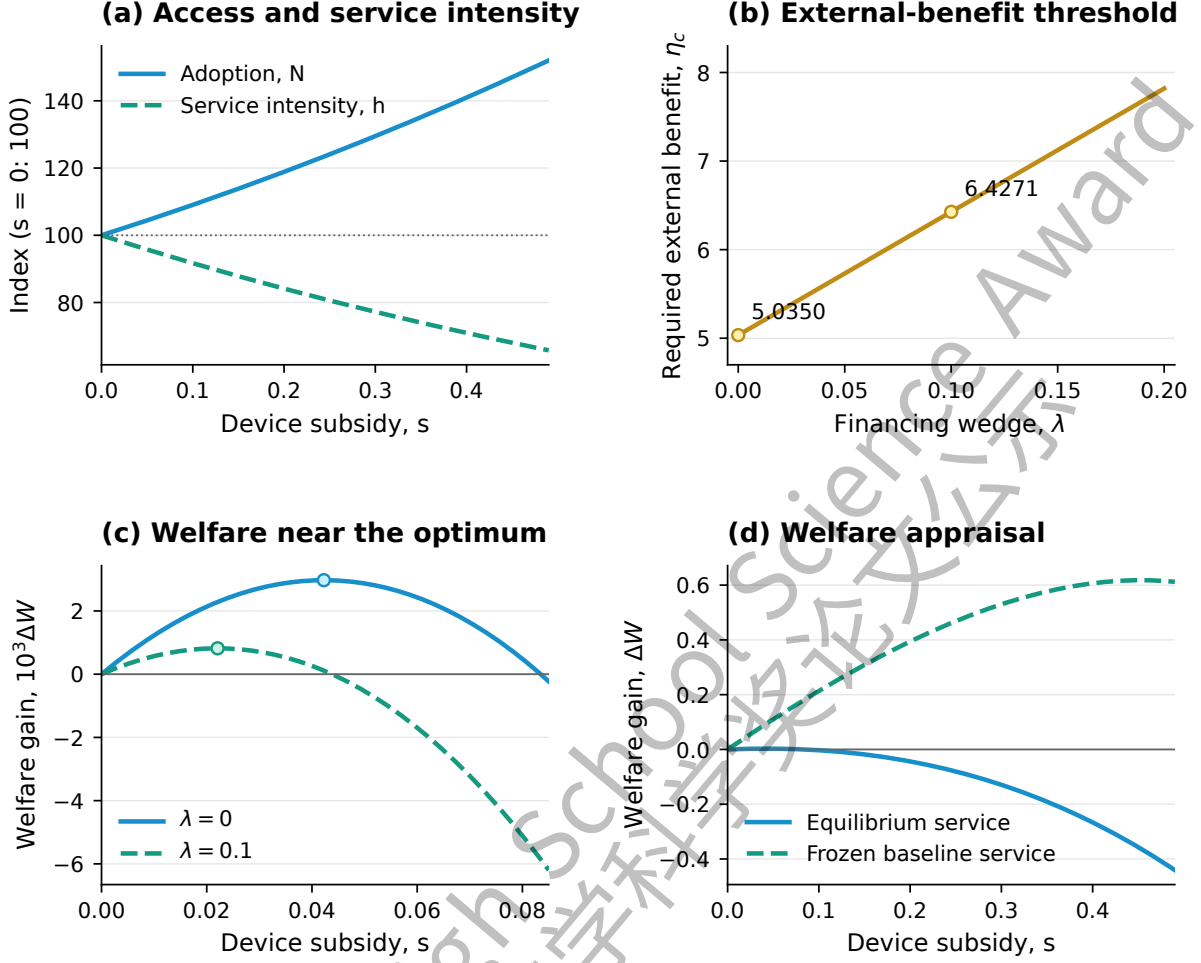


**Figure 4:** Device subsidies at fixed capacity: uniform types on  $[0, 1]$ ,  $p = c_D = 0.5$ ,  $K = 1$ , and  $q(h) = 0.2 + \sqrt{h}$ . (a) Adoption and service intensity, normalized to 100 at  $s = 0$ . Increasing  $s$  to 0.49 raises adoption by 52.0% and reduces intensity by 34.2%. (b) The local external-benefit threshold  $\eta_c$  as the financing wedge varies, holding baseline  $\rho = 0.8343$  and  $\delta = 2.3067$  fixed. (c) Welfare changes near the optimum at  $\eta = 8$ ; dots mark optimal subsidies and the vertical axis shows  $10^3 \Delta W$ . (d) At  $\eta = 8$ ,  $\lambda = 0$ , freezing service quality overstates welfare along the same adoption path. Both curves subtract their zero-subsidy values; panel (c) resolves the small true gain. All panels are uncalibrated model illustrations.

Capacity can move the model across threshold regimes. At  $K = 0.29, 1.01$ , and  $3$ , the square-root technology gives  $\rho = 0.5849, 0.8369$ , and  $1.2139$ . The saturating technology gives  $0.5987, 0.4146$ , and  $0.0420$ . Both technologies raise service intensity as capacity rises. Their different congestion-ratio paths are consistent with (31), and show why service scarcity and  $\rho$  must not be treated as synonymous.

### 6.3 The capacity correction in numbers

Table 3 evaluates capacity at  $s = 0$ ,  $\eta = 8$ , and  $\lambda = 0.1$ . At  $K = 0.5$ , the direct benefit holding adoption and subsidy fixed is  $3.4306$ , and induced adoption adds  $0.2598$ , giving  $3.6903$  after rounding the unrounded sum. At  $K = 2$ , the corresponding direct benefit is  $1.7263$ , but induced adoption subtracts  $0.0375$ , giving  $1.6888$ . These are values before the capacity unit's own cost, as in (22).

| $K$ | $\mu_K^{\text{fix}}$ | $JN_K$  | $\mu_K^{\text{eq}}$ | $\mu_K^{\text{impl}}$ |
|-----|----------------------|---------|---------------------|-----------------------|
| 0.5 | 3.4306               | 0.2598  | 3.6903              | 3.4538                |
| 1   | 2.4491               | 0.0299  | 2.4790              | 2.4631                |
| 2   | 1.7263               | -0.0375 | 1.6888              | 1.7344                |

**Table 3:** Capacity valuation under alternative fixed-adoption counterfactuals.

Square-root technology,  $s = 0$ ,  $\eta = 8$ ,  $\lambda = 0.1$ . The equilibrium value is the sum of the second and third columns before rounding; displayed components may differ from the displayed total by  $0.0001$ . The last column is a formal fixed-adoption implementation derivative. At the displayed  $s = 0$  points, a positive capacity change would require  $s < 0$  to keep adoption fixed, so that counterfactual is outside the allowed policy set.

The last column reports the fixed-adoption implementation derivative in (24). At these zero-subsidy points, holding adoption fixed after expansion would require a tax; the column is a formal comparison, not an available financing saving under  $s \geq 0$ . The equilibrium value in the preceding column remains the relevant fixed-subsidy appraisal. An alternative delivery technology can be valued in the same effective-capacity units, subject to its validated productivity and cost (Appendix B).

As Figure 5 shows, improved incumbent services and a negative induced-adoption correction can coexist.

### 6.4 Devices, capacity, and a joint policy

Table 4 compares three feasible instrument sets at the same budget ceiling. Each technology is evaluated at both  $\eta = 2$  and  $\eta = 8$ , so the table does not confound a change in technology with a change in the external-benefit wedge.

For the square-root technology at both wedge values, the numerically preferred joint policy spends the full budget on capacity. At  $\eta = 8$ , a device-only subsidy produces a small positive gain, but reallocating resources to capacity improves welfare much more under the assumed expansion cost.

For the saturating technology at  $\eta = 2$ , the joint solution combines  $s = 0.1214$  and  $x = 0.3016$ . It spends only  $0.1537$  of the  $0.6$  budget. The solution satisfies the interior first-order conditions with a slack budget: the expenditure limit need not be exhausted when additional spending lowers net welfare. For the same technology at  $\eta = 8$ , the joint solution spends the full budget and combines a larger subsidy with more capacity. These two rows illustrate distinct cases of (29): zero marginal net gains with slack expenditure and, in the latter numerical case, positive marginal gains per dollar under a valuable budget constraint.

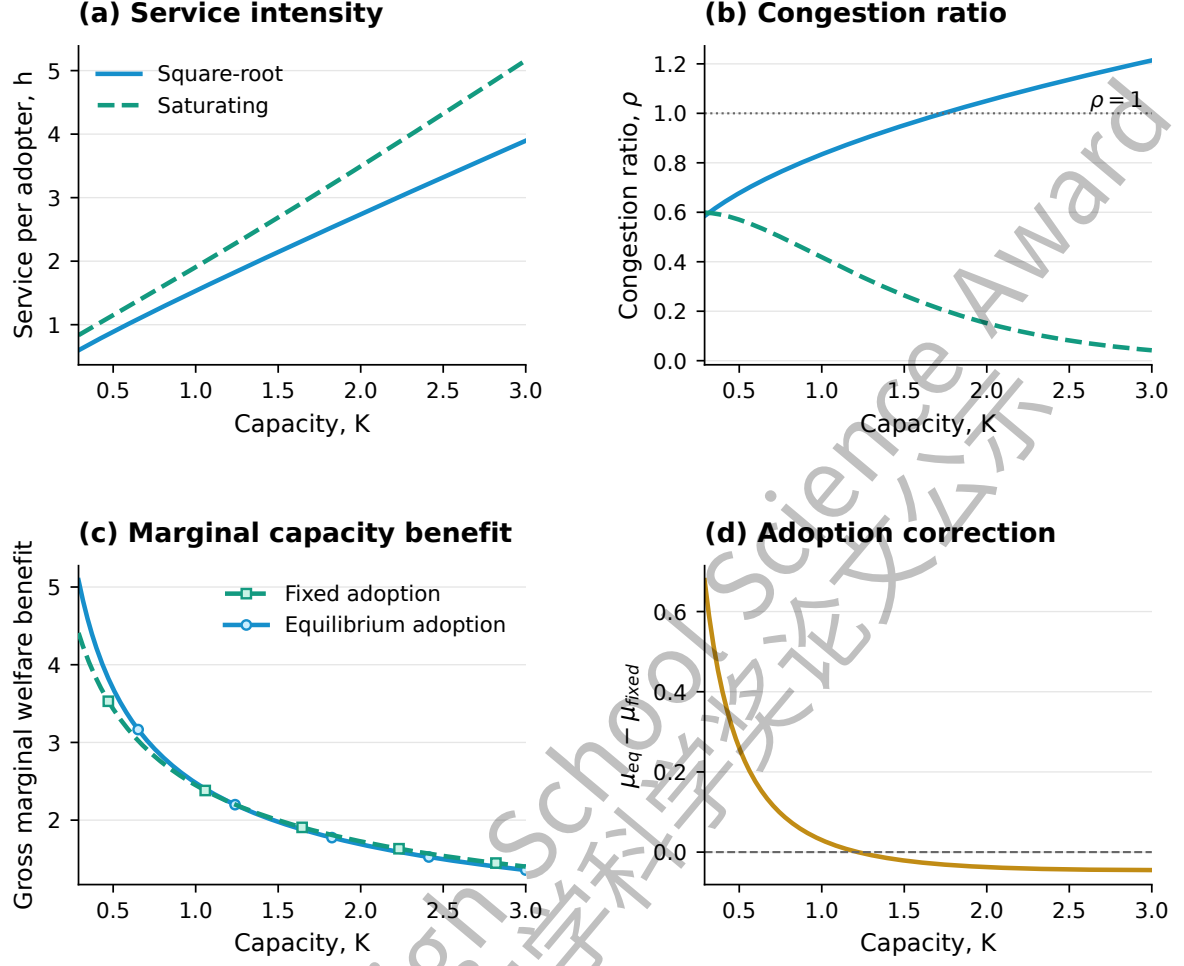


**Figure 5:** Capacity responses at  $s = 0$ , with uniform types and  $p = c_D = 0.5$ . (a)–(b) Compare the square-root and saturating technologies of Section 6. Intensity rises under both, while  $\rho = hq'(h)(1 - N/2)/p$  follows different paths; the dotted line marks  $\rho = 1$ . This comparison varies both benefit levels and curvature; Table 5 provides the matched-baseline comparison. (c)–(d) Use the square-root technology,  $\eta = 8$ , and  $\lambda = 0.1$ . The fixed-adoption value holds both adoption and subsidy constant; equilibrium adjustment adds  $JN_K$  as in (22). Values exclude the capacity unit's own cost. Panel (d) resolves the difference between the curves in panel (c).

| Technology   | $\eta$ | Policy   | $s$    | $x$    | Welfare | Spending |
|--------------|--------|----------|--------|--------|---------|----------|
| $\sqrt{h}$   | 2      | Device   | 0.0000 | 0.0000 | 1.5703  | 0.0000   |
| $\sqrt{h}$   | 2      | Capacity | 0.0000 | 2.4000 | 2.2145  | 0.6000   |
| $\sqrt{h}$   | 2      | Joint    | 0.0000 | 2.4000 | 2.2145  | 0.6000   |
| $\sqrt{h}$   | 8      | Device   | 0.0221 | 0.0000 | 5.3640  | 0.0147   |
| $\sqrt{h}$   | 8      | Capacity | 0.0000 | 2.4000 | 8.7447  | 0.6000   |
| $\sqrt{h}$   | 8      | Joint    | 0.0000 | 2.4000 | 8.7447  | 0.6000   |
| $1 - e^{-h}$ | 2      | Device   | 0.0865 | 0.0000 | 0.9684  | 0.0513   |
| $1 - e^{-h}$ | 2      | Capacity | 0.0000 | 0.1933 | 0.9668  | 0.0483   |
| $1 - e^{-h}$ | 2      | Joint    | 0.1214 | 0.3016 | 0.9856  | 0.1537   |
| $1 - e^{-h}$ | 8      | Device   | 0.2109 | 0.0000 | 3.5678  | 0.1475   |
| $1 - e^{-h}$ | 8      | Capacity | 0.0000 | 0.9540 | 3.7406  | 0.2385   |
| $1 - e^{-h}$ | 8      | Joint    | 0.3113 | 1.3608 | 4.1761  | 0.6000   |

**Table 4:** Device-only, capacity-only, and joint policies at a common budget limit.

All rows use  $B = 0.6$ ,  $c_x = 0.25$ ,  $K_0 = 1$ ,  $p = 0.5$ ,  $a = 0.2$ , and  $\lambda = 0.1$ . Spending is allowed to be below the budget. Compare policies within a technology and wedge combination; the two technologies are not calibrated to the same benefit function.

Joint policy weakly dominates the restricted policies by feasible-set inclusion. The economically informative findings are which restrictions bind, how much is spent, and the size of the gain. The examples do not establish a general rule that one technology should receive capacity-only spending or that the sign of  $d\rho/dK$  selects the instrument mix.

Unused funds in Figure 6 reflect budget slack at the selected numerical solution.

The gains in Figure 7 distinguish the small return to device-only spending from the larger improvements attainable when capacity is also available. Numerical labels support comparison across panels with different horizontal scales.

## 6.5 Separating curvature from levels and testing policy sensitivity

The preceding technology families have different benefit levels as well as different curvature. To isolate the local curvature mechanism, consider  $q(h) = a + b(1 - e^{-th})$  and match both  $q(h^0)$  and  $q'(h^0)$  to the square-root baseline. Set  $t = \kappa/h^0$ ,  $b = q'(h^0)e^\kappa/t$ , and  $a = q(h^0) - b(1 - e^{-\kappa})$ . The three choices in Table 5 retain positive intercepts and amplitudes. They therefore satisfy the baseline assumptions and share the same initial adoption, congestion ratio, and adoption responses.

Despite these common initial values,  $d \log \rho / d \log K$  is 0.3977, 0.0723, and  $-0.2530$  at  $\kappa = 0.4$ , 0.8, and 1.2. Finite differences confirm the analytical slopes. The sign reversal illustrates the curvature result without attributing an unnormalized difference in benefit levels to curvature. It is a local comparison; the functions diverge away from the matched point.

A 54-scenario exercise varies capacity cost  $c_x \in \{0.1, 0.25, 1\}$ , budget  $B \in \{0.15, 0.6, 1.2\}$ , and external benefits  $\eta \in \{0.2, 2, 8\}$  for each original technology. Square-root cases favor capacity-only spending at the two lower costs; six of nine high-cost cases select no intervention. Saturating cases include all four instrument configurations. Thus technology alone does not select a policy: costs and benefits matter. Table 15 summarizes the grid, and Table 16 gives each allocation, welfare gain, spending, and slack. This sensitivity exercise retains the baseline sharing rule, common service function, and equal welfare weights.

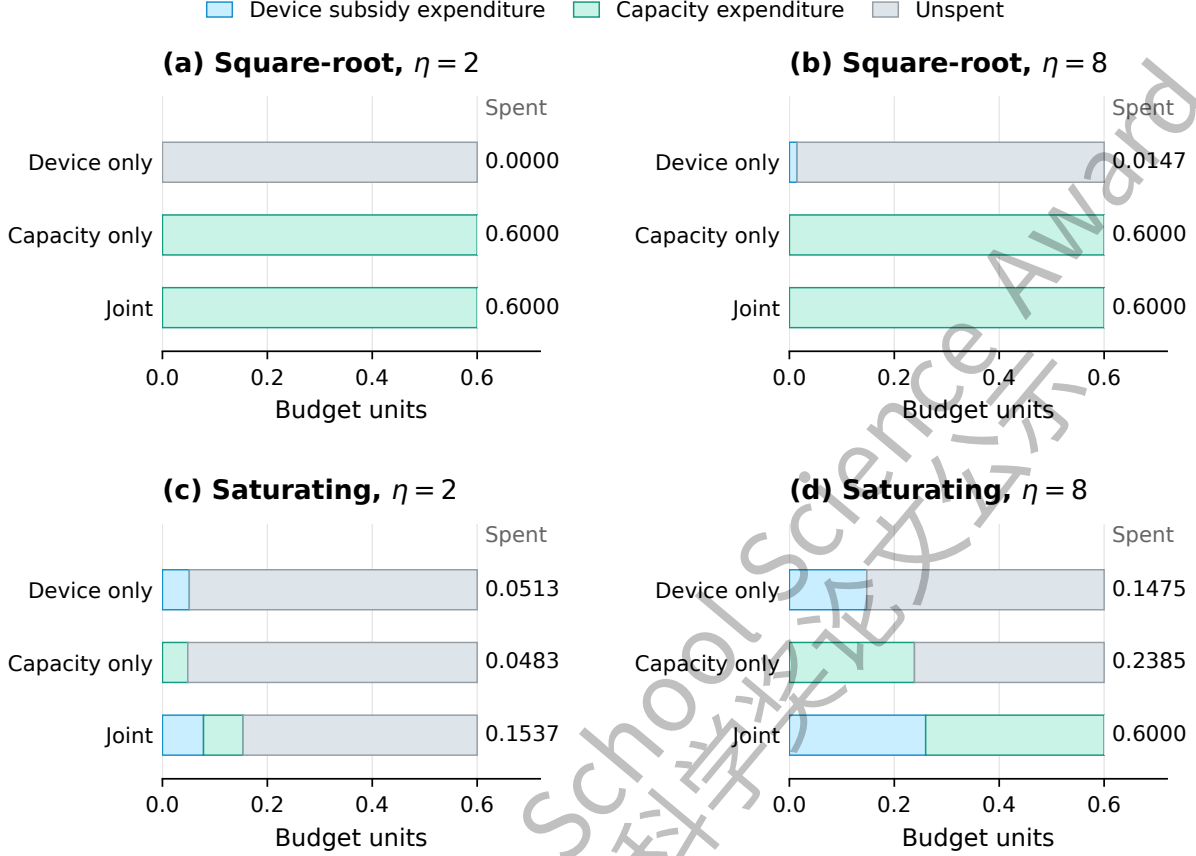


**Figure 6:** Spending composition under  $B = 0.6$ ,  $c_x = 0.25$ ,  $\lambda = 0.1$ ,  $K = 1 + x$ , and  $p = c_D = 0.5$ . Sky blue represents device subsidy expenditure  $sN$ , mint green capacity expenditure  $c_x x$ , and gray unused budget. Each full bar represents the same budget ceiling; “Spent” reports  $E$ . Device-only policy fixes  $x = 0$ , capacity-only policy fixes  $s = 0$ , and joint policy permits both. The technology and external-benefit scenarios are defined in Section 6; Table 4 reports their solutions.

| $\kappa(h^0)$ | Intercept $a$ | Amplitude $b$ | Rate $t$ | $d \log \rho / d \log K$ |
|---------------|---------------|---------------|----------|--------------------------|
| 0.4           | 0.6770        | 2.3088        | 0.2609   | 0.3977                   |
| 0.8           | 0.4898        | 1.7222        | 0.5219   | 0.0723                   |
| 1.2           | 0.2412        | 1.7128        | 0.7828   | -0.2530                  |

**Table 5:** Local curvature comparison after matching initial benefit and marginal productivity.

The alternatives are  $q(h) = a + b(1 - e^{-th})$ , with positive  $a, b, t$ . Each matches the square-root benchmark at  $K = 1, s = 0$ :  $N^0 = 0.6523$ ,  $h^0 = 1.5330$ ,  $q(h^0) = 1.4381$ ,  $q'(h^0) = 0.4038$ , and  $\rho^0 = 0.8343$ . The common initial allocation and first derivative isolate the local curvature effect; these are not clinical calibrations.

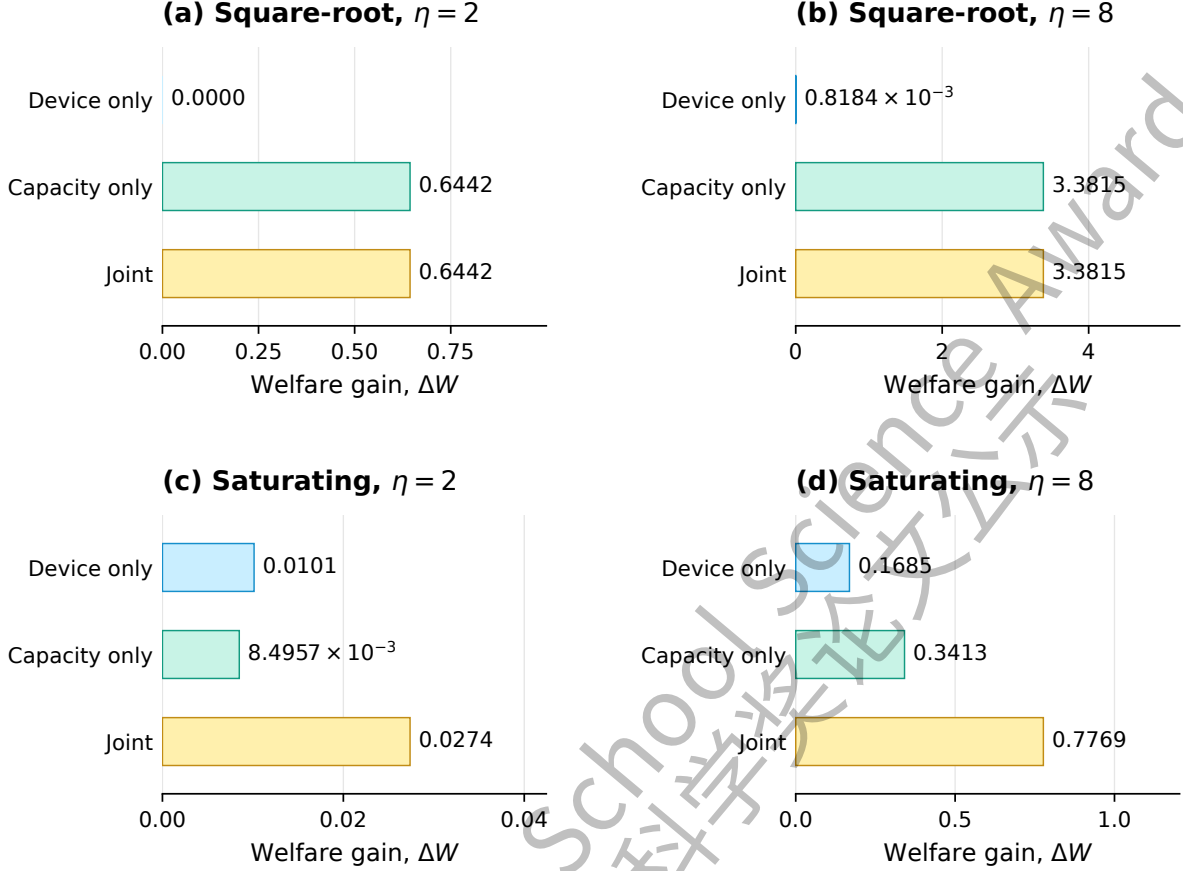


**Figure 7:** Welfare gains relative to  $s = x = 0$  within each scenario, for the policies in Figure 6. Sky blue, mint green, and gold identify device-only, capacity-only, and joint policies. Horizontal axes start at zero but have different limits; compare labels across panels. Labels use four decimals, with gains below 0.01 explicitly expressed in units of  $10^{-3}$ ; axes remain in unscaled model units. Joint policy weakly dominates by feasible-set inclusion. The differences in gain and instrument use, rather than this weak ranking, characterize the scenarios.

## 6.6 Population sensitivity and matched initial adoption

The policy choices above use uniform benefit types. We now compare  $Z \sim \text{Beta}(\alpha, \beta)$  for  $(\alpha, \beta) \in \{(1, 1), (2, 2), (0.7, 1.5), (3, 1)\}$ , retaining the same technologies,  $p = 0.5$ ,  $K_0 = 1$ ,  $\lambda = 0.1$ ,  $c_x = 0.25$ , and  $B = 0.6$ . Joint policies are evaluated at  $\eta = 2$  and 8. A fixed-primitives design sets  $\theta = Z$ ; this preserves the type support but allows the mean and initial adoption to change. The (1, 1) and (2, 2) populations additionally share a mean of 0.5.

A second design sets  $\theta = tZ$  and chooses  $t$  to match the uniform population's zero-subsidy adoption within each technology. This also matches initial service intensity and the marginal adopter's type. It does not match the average adopter type, threshold density, or aggregate welfare, and changes the type support. Thus it controls the initial coverage margin without isolating distribution shape from all benefit-level differences. Appendix D.6 specifies the scale, solver, and all 28 policy rows; four uniform rows are shared by the two designs.

Table 6 shows that matching adoption does not fix the local subsidy threshold. For the square-root technology, matched (3, 1) types reduce  $\eta_c$  from 6.4271 to 1.3612, while matched (0.7, 1.5) types

**Table 6:** Population sensitivity at the zero-subsidy equilibrium.

| Tech. | Design  | $(\alpha, \beta)$ | $t$    | $N^0$  | $\rho$ | $\delta$ | $\eta_c$ |
|-------|---------|-------------------|--------|--------|--------|----------|----------|
| Root  | Fixed   | (1, 1)            | 1.0000 | 0.6523 | 0.8343 | 2.3067   | 6.4271   |
| Root  | Fixed   | (2, 2)            | 1.0000 | 0.7052 | 0.7297 | 1.8479   | 3.3833   |
| Root  | Fixed   | (0.7, 1.5)        | 1.0000 | 0.4531 | 0.8323 | 1.7614   | 6.0119   |
| Root  | Fixed   | (3, 1)            | 1.0000 | 0.9336 | 0.8090 | 5.1080   | 6.9116   |
| Root  | Matched | (2, 2)            | 0.8758 | 0.6523 | 0.6850 | 1.5745   | 2.6750   |
| Root  | Matched | (0.7, 1.5)        | 2.3524 | 0.6523 | 1.3278 | 3.2435   | –        |
| Root  | Matched | (3, 1)            | 0.4944 | 0.6523 | 0.5318 | 1.0559   | 1.3612   |
| Sat.  | Fixed   | (1, 1)            | 1.0000 | 0.5245 | 0.4180 | 1.3723   | 0.9539   |
| Sat.  | Fixed   | (2, 2)            | 1.0000 | 0.5334 | 0.3883 | 1.0206   | 0.8015   |
| Sat.  | Fixed   | (0.7, 1.5)        | 1.0000 | 0.3133 | 0.1703 | 0.8947   | 0.3130   |
| Sat.  | Fixed   | (3, 1)            | 1.0000 | 0.8289 | 0.5913 | 2.0161   | 1.9400   |
| Sat.  | Matched | (2, 2)            | 0.9832 | 0.5245 | 0.3779 | 0.9930   | 0.7670   |
| Sat.  | Matched | (0.7, 1.5)        | 1.9969 | 0.5245 | 0.5887 | 1.9823   | 1.9134   |
| Sat.  | Matched | (3, 1)            | 0.6092 | 0.5245 | 0.3104 | 0.6370   | 0.5425   |

Root:  $q(h) = 0.2 + \sqrt{h}$ ; Sat.:  $q(h) = 0.2 + 1 - e^{-h}$ . Types are  $\theta = tZ$ , with  $Z \sim \text{Beta}(\alpha, \beta)$ . Fixed holds  $t = 1$ ; Matched scales types to reproduce the uniform population's  $N^0$  within each technology. The uniform row serves both designs. All rows use  $p = 0.5$ ,  $K_0 = 1$ , and  $\lambda = 0.1$ . A dash means  $\rho \geq 1$ , so no nonnegative proportional external-benefit wedge makes  $W_s^0$  positive. These are local thresholds at fixed capacity, not joint-policy optima.

give  $\rho = 1.3278$ , making the initial subsidy derivative negative for every  $\eta \geq 0$ . The same initial coverage can therefore coexist with sharply different entry externalities and fiscal responses. This follows from the average-to-marginal type ratio and the threshold density in (17).

The square-root technology selects capacity-only investment throughout the examined distribution cells. The saturating technology is more sensitive: at  $\eta = 2$ , fixed (0.7, 1.5) types select a device-only policy,  $(s, x) = (0.1751, 0)$ , while uniform types select (0.1214, 0.3016). Matching initial adoption for that alternative population restores a mixed policy, (0.0825, 0.4019). At  $\eta = 8$ , all saturating cells select a mixed policy. The change after rescaling shows that the fixed-primitives difference cannot be attributed to distribution shape alone.

Instrument labels are less informative than the allocations behind them. Across these saturating cases, expansion ranges from 0 to 1.8746 and welfare gains from 0.0274 to 1.5743 in model units. Each gain is calculated against its own population's no-intervention outcome; these unnormalized gains are not a welfare ranking of populations. The comparison establishes sensitivity within the specified distributions and two external-benefit values. It neither estimates the population distribution nor extends the benchmark global-concavity result to every case.

## 7 Discussion and limitations

### 7.1 What a program should evaluate

The relevant policy outcome is a benefit produced with shared services. Counting devices omits both the support lost by incumbents and the differences between their benefit types and those of new recipients. The model brings these effects into the same account as external gains, device resource costs, and public financing. Capacity must be valued along the intended policy path, including

**Table 7:** Which conclusions survive changes in the specification?

| Specification                                     | Conclusion retained                                                      | Boundary of the conclusion                                                                                     |
|---------------------------------------------------|--------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------|
| General continuous types; common service response | Unique static adoption equilibrium and the local subsidy criterion.      | The general local criterion does not establish a global optimum.                                               |
| Service standard with sufficient capacity         | The planner still compares marginal benefits and fiscal costs.           | Entry does not dilute services until the standard becomes infeasible; unused expansion has no service benefit. |
| External benefit $G(N, K)$                        | The local condition is $G_N > C + \lambda D$ .                           | The impossibility result at $\rho \geq 1$ relies on proportional external benefits.                            |
| Bounded success probability, $a = 0$              | Entry lowers individual success probability at fixed scarce capacity.    | Total expected successful use is nondecreasing under the stated concavity assumptions.                         |
| Alternative sharing or distributional objectives  | The accounting distinction between transfers and resource costs remains. | Equal-sharing losses and the baseline subsidy threshold require re-derivation.                                 |

induced adoption and its fiscal consequences. Neither spending the whole budget nor using both instruments is a policy objective in itself.

The distribution exercise asks whether the numerical instrument choice survives changes in the recipient population; the interval criterion asks whether a local subsidy sign survives uncertainty about the relevant statistics. These are different tests. A stable instrument label can conceal large differences in gains and expenditure, while an unresolved interval may reflect an overly broad envelope. Neither test supplies the missing empirical parameters. Table 7 consolidates the principal theoretical boundaries; Appendix B derives the extensions.

## 7.2 Institutional conditions and competing policy motives

The application concerns divisible follow-up support. Need-based triage, waiting lists, appointment caps, and private supplementary services can change who bears the loss from entry. The service-standard extension permits slack capacity: dilution begins only when the common entitlement becomes scarce. The empirical question is therefore whether additional entry reduces services to existing users, and through which allocation rule.

Equal welfare weights and quasilinear utility exclude important subsidy motives. A low-income person with high clinical need may have lower money-metric willingness to pay than a wealthier person with less need. Redistribution, liquidity relief, and insurance would require an explicit extension of preferences or the planner’s objective; changing the type distribution alone does not model them. Likewise, competitive pricing excludes markups and imperfect subsidy pass-through. A subsidy transfers purchasing power, whereas a technological reduction in device cost saves resources.

The static framework uses a common period for capacity, costs, and benefits. A durable-device application must separate acquisition and expansion from recurring operating costs and allow service needs to change with experience. Common service productivity also excludes type-specific learning and responses to support. These restrictions define the scope of the present policy comparison.

### 7.3 From a theoretical criterion to an empirical design

A provider or catchment study should record eligibility, new adoption, usable service capacity, actual appointments or professional time, and repeated recipient outcomes. Workforce headcounts alone do not measure usable capacity, and current use is not newly induced adoption. Capacity changes can identify service responses only when their timing is not driven by unobserved need; independently justified net-price variation is needed for  $N_s$ . Testing dilution requires linking access expansion to existing recipients' support and outcomes while accounting for provider demand changes. Shared resources create spillovers, so individual treatment assignment alone does not ensure an unaffected control group.

Acquisition, wearing, and meaningful hearing benefit require separate measurement. Valuing  $\eta$  additionally requires benefits not already internalized privately; associated healthcare expenditures are not automatically causal savings. Where point identification is unavailable, credible joint bounds can be evaluated with Corollary 2, retaining any known restrictions between parameters. Our illustrative intervals provide the calculation, not those empirical bounds.

NHANES leaves complete-case selection, reverse causality, unmeasured confounding, and sparse subgroups unresolved. Conditioning on current use can create associations if motivation, service contact, and continued use jointly determine inclusion. PTA and top-coded age enter the primary log-odds linearly; the reported sensitivities do not exhaust alternative functional forms. Consequently, the survey cannot determine a policy level, and the numerical comparisons remain conditional on hypothetical technologies, expansion costs, and external benefits.

## 8 Conclusion

Wider access changes both who receives assistance and how much support each recipient obtains. Under equal entitlement, the local subsidy trade-off depends on service productivity, the gap between marginal and average adopter benefits, and financing costs. Capacity improves services per person, but its effect on the relative congestion loss also depends on curvature and recipient selection.

The framework provides a conditional decision criterion: compare marginal welfare gains and costs, check the service-allocation rule, and establish whether uncertainty leaves the policy sign unresolved. Distribution comparisons test the stability of selected numerical policies without converting them into calibrated recommendations. For a hearing-care program, the next empirical step is to measure coverage, received support, and meaningful outcomes together, including the effects of entry on existing recipients.

## A Additional proofs

### A.1 Benchmark concavity and the fixed-capacity optimum

For uniform types on  $[0, 1]$ ,  $\theta_m = 1 - N$  and  $M = N - N^2/2$ . Under  $q = a + \sqrt{K/N}$ ,

$$\begin{aligned} Mq &= a(N - N^2/2) + \sqrt{K}(N^{1/2} - N^{3/2}/2), \\ \theta_m q N &= a(N - N^2) + \sqrt{K}(N^{1/2} - N^{3/2}). \end{aligned}$$

Their second derivatives are

$$\begin{aligned} (Mq)_{NN} &= -a - \frac{\sqrt{K}}{4N^{3/2}} - \frac{3\sqrt{K}}{8\sqrt{N}} < 0, \\ (\theta_m q N)_{NN} &= -2a - \frac{\sqrt{K}}{4N^{3/2}} - \frac{3\sqrt{K}}{4\sqrt{N}} < 0. \end{aligned}$$

All other terms of  $V$  are linear in  $N$ . Since  $1 + \eta > 0$  and  $\lambda \geq 0$ , this proves Lemma 1.

At fixed  $K$ , implementable allocations are  $N \in [N^0, 1]$ . Strict concavity implies that a nonpositive derivative at  $N^0$  makes it the unique maximizer on this interval. If the derivative at  $N^0$  is positive, the derivative at  $N = 1$  is

$$V_N(1) = -(1 + \eta)C(1, K) - \lambda q(K) - (1 + \lambda)p < 0,$$

so there is a unique interior maximizer. The increasing implementation map translates it to a unique optimal subsidy. For general technologies and distributions, the expression

$$\mathcal{Q}_{NN} = -\frac{q}{f(\theta_m)} - \frac{2\theta_m h q'}{N} + \frac{M}{N^2}(2h q' + h^2 q'') \quad (32)$$

need not be negative under concavity of  $q$  alone. This is the reason for separating the general local subsidy criterion from the benchmark global claim.

### A.2 Proof of the congestion-blind ranking

In the benchmark, true welfare and blind welfare are strictly concave in  $N$ . Blind welfare has derivative  $\tilde{W}_N = (1 + \eta)q(h^0)(1 - N) - p$ . If its optimum is above  $N^0$ , it is interior because this derivative equals  $-p < 0$  at  $N = 1$ . Denote it by  $\tilde{N}$ . At that point,

$$W_N(\tilde{N}) - \tilde{W}_N(\tilde{N}) = (1 + \eta)\{\theta_m[q(K/\tilde{N}) - q(h^0)] - C\} < 0.$$

The blind derivative is zero, so the true derivative is negative. Strict concavity places the true constrained optimum strictly below  $\tilde{N}$ . Mapping both allocations through the same increasing subsidy schedule proves Proposition 4. If the blind optimum is  $N^0$ , the true constrained optimum is also  $N^0$ .

### A.3 An explicit fiscal derivative

The aggregate net payment by adopters is  $A(N) = \theta_m(N)q(K/N)N$ . At fixed capacity,

$$A'(N) = -\frac{Nq}{f(\theta_m)} + \theta_m q - \theta_m h q'.$$

Since  $sN = pN - A(N)$ , its derivative at  $s = 0$  is

$$D = p - A'(N^0) = \frac{N^0 q}{f(\theta_m)} + \theta_m h q' = N^0 H.$$

This also verifies (13) by direct differentiation of the transformed welfare function, independent of the policy-coordinate derivation.

## B Alternative benefit and use specifications

### B.1 External benefits need not be proportional

Replace the external benefit  $\eta Q$  with a differentiable function  $G(N, K)$ , retaining private benefit  $Q$  and the same equilibrium. At an interior unsubsidized equilibrium,

$$W_s^0 = [G_N(N^0, K) - C]N_s^0 - \lambda N^0. \quad (33)$$

Thus the general local condition is  $G_N > C + \lambda D$ . The proportional case  $G = \eta Q$  gives  $G_N = \eta(p - C)$  and recovers (16). For  $G = vN$ , where  $v \geq 0$  is an external benefit per adopter independent of service intensity, the condition becomes

$$v > C + \lambda D.$$

There is no impossibility region indexed only by  $C/p$  in this additive case. Conversely, calling all societal benefits proportional to private benefit would obscure this assumption. The extension is a sensitivity of the externality specification, not a model of redistribution or liquidity relief.

### B.2 A bounded-use interpretation within a restricted subclass

The baseline does not specify wearing behavior. A possible interpretation for the subclass  $a = 0$  is to let  $r(h) \in [0, 1]$  be a success or use probability, with  $r(0) = 0$ ,  $r' > 0$ , and  $r'' \leq 0$ . A successful adopter of type  $\theta$  receives benefit  $b\theta$ , for  $b > 0$ , so expected benefit is  $\theta q(h)$  with  $q(h) = br(h)$ . All baseline results then apply to expected welfare.

At fixed capacity, a subsidy lowers the individual success probability because  $h_s < 0$ . Nevertheless, the expected mass of successful users,  $L(N) = Nr(K/N)$ , satisfies

$$L_N = r(h) - hr'(h) \geq 0. \quad (34)$$

The inequality follows from concavity and  $r(0) = 0$ : the tangent at  $h$  lies above the point  $(0, 0)$ . Consequently even this explicit use interpretation does not imply that total successful use falls when access expands. What can fall is success per recipient, average type, or net welfare. No probability function is estimated from NHANES, and the unbounded square-root illustration is not a probability model.

### B.3 Scalable delivery as an optional extension

Suppose a new delivery technology provides  $\alpha z$  effective capacity units at cost  $c_z z$ , with  $\alpha > 0$ . Replace  $K$  by  $K_0 + x + \alpha z$  and include  $c_z z$  in both resource costs and public expenditure. At fixed  $s$ , the marginal net welfare effect is

$$W_z = \alpha \mu_K^{\text{eq}} - (1 + \lambda)c_z. \quad (35)$$

Without a binding budget, the break-even condition is  $c_z < \alpha\mu_K^{\text{eq}}/(1 + \lambda)$ . With multiplier  $\tau$ , the relevant marginal Lagrangian gain is

$$\alpha\mu_K^{\text{eq}} - (1 + \lambda)c_z - \tau(c_z + s\alpha N_K).$$

This extension can represent validated remote or automated support, but specifies no clinical effectiveness of an AI system. Human and alternative capacity are perfect substitutes in this formulation. If  $c_x \neq c_z/\alpha$ , any positive expansion uses the source with the lower resource cost per effective unit. At equality, every mix supplying the same effective capacity has the same cost and welfare. A strictly preferred diversified mix would require effectiveness heterogeneity, rising marginal costs, or technological complementarity beyond this extension.

#### B.4 A service standard and slack capacity

Suppose each recipient can use at most a service standard  $\bar{h} > 0$ , so delivered intensity is  $h = \min\{\bar{h}, K/N\}$ . Let  $N^u = 1 - F((p - s)/q(\bar{h}))$ , with  $F$  extended by zero and one outside its support, be adoption with sufficient service. If  $K \geq \bar{h}N^u$ , the unique equilibrium is  $N^u$  and capacity is sufficient. If  $K < \bar{h}N^u$ , the equilibrium lies above  $K/\bar{h}$  and is determined by the congested inverse-demand schedule, or the full-adoption corner. Uniqueness follows because  $\theta_m(N)q(\min\{\bar{h}, K/N\})$  remains strictly decreasing on the interior: its service factor is constant or decreasing, while its type factor is strictly decreasing.

Away from the boundary, an interior sufficient-capacity equilibrium has  $N_s = f(\theta_m)/q(\bar{h})$ ,  $N_K = 0$ , and no dilution loss. At  $s = 0$ , the local welfare derivative becomes  $W_s = \eta p N_s - \lambda N$ , and an unused marginal capacity expansion has net value  $-(1 + \lambda)c_x$ . Once the standard becomes capacity-constrained, the baseline formulas apply. At  $K = \bar{h}N^u$ , one-sided derivatives replace a smooth first-order condition. Thus scarcity is an empirically testable condition for the mechanism, not a necessary property of every program size.

#### B.5 Existence at zero adoption

For bounded types,  $M(N) \leq \bar{\theta}N$ . The sublinear-growth assumption implies

$$0 \leq Q(N, K) \leq \bar{\theta}Nq(K/N) = \bar{\theta}K \frac{q(K/N)}{K/N} \rightarrow 0 \quad \text{as } N \downarrow 0.$$

Defining  $Q(0, K) = 0$  therefore gives the continuous extension used in the planner problem. For bounded  $q$ , zero adoption arises when the net device price is at least  $\bar{\theta}q_\infty$ . With a square-root technology and positive capacity, this corner is unavailable at any finite positive net price. At full adoption the primitive accounting remains valid even when the marginal-type equality does not bind.

### C Empirical methods and replication details

#### C.1 Coding, eligibility, and denominator construction

The analysis merges the three public XPT files by SEQN after verifying one-to-one keys. Appendix C.6 supplies the complete operational variable mapping and Algorithms 1–2 specify the processing and estimation steps. For each required audiometric threshold, values outside  $[-10, 110]$

are treated as unavailable, including the nonresponse codes 666 and 888. The first 1-kHz reading is used, with the second reading as a fallback when the first is unavailable. All four frequencies must be observed in each ear; the lower of the two complete ear means defines better-ear PTA. No single-ear substitution or outcome imputation is performed. Nonresponse at the equipment limit can be related to severity, so complete-case audiometry may select the sample.

| Sequential inclusion criterion                         | $n$      |
|--------------------------------------------------------|----------|
| Participants in the demographic file                   | 15,560   |
| Age at least 70, examined, positive examination weight | 1,493    |
| Valid bilateral four-frequency audiometry              | 1,230    |
| Reports current use of a hearing device                | 205      |
| Identifies a hearing aid; valid wearing category       | 199      |
| Valid specialist-test recency for contact analysis     | 197      |
| Recent testing / earlier or never                      | 92 / 105 |

**Table 8:** Sequential empirical inclusion flow derived from the public-use data.

The distinction between a valid negative answer and a structural skip is explicit. A respondent who reports no current device use is a non-hearing-aid user. A respondent who identifies another device but no hearing aid is also a non-hearing-aid user, and remains in the severity-band denominator. AUQ149A is a multiple-response device indicator; “hearing-aid users” does not mean that all other device types have been excluded from their personal use. The binary recency comparison accepts only substantive AUQ144 responses; refusal and unknown codes are missing. The released questionnaire wording and code meanings are documented by NCHS [19].

The primary contact model uses PTA, recorded age, and sex. The model with an additional age-80+ indicator acknowledges that the top-coded age should not be read as exact chronological age [21]. Income is the income-to-poverty ratio. Education is categorical, preserving its five released substantive codes; refused and unknown codes are excluded. Race/ethnicity is likewise categorical, with all released substantive groups retained. Small categories are retained and checked for separation.

## C.2 Survey-weighted estimation

Let  $w_i$  denote examination weights,  $d_i$  indicate domain inclusion, and  $y_i$  be a binary response. The reported proportion and its linearized contribution are

$$\hat{P} = \frac{\sum_i w_i d_i y_i}{\sum_i w_i d_i}, \quad u_i = \frac{w_i d_i (y_i - \hat{P})}{\sum_j w_j d_j}.$$

Within stratum  $h$ , aggregate contributions over sampled PSU  $j$  as  $U_{hj} = \sum_{i \in (h,j)} u_i$ . With  $m_h$  sampled PSUs, the with-replacement Taylor variance is

$$\widehat{\text{Var}}(\hat{P}) = \sum_h \frac{m_h}{m_h - 1} \sum_{j=1}^{m_h} (U_{hj} - \bar{U}_h)^2. \quad (36)$$

Every sampled PSU remains in the sum, including zero contributions from a domain with no participants in that PSU. The full design has 24 strata and 49 PSUs, giving 25 design degrees of freedom. All full sampling strata contain at least two PSUs. A stratum with only one contributing domain PSU is therefore not treated as a genuinely singleton sampling stratum; the other sampled

PSU remains with zero contribution. No additional lonely-PSU correction is needed under this implementation.

For a domain proportion we use  $\nu = \text{contributing PSUs} - \text{contributing strata}$ , and a logit interval

$$\text{logit}^{-1} \left[ \text{logit}(\hat{P}) \pm t_{0.975, \nu} \frac{\text{SE}(\hat{P})}{\hat{P}(1 - \hat{P})} \right].$$

The severity bands use 25, 25, 21, and 3 design degrees of freedom, while recent and earlier/never-testing groups use 14 and 16. The pooled contact domain has 20. The low degrees of freedom in the highest severity band explain its especially wide interval.

Logistic coefficients solve the survey-weighted score equation  $\sum_i w_i d_i X_i (y_i - \pi_i) = 0$ , where  $\pi_i = \text{logit}^{-1}(X_i' \beta)$ . The information matrix is  $A = \sum_i w_i d_i \pi_i (1 - \pi_i) X_i X_i'$ . Coefficient influence contributions are  $A^{-1} w_i d_i X_i (y_i - \pi_i)$ . Their PSU totals enter the multivariate version of (36). Wald intervals and tests use  $\nu_{\text{domain}} - (r - 1)$  residual degrees of freedom for model rank  $r$ , including the intercept. Thus the primary adjusted model has 16 residual degrees of freedom. Rescaling all weights by the same constant leaves estimates and variances unchanged.

### C.3 Sensitivity analyses and unresolved confounding

A linear sensitivity using the ordered wearing score (zero for no wearing through four for the highest category) gives an adjusted difference between the recent and earlier/never testing groups of 0.412 (95% CI 0.152–0.673;  $p = 0.004$ ), using the same primary covariates. This score is ordinal and not measured in hours; assigning equally spaced category scores is a sensitivity assumption.

The income/education extension has 177 complete cases and 10 residual degrees of freedom. Its recent-testing odds ratio is 2.93 (95% CI 1.03–8.33). The additional race-adjusted model has 15 design-matrix columns but exhibits separation in a six-person race category. We report no conventional finite maximum-likelihood estimate for it. This sparse-data limitation is not evidence that adjusting for race would leave the main estimate unchanged. All sensitivity analyses are exploratory and do not resolve confounding or selection. The reported intervals and  $p$ -values are nominal, without multiplicity adjustment; the set of positive sensitivity estimates is not a family-wise confirmatory test.

Additional measurement checks exclude three users reporting another device and redefine regular wearing as at least eight hours per day (Table 12). For the retest sensitivity, a valid insert-earphone reading replaces the initial value at the same ear and frequency; the second 1-kHz reading is a fallback within each measurement type. Only finite thresholds in  $[-10, 110]$  are substituted. A nonresponse retest does not overwrite an observed initial value. This is an alternative finite-measurement rule, not a claim to resolve the censoring of unmeasured severe thresholds. NCHS describes retesting for potential crossover between ears [20]. Of 1,230 eligible participants, 59 have a relevant valid retest; only one better-ear PTA changes. Among the 197 primary participants, 18 have retests but no better-ear PTA changes, leaving the regression identical. No new participants enter this same-sample sensitivity.

Table 13 compares availability of complete bilateral audiometry. The 263 excluded participants have a weighted recorded mean age of 77.37 versus 75.55 years and reported hearing-aid use of 32.7% versus 19.2%. These differences motivate caution about additional selection; they do not by themselves identify its direction or magnitude for the contact–wearing association. Missingness is not repaired simply by retaining MEC weights.

The unadjusted probability difference uses a single influence function with the recent-group contribution minus the earlier/never-group contribution. Applying (36) to that difference preserves the covariance arising from shared PSUs; the interval uses the pooled domain’s 20 degrees of freedom.

For precision assessment, we also compute a Korn–Graubard-type interval with a design- and degrees-of-freedom-adjusted effective sample size. Let  $v$  be the Taylor variance,  $n$  the domain count, and  $\nu$  its reported degrees of freedom. Set

$$n_e = \min\{n, \hat{P}(1 - \hat{P})/v\}, \quad n_e^* = \min \left\{ n, \frac{\hat{P}(1 - \hat{P})}{v} \left( \frac{t_{0.975, n-1}}{t_{0.975, \nu}} \right)^2 \right\}.$$

With  $k^* = n_e^* \hat{P}$ , the interval uses beta quantiles  $\text{Beta}_{.025}(k^*, n_e^* - k^* + 1)$  and  $\text{Beta}_{.975}(k^* + 1, n_e^* - k^*)$ . For the highest PTA band this gives 51.5–95.5%, a width of 44.0 percentage points, with  $\nu = 3$ . The primary figure retains the originally specified logit interval but marks low precision. NCHS guidance evaluates sample size, interval width and degrees of freedom; our flag is an independent analytical disclosure, not an official clearance determination [22]. The unadjusted effective size  $n_e$  and the interval parameter  $n_e^*$  are distinct.

#### C.4 Detailed descriptive and regression results

The following tables provide the estimates underlying Figure 3, including the additional sensitivity specification.

| Better-ear PTA (dB HL) | $n$ | Current hearing-aid use (%) | 95% CI    |
|------------------------|-----|-----------------------------|-----------|
| $\leq 25$              | 520 | 2.6                         | 1.1–5.8   |
| (25, 40]               | 423 | 18.5                        | 12.1–27.1 |
| (40, 60]               | 255 | 53.0                        | 43.9–61.9 |
| $> 60^*$               | 32  | 79.6                        | 51.3–93.5 |

**Table 9:** Current hearing-aid use by audiometric band, ages 70 and older.

Source: reanalysis of the public NHANES 2017–March 2020 public-use files. Denominators include all 1,230 participants with valid bilateral audiometry, including users of other devices. Bands are analytic intervals, not diagnostic categories.

\* Low precision: 32 observations and three domain design degrees of freedom; interpret the point estimate cautiously.

| Specialist testing             | $n$ | Regular wearing (%) | 95% CI    |
|--------------------------------|-----|---------------------|-----------|
| Within the past year           | 92  | 91.9                | 85.1–95.8 |
| At least one year ago or never | 105 | 77.1                | 65.7–85.6 |

**Table 10:** Specialist-test recency and regular device wearing among hearing-aid users.

Regular device wearing denotes the two response categories corresponding to at least four hours per day. Two unknown-recency responses are excluded, leaving 197 participants. Percentages are survey-weighted; counts are unweighted. This is an association, not a treatment effect.

| Specification                    | <i>n</i> | OR   | 95% CI    | <i>p</i> |
|----------------------------------|----------|------|-----------|----------|
| Contact only                     | 197      | 3.38 | 1.63–6.99 | 0.002    |
| Contact + PTA, recorded age, sex | 197      | 2.82 | 1.18–6.73 | 0.023    |
| Additional age-80+ indicator     | 197      | 2.80 | 1.18–6.67 | 0.023    |
| Additional income and education  | 177      | 2.93 | 1.03–8.33 | 0.045    |

**Table 11:** Association of recent testing with regular device wearing: survey-weighted logistic models. All odds ratios compare recent with earlier/never testing. The second row is the primary adjusted specification. The last two rows are exploratory sensitivities. The income/education row adds those covariates to the primary model; it does not include the age-80+ indicator or race. The full race-adjusted extension encounters separation and has no finite ordinary maximum-likelihood estimate.

| Measurement choice                   | <i>n</i> | OR   | 95% CI    |
|--------------------------------------|----------|------|-----------|
| Exclude mixed-device users           | 194      | 2.85 | 1.19–6.78 |
| At least eight hours per day         | 197      | 2.42 | 1.10–5.32 |
| Prefer valid insert-earphone retests | 197      | 2.82 | 1.18–6.73 |

**Table 12:** Exploratory measurement sensitivities for recent testing and wearing. Each model adjusts for PTA, recorded age, and sex. The original four-hour outcome applies except in the eight-hour row. Retest preference changes no primary-sample better-ear PTA. These analyses retain the original survey-variance and degrees-of-freedom conventions; none identifies a causal effect.

| Characteristic                          | Included | Excluded |
|-----------------------------------------|----------|----------|
| Unweighted sample size                  | 1,230    | 263      |
| Recorded age (weighted mean)            | 75.55    | 77.37    |
| Female (weighted %)                     | 57.0     | 54.1     |
| Current hearing-aid use (weighted %)    | 19.2     | 32.7     |
| Current hearing-aid users (unweighted)  | 199      | 66       |
| Income-to-poverty ratio (weighted mean) | 3.07     | 2.76     |
| Valid income-to-poverty observations    | 1,070    | 206      |

**Table 13:** Observed characteristics by availability of complete bilateral audiometry. The comparison starts with 1,493 examined participants aged at least 70. Means and percentages use MEC weights; counts are unweighted. Age is top-coded at 80. Income means use the stated nonmissing denominators. These are descriptive selection diagnostics, not tests of a missing-at-random assumption.

## C.5 Data provenance and availability

The empirical analysis uses public-use, de-identified NHANES records and involves no new participant contact. The release, file identifiers, variables, domain rules, estimation equations, and pseudocode are specified here so the analysis can be reconstructed from the official public source without an author-provided data or code supplement. Record-level data are not reproduced in the paper. The documentation is for the combined pre-pandemic release, not the separate 2017–2018 two-year codebooks. The sum of complete-case survey weights is not presented as an unbiased estimate of all older hearing-aid owners.

## C.6 Source files and operational variable dictionary

The data source is the public NHANES 2017–March 2020 pre-pandemic release. Obtain the SAS transport files `P_DEMO.xpt` (15,560 rows), `P_AUQ.xpt` (14,986 rows), and `P_AUX.xpt` (5,147 rows) from the NCHS release documented in [19], [20], [21]. These are respectively the demographic/design, hearing-questionnaire, and audiometric-examination components; their row counts are not analytic sample sizes. Use the combined-release examination weight, not a two-year or interview-only weight. The variable names below use the uppercase spelling read from the transport files; codebook headings may use lowercase suffixes.

**Table 14:** Operational data dictionary for the reported analysis.

| Variable(s)        | Definition and treatment                                                                                                                                                                                                                                                                        |
|--------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| SEQN               | Respondent key. Require unique nonmissing keys in each file. Left-join both measurement files onto the demographic file; retain unmatched rows as missing measurements, never duplicate a participant.                                                                                          |
| RIDSTATR, WTMECPRP | Examination status equals 2 and examination weight is positive. The positive-weight examined population defines the retained sampling-PSU universe.                                                                                                                                             |
| SDMVSTRA, SDMVPSU  | Sampling stratum and PSU identifiers. Use their pair as the cluster key. Keep all sampled PSUs in variance calculations, including zero domain contributions.                                                                                                                                   |
| RIDAGEYR, RIAGENDR | Restrict age to at least 70. Recorded age is continuous in the regression and top-coded at 80. Female indicator equals 1 for sex code 2, with code 1 as reference. The age-80+ sensitivity adds $1\{\text{age} \geq 80\}$ .                                                                     |
| AUQ147, AUQ149A    | Hearing-aid indicator equals 1 when current-device use is confirmed (147 = 1) and a hearing aid is selected (149A = 1). It equals 0 for a valid no-device response (147 = 2) or another identified device without a hearing aid. Structural nonselection is distinguished from refusal/unknown. |
| AUQ144             | Recent-test indicator: code 1 $\mapsto$ 1; codes 2–5 $\mapsto$ 0. Codes 1–5 mean less than one year, 1–4 years, 5–9 years, at least ten years, and never, respectively. Codes 7, 9, and missing are excluded from this comparison.                                                              |

Table 14 continued

| Variable(s)      | Definition and treatment                                                                                                                                                                                                                                                                                                                                                          |
|------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| AUQ153           | Codes 1–5 mean less than one hour, 1–3 hours, 4–7 hours, at least eight hours daily, and no wearing, respectively. Regular wearing is 1 for codes 3 or 4, and 0 for 1, 2, or 5. Other codes are missing. Ordered score maps 5 to 0 and retains 1–4; the eight-hour sensitivity uses code 4 only.                                                                                  |
| AUQ149B, AUQ149C | The mixed-device sensitivity excludes a hearing-aid user if a sound amplifier is selected (149B = 2) or an implant is selected (149C = 3). A nonselected device field is not itself a missing hearing-aid outcome.                                                                                                                                                                |
| INDFMPIR         | Continuous income-to-poverty ratio, used without imputation. Observations missing this value are excluded only from analyses requiring it.                                                                                                                                                                                                                                        |
| DMDEDUC2         | Education codes 1–5 denote below grade 9; grades 9–11 or grade 12 without a diploma; high-school completion or equivalency; some college or an associate degree; and college completion or higher. Use indicators for 2–5, with 1 as reference. Refusal/unknown or absent values are excluded from the socioeconomic specification. Codes are categories, not years of schooling. |
| RIDRETH3         | The attempted race extension uses Mexican American (1, reference), other Hispanic (2), non-Hispanic White (3), Black (4), Asian (6), and other/multiracial (7), with five indicators. It fails because of separation; no finite ordinary logistic estimate is reported.                                                                                                           |

**Audiometry mapping.** For each suffix  $S \in \{R, L\}$ , the four primary fields are  $AUXU500S$ ,  $AUXU1K1S$ ,  $AUXU2KS$ , and  $AUXU4KS$ . Replace an unavailable initial 1-kHz value with  $AUXU1K2S$  only when that fallback is valid. Validity means a finite value in  $[-10, 110]$  dB HL. The corresponding insert-earphone retests are  $AUXR5CS$ ,  $AUXR1K1S$ ,  $AUXR2KS$ , and  $AUXR4KS$ , with  $AUXR1K2S$  as the within-retest 1-kHz fallback. The same range check applies. These retests are used only in the stated sensitivity; they do not alter the primary definition.

**Algorithm 1:** Construct the empirical domains.

```

Read the three public transport files; validate one-to-one SEQN keys.
D = left_join(DEMO, AUQ, AUX, key=SEQN).
Keep the complete examined-PSU index for later variance calculations.
For each person and each ear:
  Mark nonfinite/out-of-range thresholds unavailable.
  Use the second 1-kHz threshold only if the first is unavailable.
  If all four frequencies are valid, PTA_ear = their mean;
  otherwise PTA_ear is missing.
PTA = min(PTA_R, PTA_L), requiring both ear means to be observed.
BASE = examined AND positive MEC weight AND age >= 70 AND valid PTA.
Construct hearing-aid status using the dictionary, respecting skips.
HA = BASE AND hearing-aid status = 1 AND AUQ153 in {1,2,3,4,5}.
PRIMARY = HA AND AUQ144 in {1,2,3,4,5}.
For each model, additionally require its specified covariates.
For the retest sensitivity, replace valid same-ear/frequency values,
recompute PTA, and refit within the original PRIMARY domain.

```

The observed BASE domain has 1,025 valid no-device responses and 205 current-device responses, with no unknown AUQ147 values. Of the 205, 199 identify a hearing aid and all have valid wearing categories. Their unweighted AUQ153 counts for codes 1–5 are 11, 19, 36, 120, and 13. Their AUQ144 counts for codes 1–5 are 92, 82, 14, 8, and 1; two additional responses are code 9 and are excluded. These checks explain the 199-to-197 transition without treating unknown recency as an earlier test. All primary covariates are observed for those 197 participants. The income/education extension has 177 complete cases. No imputation, winsorization, or rescaling of audiometric units is used.

## C.7 Implementable estimation algorithm

Use the definitions of  $u_i$ ,  $U_{hj}$ ,  $A$ , and the Taylor covariance in Section C. The primary design vector is

$$X_i = (1, R_i, \text{PTA}_i, \text{age}_i, \text{female}_i)',$$

where  $R_i$  is the recent-test indicator. The unadjusted model retains only the intercept and  $R_i$ . The age-indicator and socioeconomic models separately extend the primary vector. The latter adds the income-to-poverty ratio and the four education indicators. The ordinal-score sensitivity uses this same primary vector in weighted linear regression.

### Algorithm 2: Weighted estimation with design-based uncertainty.

```

Input: domain mask, outcome y, covariate matrix X, MEC weights w,
      and the complete stratum/PSU index from Algorithm 1.
Proportion: P = sum(w*y)/sum(w) within the domain.
Influence_i = w_i*(y_i-P)/sum(w); zero outside the domain.
Logistic model:
  Divide domain weights by their mean; include an intercept.
  Initialize slopes at 0 and intercept at logit(weighted mean y).
  Repeat at most 200 times:
    pi = logistic(X*beta)
    A = X' diag(w*pi*(1-pi)) X
    delta = solve(A, X' (w*(y-pi)))
    beta = beta + delta
    Stop if max(abs(delta)) < 1e-10.
  Recompute pi and A at the final beta.
  Influence_i = inverse(A) X_i*w_i*(y_i-pi_i).
For either estimator:
  Sum influences within each sampled stratum/PSU pair.
  Reinsert zero totals for sampled PSUs absent from the domain.
  Sum m/(m-1) times centered PSU outer products within strata.
  Use the square root of the appropriate diagonal as the SE.
Report logit-scale proportion CIs and exponentiated coefficient CIs,
using the domain/residual t degrees of freedom specified in the text.
Do not report ordinary logistic Wald inference after nonconvergence,
singular information, or separation.

```

For the recent-testing coefficient, the odds ratio is  $\exp(\hat{\beta}_R)$ , its interval is  $\exp[\hat{\beta}_R \pm t_{0.975, \nu} \text{SE}(\hat{\beta}_R)]$ , and the two-sided  $p$ -value is  $2 \Pr\{T_\nu \geq |\hat{\beta}_R / \text{SE}(\hat{\beta}_R)|\}$ . The linear-score calculation replaces  $\pi$  by  $X\beta$  and  $\pi(1-\pi)$  by 1, retaining the same PSU covariance calculation. For the probability difference, use the recent-domain influence minus the earlier/never-domain influence before aggregating to PSUs. This preserves the covariance induced by the shared survey design:

$$\text{Var}(\hat{P}_1 - \hat{P}_0) = \text{Var}(\hat{P}_1) + \text{Var}(\hat{P}_0) - 2 \text{Cov}(\hat{P}_1, \hat{P}_0).$$

Treating the estimates as independent and adding their variances would omit the final term. The low-precision check uses the effective-size and beta-quantile formulas above; the disclosure flag is triggered if  $n < 30$ ,  $n_e < 30$ , interval width is at least 0.30, or domain degrees of freedom are below 8.

## D Numerical verification and reproducibility

The numerical procedure solves the equilibrium using Brent’s method with absolute root tolerance  $10^{-14}$ , checks finite-difference policy derivatives against the analytical expressions, and compares the two capacity-coordinate decompositions. The maximum discrepancy in those derivative checks is approximately  $2.1 \times 10^{-9}$ . A separate 72-cell implementation check varies the type distribution across four beta distributions while retaining a common service function; it verifies the inverse adoption response and the congestion identity. The largest identity discrepancy is below  $10^{-7}$ . These are implementation checks, not substitutes for proofs or an empirical validation of the model.

Fixed-capacity optima are computed by bounded scalar optimization with endpoints evaluated separately. Joint policies use constrained optimization from 16 starting points. Candidate optima are checked for budget feasibility, complementary slackness, and one-sided instrument conditions. A separate 81-by-101 feasible grid does not improve any reported joint solution. The largest reported stationarity residual is below  $2 \times 10^{-7}$ . This supports numerical accuracy in the examined parameter cells without claiming a general proof of global joint concavity.

### D.1 Equilibrium and policy algorithms

The original policy scenarios in Tables 4 and 16 use uniform types,  $p = 0.5$ ,  $K_0 = 1$ , and either  $q(h) = 0.2 + \sqrt{h}$  or  $q(h) = 0.2 + 1 - e^{-h}$ . For any trial  $(s, x)$ , set  $K = 1 + x$  and compute  $N$  before evaluating expenditure or welfare. The exact functions used are

$$\begin{aligned} E(s, x) &= sN + c_x x, \\ W(s, x) &= (1 + \eta)(N - N^2/2)q((1 + x)/N) - 0.5N - (1 + \lambda)c_x x - \lambda sN. \end{aligned}$$

At  $N = 0$ , define aggregate benefit by its zero limit and evaluate welfare as  $-(1 + \lambda)c_x x$ , without dividing by zero. At  $s = p$ , set  $N = 1$ . No reported cost–budget case has zero adoption.

**Algorithm 3:** Equilibrium solution and policy search.

```

Equilibrium(s, K, technology):
    If s >= p, return N = 1.
    For the saturating technology, if p-s >= 1.2, return N = 0.
    Otherwise solve (1-N)*q(K/N) - p + s = 0 by Brent's method
    on [1e-14, 1], with absolute root tolerance 1e-14.
Evaluate(s,x):
    N = Equilibrium(s,1+x,technology).
    Return W(s,x), E(s,x), and the analytical policy derivatives.
Fixed-capacity search:
    Maximize W(s,0) on [0,p] by bounded scalar optimization
    with x tolerance 1e-13; compare the result with both endpoints.
Budget-constrained search:
    Bounds: 0 <= s <= p, 0 <= x <= B/c_x; require E(s,x) <= B.
    For device-only policy fix x=0; for capacity-only policy fix s=0.
    Run SLSQP from every stated starting point (settings below).
    Retain successful candidates with spending <= B + 1e-8.
    Select the feasible candidate with the highest unrounded W.
    Check one-sided KKT conditions and compare with the stated
    independent feasible grid; report W-W(0,0), spending and slack.

```

For Table 4, the 16 starting pairs are  $s_0 \in \{0, 0.1, 0.3, 0.49\}$  and  $x_0 \in \{0, 0.3, 1.2, 2.4\}$ . Fix any excluded instrument at zero. SLSQP uses numerical gradient approximations; analytical derivatives are reserved for the KKT checks. It uses function tolerance  $10^{-12}$  and at most 400 iterations. The validation grid has 81 evenly spaced subsidy values on  $[0, 0.5]$  and 101 expansion values on  $[0, 2.4]$ ; retain points satisfying  $E \leq 0.6 + 10^{-12}$ . No grid value exceeds a selected candidate by more than  $10^{-8}$ .

For the 54-case exercise, use the Cartesian product of both technologies,  $c_x \in \{0.1, 0.25, 1\}$ ,  $B \in \{0.15, 0.6, 1.2\}$ , and  $\eta \in \{0.2, 2, 8\}$ , with  $\lambda = 0.1$ . Its nine starts are  $s_0 \in \{0, 0.15, 0.4\}$  and  $x_0 \in \{0, 0.4, 1\} B/c_x$ . Include  $(0, 0)$  as an explicit feasible candidate. Use the same function tolerance with 500 maximum iterations. The separate grid has 21 evenly spaced subsidy values and 25 expansion values across their bounds, retaining  $E \leq B + 10^{-12}$ . The comparison tolerance is  $10^{-7}$ . Starting points need not initially satisfy the nonlinear spending constraint; only feasible successful endpoints are retained.

For KKT diagnostics, define  $r_j = W_j - \tau E_j$ . At a lower bound test  $\max(r_j, 0)$ , at an upper bound test  $\max(-r_j, 0)$ , and in an interior direction test  $|r_j|$ , excluding any fixed instrument. Set  $\tau = 0$  for budget slack greater than  $10^{-6}$ . At an active budget use the nonnegative marginal welfare-to-expenditure ratio in an available interior direction; the 54-case check averages the ratios if both instruments supply that direction. Bound comparisons use  $10^{-7}$ . The reported maximum residual is below  $2.5 \times 10^{-7}$ . Policy labels use the separate economic reporting threshold  $10^{-6}$ , as specified in Table 15. These are numerical diagnostics of the reported candidates, not a proof of global concavity.

## D.2 Curvature matching and numerical checks

To reproduce the matched-curvature comparison, first calculate  $(N^0, h^0, q_0, q'_0)$  at the square-root baseline,  $s = 0$ ,  $K = 1$ . For each  $\kappa \in \{0.4, 0.8, 1.2\}$  construct

$$t = \frac{\kappa}{h^0}, \quad b = \frac{q'_0 e^\kappa}{t}, \quad a = q_0 - b(1 - e^{-\kappa}), \quad q_\kappa(h) = a + b(1 - e^{-th}).$$

Here  $a$  denotes the matched intercept, not the fixed 0.2 intercept used in the original two families. These choices give  $q_\kappa(h^0) = q_0$ ,  $q'_\kappa(h^0) = q'_0$ , and  $-h^0 q''_\kappa(h^0)/q'_\kappa(h^0) = \kappa$ . All three matched intercepts are positive. The common initial adoption follows from the identical initial willingness-to-pay equation and uniqueness. Re-solve the equilibrium at  $K = 1 \pm 10^{-5}$  and compare the log difference quotient for  $\rho$  with (31). Table 5 reports all three parameter sets and slopes.

The original derivative check evaluates both technologies over  $K \in \{0.2, 0.5, 1, 2, 3\}$ ,  $s \in \{0, 0.12, 0.4\}$ ,  $\eta \in \{0, 0.2, 2, 8\}$ , and  $\lambda \in \{0, 0.1, 0.35\}$ . Centered differences use step  $10^{-6}$  and  $c_x = 1$ ; at  $s = 0$  the centered diagnostic uses the local mathematical continuation to a small negative subsidy, whereas policy choices remain nonnegative. The two capacity-coordinate formulas are also compared directly. For the heterogeneous-type check use beta shape pairs (1, 1), (2, 2), (0.7, 1.5), and (3, 1), both technologies,  $K \in \{0.2, 1, 3\}$ , and  $s \in \{0, 0.2, 0.4\}$ . Solve  $F^{-1}(1 - N)q(K/N) = p - s$ , with

$$M(N) = \frac{\alpha}{\alpha + \beta} [1 - F_{\alpha+1, \beta}(\theta_m(N))].$$

Check the congestion identity by the centered five-point derivative of  $M(N)q(K/N)$  using  $d = \min\{10^{-5}, N/10, (1 - N)/10\}$ . Check  $N_s = 1/H$  by centered differences in  $s$  with the same  $d$ . The reported 72 cells count the four distributions, two technologies, three capacities, and three subsidies.

### D.3 Reconstructing the numerical figures

Figure 4(a) evaluates 301 evenly spaced subsidies from 0 to 0.49, normalizing  $N(s)$  and  $h(s)$  to 100 at  $s = 0$ . Panel (b) evaluates 101 financing wedges from 0 to 0.2, holding the zero-subsidy values of  $\rho$  and  $\delta$  fixed. Panel (c) evaluates 301 subsidies from 0 to 0.085 at  $\eta = 8$ , separately for  $\lambda = 0$  and 0.1, and plots  $10^3[W(s, 0) - W(0, 0)]$ . Panel (d) uses the panel-(a) subsidy grid,  $\eta = 8$ , and  $\lambda = 0$ . Its frozen-service appraisal is  $9(N - N^2/2)q(h^0) - 0.5N$ , evaluated on the actual equilibrium adoption path; subtract its value at  $s = 0$ . The true curve uses equilibrium  $q(K/N)$  instead. This specifies the plotted curves without requiring individual saved plotting rows.

Figure 5 evaluates 301 evenly spaced capacity values between 0.29 and 3 at  $s = 0$ , using the analytical quantities in its caption. Figures 6 and 7 use the 12 policy rows in Table 4. Reconstruct each budget bar as  $(sN, c_x x, B - E)$  and each welfare bar as  $W(s, x) - W(0, 0)$  within its own technology and external-benefit scenario. All points and curves are deterministic model outputs. Figure 3 instead uses the estimates and intervals reported in the empirical appendix; its confidence intervals are survey uncertainty, not numerical-solver uncertainty.

### D.4 Numerical reporting conventions

Counts are integers, with thousands separators when applicable. Empirical percentages and percentage-point differences, including their confidence limits, use one decimal; odds ratios and their limits use two decimals; descriptive means use two decimals, and ordinal-score differences use three. Nominal  $p$ -values use three decimals (values below 0.001 would be reported as  $p < 0.001$ ). Computed model levels, policy allocations, and dimensionless statistics use four decimals. The small fixed-capacity welfare gains are reported in units of  $10^{-3}$  to retain resolution; Figure 7 and Table 16 use the same scaling for gains below 0.01. Exact input parameters, axis ticks, and numerical tolerances follow their stated units and purpose rather than this display rule. Calculations use unrounded values. Consequently, rounded components need not sum exactly to a rounded total. The displayed precision does not imply empirical identification or comparable measurement accuracy.

## D.5 Additional numerical scenarios

The 54 cost–budget–external-benefit cases use nine constrained-optimizer starting points per case and a separate 21-by-25 feasible grid. No grid point improves a reported solution, and the largest one-sided KKT residual is below  $2.5 \times 10^{-7}$ . These finite checks support the reported candidate optima without establishing joint concavity. The matched-curvature slopes are independently checked by centered differences of the re-solved equilibrium, with discrepancies below  $2.0 \times 10^{-11}$ . Table 16 reports every scenario, including both instruments, the welfare gain, spending, and unused budget. The maximum residual stated here summarizes the numerical diagnostics across these rows.

| Technology  | $c_x$ | None | Device | Capacity | Mixed |
|-------------|-------|------|--------|----------|-------|
| Square-root | 0.1   | 0    | 0      | 9        | 0     |
| Square-root | 0.25  | 0    | 0      | 9        | 0     |
| Square-root | 1     | 6    | 0      | 3        | 0     |
| Saturating  | 0.1   | 0    | 0      | 3        | 6     |
| Saturating  | 0.25  | 3    | 0      | 0        | 6     |
| Saturating  | 1     | 3    | 3      | 0        | 3     |

**Table 15:** Instrument composition across 54 illustrative cost–budget–external-benefit scenarios.

Each row contains the nine combinations  $B \in \{0.15, 0.6, 1.2\}$  and  $\eta \in \{0.2, 2, 8\}$ , at  $\lambda = 0.1$ ,  $K_0 = 1$ , and  $p = 0.5$ . An instrument is counted as used above  $10^{-6}$  model units. Counts summarize this chosen scenario grid, not frequencies or probabilities for real programs. Individual solutions, expenditure and slack appear in Table 16; Appendix D.1 gives the diagnostic procedure and residual bound.

**Table 16:** Complete cost–budget–external-benefit scenario results.

| Tech. | $c_x$ | $B$  | $\eta$ | $s$    | $x$     | $\Delta W$              | $E$    | $B - E$ |
|-------|-------|------|--------|--------|---------|-------------------------|--------|---------|
| Root  | 0.1   | 0.15 | 0.2    | 0.0000 | 1.5000  | 0.1655                  | 0.1500 | 0.0000  |
| Root  | 0.1   | 0.15 | 2      | 0.0000 | 1.5000  | 0.7365                  | 0.1500 | 0.0000  |
| Root  | 0.1   | 0.15 | 8      | 0.0000 | 1.5000  | 2.6399                  | 0.1500 | 0.0000  |
| Root  | 0.1   | 0.6  | 0.2    | 0.0000 | 6.0000  | 0.2939                  | 0.6000 | 0.0000  |
| Root  | 0.1   | 0.6  | 2      | 0.0000 | 6.0000  | 1.8641                  | 0.6000 | 0.0000  |
| Root  | 0.1   | 0.6  | 8      | 0.0000 | 6.0000  | 7.0981                  | 0.6000 | 0.0000  |
| Root  | 0.1   | 1.2  | 0.2    | 0.0000 | 6.2127  | 0.2940                  | 0.6213 | 0.5787  |
| Root  | 0.1   | 1.2  | 2      | 0.0000 | 12.0000 | 2.6554                  | 1.2000 | 0.0000  |
| Root  | 0.1   | 1.2  | 8      | 0.0000 | 12.0000 | 10.8305                 | 1.2000 | 0.0000  |
| Root  | 0.25  | 0.15 | 0.2    | 0.0000 | 0.0230  | $0.0317 \times 10^{-3}$ | 0.0057 | 0.1443  |
| Root  | 0.25  | 0.15 | 2      | 0.0000 | 0.6000  | 0.2484                  | 0.1500 | 0.0000  |
| Root  | 0.25  | 0.15 | 8      | 0.0000 | 0.6000  | 1.1296                  | 0.1500 | 0.0000  |
| Root  | 0.25  | 0.6  | 0.2    | 0.0000 | 0.0230  | $0.0317 \times 10^{-3}$ | 0.0057 | 0.5943  |
| Root  | 0.25  | 0.6  | 2      | 0.0000 | 2.4000  | 0.6442                  | 0.6000 | 0.0000  |
| Root  | 0.25  | 0.6  | 8      | 0.0000 | 2.4000  | 3.3815                  | 0.6000 | 0.0000  |
| Root  | 0.25  | 1.2  | 0.2    | 0.0000 | 0.0230  | $0.0317 \times 10^{-3}$ | 0.0057 | 1.1943  |
| Root  | 0.25  | 1.2  | 2      | 0.0000 | 4.8000  | 0.8439                  | 1.2000 | 0.0000  |
| Root  | 0.25  | 1.2  | 8      | 0.0000 | 4.8000  | 5.3439                  | 1.2000 | 0.0000  |
| Root  | 1     | 0.15 | 0.2    | 0.0000 | 0.0000  | 0.0000                  | 0.0000 | 0.1500  |
| Root  | 1     | 0.15 | 2      | 0.0000 | 0.0000  | 0.0000                  | 0.0000 | 0.1500  |
| Root  | 1     | 0.15 | 8      | 0.0000 | 0.1500  | 0.1923                  | 0.1500 | 0.0000  |
| Root  | 1     | 0.6  | 0.2    | 0.0000 | 0.0000  | 0.0000                  | 0.0000 | 0.6000  |
| Root  | 1     | 0.6  | 2      | 0.0000 | 0.0000  | 0.0000                  | 0.0000 | 0.6000  |
| Root  | 1     | 0.6  | 8      | 0.0000 | 0.6000  | 0.6346                  | 0.6000 | 0.0000  |

Table 16 continued

| Tech. | $c_x$ | $B$  | $\eta$ | $s$    | $x$    | $\Delta W$              | $E$    | $B - E$ |
|-------|-------|------|--------|--------|--------|-------------------------|--------|---------|
| Root  | 1     | 1.2  | 0.2    | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 1.2000  |
| Root  | 1     | 1.2  | 2      | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 1.2000  |
| Root  | 1     | 1.2  | 8      | 0.0000 | 1.2000 | 1.0204                  | 1.2000 | 0.0000  |
| Sat.  | 0.1   | 0.15 | 0.2    | 0.0000 | 0.0382 | $0.1139 \times 10^{-3}$ | 0.0038 | 0.1462  |
| Sat.  | 0.1   | 0.15 | 2      | 0.1066 | 0.8033 | 0.1201                  | 0.1500 | 0.0000  |
| Sat.  | 0.1   | 0.15 | 8      | 0.0904 | 0.9184 | 0.6867                  | 0.1500 | 0.0000  |
| Sat.  | 0.1   | 0.6  | 0.2    | 0.0000 | 0.0382 | $0.1139 \times 10^{-3}$ | 0.0038 | 0.5962  |
| Sat.  | 0.1   | 0.6  | 2      | 0.1940 | 1.0971 | 0.1322                  | 0.2517 | 0.3483  |
| Sat.  | 0.1   | 0.6  | 8      | 0.3819 | 2.4269 | 1.0965                  | 0.5863 | 0.0137  |
| Sat.  | 0.1   | 1.2  | 0.2    | 0.0000 | 0.0382 | $0.1139 \times 10^{-3}$ | 0.0038 | 1.1962  |
| Sat.  | 0.1   | 1.2  | 2      | 0.1940 | 1.0971 | 0.1322                  | 0.2517 | 0.9483  |
| Sat.  | 0.1   | 1.2  | 8      | 0.3819 | 2.4269 | 1.0965                  | 0.5863 | 0.6137  |
| Sat.  | 0.25  | 0.15 | 0.2    | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 0.1500  |
| Sat.  | 0.25  | 0.15 | 2      | 0.1189 | 0.2945 | 0.0274                  | 0.1500 | 0.0000  |
| Sat.  | 0.25  | 0.15 | 8      | 0.0871 | 0.3833 | 0.3898                  | 0.1500 | 0.0000  |
| Sat.  | 0.25  | 0.6  | 0.2    | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 0.6000  |
| Sat.  | 0.25  | 0.6  | 2      | 0.1214 | 0.3016 | 0.0274                  | 0.1537 | 0.4463  |
| Sat.  | 0.25  | 0.6  | 8      | 0.3113 | 1.3608 | 0.7769                  | 0.6000 | 0.0000  |
| Sat.  | 0.25  | 1.2  | 0.2    | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 1.2000  |
| Sat.  | 0.25  | 1.2  | 2      | 0.1214 | 0.3016 | 0.0274                  | 0.1537 | 1.0463  |
| Sat.  | 0.25  | 1.2  | 8      | 0.3402 | 1.5202 | 0.7822                  | 0.6729 | 0.5271  |
| Sat.  | 1     | 0.15 | 0.2    | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 0.1500  |
| Sat.  | 1     | 0.15 | 2      | 0.0865 | 0.0000 | 0.0101                  | 0.0513 | 0.0987  |
| Sat.  | 1     | 0.15 | 8      | 0.1871 | 0.0227 | 0.1724                  | 0.1500 | 0.0000  |
| Sat.  | 1     | 0.6  | 0.2    | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 0.6000  |
| Sat.  | 1     | 0.6  | 2      | 0.0865 | 0.0000 | 0.0101                  | 0.0513 | 0.5487  |
| Sat.  | 1     | 0.6  | 8      | 0.2321 | 0.2107 | 0.1994                  | 0.3811 | 0.2189  |
| Sat.  | 1     | 1.2  | 0.2    | 0.0000 | 0.0000 | 0.0000                  | 0.0000 | 1.2000  |
| Sat.  | 1     | 1.2  | 2      | 0.0865 | 0.0000 | 0.0101                  | 0.0513 | 1.1487  |
| Sat.  | 1     | 1.2  | 8      | 0.2321 | 0.2107 | 0.1994                  | 0.3811 | 0.8189  |

Root:  $q(h) = 0.2 + \sqrt{h}$ ; Sat.:  $q(h) = 0.2 + 1 - e^{-h}$ . All rows use  $p = 0.5$ ,  $K_0 = 1$ , and  $\lambda = 0.1$ .  $\Delta W = W(s, x) - W(0, 0)$  uses each row's own technology and  $\eta$ . Spending is  $E = sN + c_x x$ , with  $N$  determined by Algorithm 3. Gains with absolute magnitude below  $10^{-10}$  are displayed as numerical zero. Other positive welfare gains below 0.01 use explicit  $10^{-3}$  scaling so small improvements are not rounded to zero. Values are rounded only for display; numerical negative zeros in slack are displayed as zero. The instrument counts in Table 15 use the unrounded  $10^{-6}$  threshold. Algorithm 3 and Appendix D.1 specify the reconstruction and numerical checks for these candidates.

## D.6 Distribution sensitivity: construction and verification

Let  $F_{\alpha, \beta}$  be the beta CDF on  $[0, 1]$ . For  $\theta = tZ$ , the functions needed to reproduce the distribution comparison are

$$z_m(N) = F_{\alpha, \beta}^{-1}(1 - N), \quad \theta_m(N) = tz_m(N), \quad (37)$$

$$M(N) = t \frac{\alpha}{\alpha + \beta} [1 - F_{\alpha+1, \beta}(z_m(N))], \quad (38)$$

$$f(\theta_m) = \frac{z_m^{\alpha-1}(1 - z_m)^{\beta-1}}{t B(\alpha, \beta)}. \quad (39)$$

Here  $B(\alpha, \beta)$  is the beta function, distinct from the spending ceiling  $B$ . In the fixed-primitives design  $t = 1$ . In the matched design, first solve the uniform equilibrium  $N_U^0$  at  $s = 0$ ,  $K = 1$  for each technology and set

$$t = \frac{1 - N_U^0}{F_{\alpha, \beta}^{-1}(1 - N_U^0)}. \quad (40)$$

Then  $\theta_m(N_U^0) = 1 - N_U^0$  and the same equilibrium equation holds. Uniqueness establishes the adoption match. Neither  $M$  nor  $f(\theta_m)$  is constrained by this matching condition. The parameters and initial statistics are in Table 6.

We optimize in allocation coordinates  $(N, x)$ , computing  $s = p - \theta_m q((1+x)/N)$ ,  $E = sN + c_x x$ , and  $V$  from (9). Impose  $s \geq 0$ ,  $E \leq B$ ,  $0 < N \leq 1$ , and  $x \geq 0$ ;  $s \leq p$  follows from nonnegative types and benefits. With  $C = hq'M/N$ , the analytical gradients are

$$\begin{aligned} s_N &= H, & s_x &= -\theta_m q'/N, \\ E_N &= s + NH, & E_x &= c_x - \theta_m q', \\ V_N &= (1 + \eta)(\theta_m q - C) - p - \lambda(s + NH), & V_x &= (1 + \eta)(M/N)q' + \lambda\theta_m q' - (1 + \lambda)c_x. \end{aligned}$$

At  $N = 1$ ,  $M = t\alpha/(\alpha + \beta)$  and the implementing subsidy is  $p$ . Zero adoption is unavailable in these cells: the square-root technology is unbounded, and every saturating cell has  $1.2t > p$ .

**Algorithm 4:** Distribution comparison and independent verification.

```

For each technology and beta shape pair:
  Fixed design: t=1. Matched: t=(1-N_U^0)/BetaQuantile(1-N_U^0).
  Reuse the uniform row in both designs.
  Solve the zero-subsidy equilibrium and compute rho, delta, eta_c.
  For eta in {2,8}, maximize V(N,x) with the constraints above.
  Use 16 starts: x in {0,0.2,0.6,1}*(B/c_x), and
  N=N(0,1+x)+r*(1-N(0,1+x)), r in {0,0.25,0.6,0.95}.
  Include no intervention and exact full-adoption candidates.
  Compare feasible successful endpoints using unrounded welfare.
  Check a 121-by-101 feasible grid in (N,x).
  Re-optimize in (s,x), solving adoption anew at each evaluation.
  Check welfare agreement, feasibility, and one-sided KKT signs.
  Report the candidate, its own-baseline gain, spending and policy.

```

The allocation solver uses SLSQP with the analytical gradients, function tolerance  $10^{-12}$ , at most 600 iterations,  $N \in [10^{-9}, 1 - 10^{-9}]$ , and  $x \in [0, B/c_x]$ . Starts may violate the nonlinear constraints; retain successful endpoints with  $s \geq -10^{-9}$  and  $E \leq B + 10^{-9}$ . At full adoption, separately maximize over  $x \in [0, (B - p)/c_x]$  with bounded scalar tolerance  $10^{-13}$  and compare both endpoints. Policy coordinates  $s$  and  $x$  below  $10^{-8}$  are set to zero before re-solving adoption and re-evaluating all reported quantities.

The independent primitive-coordinate search uses numerical gradients, the same function tolerance and iteration limit, and starts  $(0, 0)$ ,  $(0.1, 0.3)$ ,  $(0.3, 1.2)$ , and the allocation candidate. Bounds are  $s \in [0, p - 10^{-9}]$  and  $x \in [0, B/c_x]$ ; the exact full-adoption boundary is covered by the separate check above. Adoption solves  $\theta_m(N)q((1+x)/N) = p - s$  by Brent's method on  $[10^{-12}, 1]$  with absolute tolerance  $10^{-14}$ . The grid uses 121 equally spaced  $N$  values from the zero-subsidy, zero-expansion equilibrium to  $1 - 10^{-9}$  and 101  $x$  values over  $[0, B/c_x]$ , retaining  $s \geq 0$  and  $E \leq B + 10^{-12}$ . No grid point improves a selected result; welfare agreement across the two coordinate searches is within  $1.3 \times 10^{-14}$ .

For KKT checks, transform the allocation gradients using  $N_s = 1/H$  and  $N_x = -s_x/H$ . Use the one-sided residual rule in Appendix D.1. The auxiliary bound  $x \leq B/c_x$  is implied by the

budget; it is not an additional economic expansion limit. At a binding budget, average the marginal welfare-to-expenditure ratios of positive instruments, excluding the subsidy if it is at its upper bound. The maximum residual is below  $10^{-7}$ . No selected cell is at full adoption.

Centered differences of the allocation gradients at  $N = N^0$ ,  $x = 0.1$ , and  $\eta \in \{2, 8\}$  use step  $10^{-6}$ ; the maximum discrepancy is below  $1.7 \times 10^{-9}$ . Direct quadrature of the beta tail moment agrees with (38) within  $3.4 \times 10^{-16}$ . The four uniform welfare values reproduce Table 4 within  $3.6 \times 10^{-15}$ . These numerical checks support the listed candidates without establishing global joint concavity. For the interval example in Table 1, evaluating  $T$  at all 16 endpoint combinations reproduces the analytical bounds.

**Table 17:** All distribution-policy comparisons.

| Tech. | Design  | $(\alpha, \beta)$ | $\eta$ | $s$    | $x$    | $N$    | $\Delta W$ | $E$    | Policy   |
|-------|---------|-------------------|--------|--------|--------|--------|------------|--------|----------|
| Root  | Fixed   | (1, 1)            | 2      | 0.0000 | 2.4000 | 0.7813 | 0.6442     | 0.6000 | Capacity |
| Root  | Fixed   | (1, 1)            | 8      | 0.0000 | 2.4000 | 0.7813 | 3.3815     | 0.6000 | Capacity |
| Root  | Fixed   | (2, 2)            | 2      | 0.0000 | 2.4000 | 0.8664 | 0.5972     | 0.6000 | Capacity |
| Root  | Fixed   | (2, 2)            | 8      | 0.0000 | 2.4000 | 0.8664 | 3.2729     | 0.6000 | Capacity |
| Root  | Fixed   | (0.7, 1.5)        | 2      | 0.0000 | 2.4000 | 0.5869 | 0.2285     | 0.6000 | Capacity |
| Root  | Fixed   | (0.7, 1.5)        | 8      | 0.0000 | 2.4000 | 0.5869 | 2.1392     | 0.6000 | Capacity |
| Root  | Fixed   | (3, 1)            | 2      | 0.0000 | 2.4000 | 0.9856 | 1.2228     | 0.6000 | Capacity |
| Root  | Fixed   | (3, 1)            | 8      | 0.0000 | 2.4000 | 0.9856 | 5.0404     | 0.6000 | Capacity |
| Root  | Matched | (2, 2)            | 2      | 0.0000 | 2.4000 | 0.8353 | 0.4440     | 0.6000 | Capacity |
| Root  | Matched | (2, 2)            | 8      | 0.0000 | 2.4000 | 0.8353 | 2.8349     | 0.6000 | Capacity |
| Root  | Matched | (0.7, 1.5)        | 2      | 0.0000 | 2.4000 | 0.7489 | 1.3535     | 0.6000 | Capacity |
| Root  | Matched | (0.7, 1.5)        | 8      | 0.0000 | 2.4000 | 0.7489 | 5.4771     | 0.6000 | Capacity |
| Root  | Matched | (3, 1)            | 2      | 0.0000 | 2.4000 | 0.8957 | 0.2822     | 0.6000 | Capacity |
| Root  | Matched | (3, 1)            | 8      | 0.0000 | 2.4000 | 0.8957 | 2.4099     | 0.6000 | Capacity |
| Sat.  | Fixed   | (1, 1)            | 2      | 0.1214 | 0.3016 | 0.6452 | 0.0274     | 0.1537 | Mixed    |
| Sat.  | Fixed   | (1, 1)            | 8      | 0.3113 | 1.3608 | 0.8346 | 0.7769     | 0.6000 | Mixed    |
| Sat.  | Fixed   | (2, 2)            | 2      | 0.1352 | 0.3790 | 0.7226 | 0.0436     | 0.1925 | Mixed    |
| Sat.  | Fixed   | (2, 2)            | 8      | 0.2808 | 1.3874 | 0.9016 | 0.9717     | 0.6000 | Mixed    |
| Sat.  | Fixed   | (0.7, 1.5)        | 2      | 0.1751 | 0.0000 | 0.4517 | 0.0453     | 0.0791 | Device   |
| Sat.  | Fixed   | (0.7, 1.5)        | 8      | 0.3536 | 0.8242 | 0.6818 | 0.5134     | 0.4471 | Mixed    |
| Sat.  | Fixed   | (3, 1)            | 2      | 0.0636 | 1.0191 | 0.9349 | 0.2032     | 0.3143 | Mixed    |
| Sat.  | Fixed   | (3, 1)            | 8      | 0.1357 | 1.8746 | 0.9681 | 1.5743     | 0.6000 | Mixed    |
| Sat.  | Matched | (2, 2)            | 2      | 0.1370 | 0.3571 | 0.7159 | 0.0427     | 0.1874 | Mixed    |
| Sat.  | Matched | (2, 2)            | 8      | 0.2840 | 1.3764 | 0.9010 | 0.9642     | 0.6000 | Mixed    |
| Sat.  | Matched | (0.7, 1.5)        | 2      | 0.0825 | 0.4019 | 0.5910 | 0.0357     | 0.1492 | Mixed    |
| Sat.  | Matched | (0.7, 1.5)        | 8      | 0.3106 | 1.4474 | 0.7668 | 0.7649     | 0.6000 | Mixed    |
| Sat.  | Matched | (3, 1)            | 2      | 0.1525 | 0.4181 | 0.8254 | 0.0781     | 0.2304 | Mixed    |
| Sat.  | Matched | (3, 1)            | 8      | 0.2595 | 1.4076 | 0.9561 | 1.1854     | 0.6000 | Mixed    |

All rows use  $c_x = 0.25$ ,  $B = 0.6$ , and  $\lambda = 0.1$ . Definitions and type scales are in Table 6. Each gain is relative to the same row's no-intervention equilibrium:  $\Delta W = W(s, x) - W(0, 0)$ . Budget slack is  $0.6 - E$ ; numerical rounding can affect the last digit. Instrument labels use a  $10^{-6}$  threshold. The uniform rows are reproduced from the original comparison and serve both designs. All allocations are numerically supported candidates, not a general global-optimality theorem.

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