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论文题目: Inertia Indices of Entanglement
Witnesses in $C^2 \otimes C^n$

Inertia Indices of Entanglement Witnesses in $\mathbb{C}^2 \otimes \mathbb{C}^n$

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First of all, according to NP-hard problem, entanglement witness is a really powerful and useful tool that helps to detect entanglement. A Hermitian operator that has non-negative expectation values on all product states is one entanglement witness. However, it attains a negative value on more than one entangled state. Since witness can help detect entanglement, the relationship between inertia of the matrix and geometry of the underlying bipartite Hilbert space still remained a large amount unexplored, even the simplest bipartite configuration. As we researched, until now, although many 2×2 entanglement witness all have inertia $(1,0,3)$, but they are not proved yet.

During our work, we first focused on 2×2 case, and we successfully proved that for any 2 dimension subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$, they must contain only one negative eigenvalue. While proving, we used fundamentally geometric to establish connection between dimension if the negative eigenspace of a witness and the product vectors. After that, we come up with a assumption that for any $2 \times n$ entanglement witness, they all have at most $n-1$ negative eigenvalues. During our proof, we used a rank argument on the block decomposition of the negative eigenspace, which shows that every subspace of dimension n have a product vector, which violate the witness condition. Moreover, we suppose that for every 2×2 entanglement witness, they all have $(1,0,3)$ inertia index since according to statistical data, all 2×2 entanglement witness established point to the same result of inertia index, which is $(1,0,3)$, but without proof. Therefore, we tried to prove it. We find hat the inertia index will be $(1,a,3-a)$ and we proved both $a=1$ and $a=2$ can not satisfy. Therefore a can only be 0 and therefore we successfully proved $(1,0,3)$.

Our main results contains mainly three parts. First we successfully proved the largest number of negative eigenvalues for a $2 \times n$ entanglement witness is $n-1$. In addition, we used our result to construct a $2N \times 2N$ diagonal unitary matrix and we advanced it by constructing a we provide an example of a $2 \times N$ subspace of dimension $N-1$ containing no product vector, which provides an useful system for the exploration and experiment of quantum physics. Moreover, we successfully proved the assumption that all 2×2 entanglement witness have the inertia index of $(1,0,3)$, which fill up the loss of proof around the world.

Keywords: Hermitian matrix, eigenvalue, entanglement witness

Contents

I. Introduction	3
II. Notations and Definitions	4
A. Matrix	4
1. Hermitian matrix	4
2. Kronecker product	4
3. Unitary matrix	4
4. Positive semi-definite matrix	4
5. Trace	5
6. Normal matrix	5
7. Schur's theorem	5
8. Symmetric Matrix	5
9. Basic Operational Properties	5
B. Inertia index	5
C. Abelian group	6
D. Ring	6
E. Linear space (vector space)	6
III. Result	7
A. properties for EW	7
B. Proof of $2 \times n$ EW matrix have at most $n - 1$ negative eigenvalue	9
C. Proof of $2 \times N$ space without product vector	10
D. Construction of entanglement witnesses on $\mathbb{C}^2 \otimes \mathbb{C}^n$ with k negative eigenvalues, $1 \leq k \leq n - 1$	11
E. An example of a $2 \times N$ subspace of dimension $N - 1$ containing no product vector	14
F. proof of 2×2 EW have a (1,0,3) inertia index	15
IV. Conclusion	17
V. appendix	18
A. Proof of Properties of Hermitian Matrices	18
Proof for real eigenvalues:	18
Proof for orthogonal eigenvectors:	18
Proof for unitary diagonalization:	19
B. Proof of $U^*U = I_2 = UU^*$	19
C. Proof for normal matrix	20
D. Proof for EW	21
E. Proof for Sylvester's law of inertia	22
References	23
Acknowledgment	24

I. INTRODUCTION

According to quantum information theory, we know that the quantum separability problem is a NP-hard problem¹. Later on, people showed it has a close connection with Edmonds' problem, and it turns out that under some specific conditions, the separability problem can be turned into an instance of Edmonds' problem. This helps explain why it is very unlikely we will get a simple, efficient, and universal criterion that always works that can be used every time. Once researchers realized this point, they started focusing more on working out practical². Separable state having positive partial transpose gave the first easy-to-compute necessary condition for checking separability³. Not long after that, the positive map criterion was suggested. It says that a state is separable if and only if it stays positive semidefinite no matter what positive map you apply to any one of its subsystems. The positive partial transpose is actually just one special case of this more general theoretical framework. Past work has shown that in $2 \otimes 2$ and $2 \otimes 3$ systems, the positive partial transpose is a necessary and sufficient condition, which is different from the PPT criterion⁴. This only gives a necessary condition, and other sufficient criteria: when a state meets an inequality that comes from a specific positive linear map, we can be sure that the state will be separable⁵. Another different method has shown that we can check if a state is PPT just by looking at its spectrum, without needing the full density matrix for this work⁶. Later on, this spectral method was extended when researchers worked out specific spectral rules that can confirm separability for qubit-qudit states, and it goes beyond PPT to tell us directly if a state is separable only from its spectrum⁷. When people went further in studying structured states, they looked into the spectral features of symmetric quantum states and how these connect to symmetric entanglement witnesses, and the results showed that symmetry constraints can make the separability problem much easier to handle⁸. Even with all these improvements, the complicated nature of the separability problem is highlighted by strange effects like bound entanglement. Previous work has shown that some entangled states that have zero distillable entanglement can still get activated when you look at certain multi-copy cases; this shows that the line between separability and entanglement is actually more complex than what people thought at first⁹. One recent paper includes the work to make a universal practical testing rule as one of the main still unsolved problems in this area of research¹⁰.

Entanglement witnesses are Hermitian operators that detect entanglement by taking negative values on some entangled states but remaining non-negative on all separable states. A basic question is: how many negative eigenvalues can an entanglement witness possess for a given bipartite system? Our main result shows that any $2 \times n$ entanglement witness has at most $n - 1$ negative eigenvalues. Moreover, we construct examples achieving this bound. First, we prove that for a 2×2 entanglement witness, it has only one negative eigenvalue. Suppose there were two negative eigenvalues $\lambda_3, \lambda_4 < 0$ with corresponding orthonormal eigenvectors u_3, u_4 . Let $V = \text{span}\{u_3, u_4\}$ be the subspace spanned by these eigenvectors. Under the natural isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong M_2(\mathbb{C})$, any two-dimensional subspace contains a rank-one matrix corresponding to a product vector $a \otimes b$: for basis matrices A, B , the equation $\det(xA + yB) = 0$ is a homogeneous quadratic equation in x, y , which always has a nontrivial solution over $\mathbb{C}$. Since $a \otimes b \in V$, we have $(a \otimes b)^* W (a \otimes b) = |x|^2 \lambda_3 + |y|^2 \lambda_4 < 0$, contradicting the witness condition. Hence exactly one negative eigenvalue exists. For the $2 \times n$ case, we extend this result by a dimensionality argument. Suppose the negative eigenspace has dimension at least n . Choosing n orthonormal negative eigenvectors and forming the matrix $V = \begin{pmatrix} X \\ Y \end{pmatrix}$ with $X, Y \in \mathbb{C}^{n \times n}$, we show that there always exists a nonzero vector $a \in \mathbb{C}^n$ such that Va is a product vector. This is established by considering three cases: if X is singular, a nonzero solution to $Xa = 0$ yields a product vector $(0; Ya)$; if Y is singular, the argument is symmetric; if both are invertible, set $T = YX^{-1}$ and take an eigenvector w of T with $Tw = \mu w$. Let $a = X^{-1}w$, then $Ya = \mu Xa$, hence Va is a product vector. For this product vector, the negative definiteness of $V^* H V$ yields $(Va)^* H (Va) < 0$, contradicting the witness condition. Therefore the negative eigenspace has dimension at most $n - 1$. Finally, we provide a constructive method to generate $2 \times n$ entanglement witnesses with any prescribed number of negative eigenvalues between 1 and $n - 1$. Starting from a witness with $n - 1$ negative eigenvalues, we add rank-one operators of the form $|\lambda_{N+i}| x_{N+i} x_{N+i}^*$, where x_{N+i} are the corresponding negative eigenvectors. Each addition lifts one negative eigenvalue to zero while leaving the remaining negative eigenvalues unchanged, yielding a new witness. After that, we did some research online and we found that no one had proved this before. All we can find is the guess of inertia index for 2×2 entanglement witness, which is (1,0,3), which comes from the collecting of data of numerous entanglement witness. Therefore, we tried to prove it. According to the previous proof, it has one negative. Therefore, the inertia index will be (1,a,3-a), and we break the topic into some circumstances, which are (1,0,3), (1,1,2), (1,2,1), and (1,3,0). During our proof, we found that except (1,0,3), the other three can not suit. Therefore, we proved the inertia index of 2×2 entanglement witness is (1,0,3).

II. NOTATIONS AND DEFINITIONS

A. Matrix

1. Hermitian matrix

Definition 1 A complex field square matrix $H \in \mathbb{C}^{d \times d}$ that satisfies $H^* = H$, then we call H as a Hermitian matrix. (H^* means performing complex conjugation on H and then transpose it)

There are three properties for Hermitian matrix. First, Hermitian matrix has real eigenvalues. Second, eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal to each other. Third, every Hermitian matrix is unitary diagonalizable. See the proof in Appendix A.

2. Kronecker product

Definition 2 The Kronecker product of two complex matrices $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$, denoted $A \otimes B$, is an $mp \times nq$ block matrix where the (i, j) -th block is formed by multiplying the scalar entry a_{ij} with the entire matrix B .

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

■

This operation is distributive with respect to matrix addition, namely $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$ and $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$, and it is compatible with scalar multiplication, so that $a \otimes A = A \otimes a = aA$ and $aA \otimes bB = ab A \otimes B$ for any scalars a and b . For conforming matrices with products AC and BD defined, one has the mixed-product rule $(A \otimes B)(C \otimes D) = AC \otimes BD$ and transposition and conjugate transposition of Kronecker product yield $(A \otimes B)^\top = A^\top \otimes B^\top$ and $(A \otimes B)^* = A^* \otimes B^*$, respectively.

It satisfies the non-vanishing identity $A \neq 0, B \neq 0$ iff $A \otimes B \neq 0$, and it is not commutative in general, so $A \otimes B \neq B \otimes A$ for arbitrary matrices, but trivially equal when the two operators are the same. If A and B are both Hermitian, $A = A^*, B = B^*$, then the Kronecker product $A \otimes B$ is also Hermitian, which is obvious from the conjugate transpose identity. For column vectors a and b , there holds the vector identity $a^\top \otimes b = ba^\top = b \otimes a^\top$.

3. Unitary matrix

Definition 3 A matrix $U \in M_n(\mathbb{C})$ is called a unitary matrix if it satisfies:

$$U^*U = UU^* = I_n.$$

All unitary matrices form a group. ■

Pauli matrices X, Y, Z satisfy two key identities:

$$X^* = X, \quad Y^* = Y, \quad Z^* = Z,$$

$$X^*X = Y^*Y = Z^*Z = I_2.$$

Where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $UU^* = I_2, U^*U = I_2$. Proof. See Appendix B.

4. Positive semi-definite matrix

Definition 4 For the Hermitian square matrix X , if for any complex vector $y \in \mathbb{C}^n$ satisfies $y^*Xy \geq 0$ then X is called a positive semi-definite matrix, denoted as $X \succeq 0$. ■

5. Trace

Definition 5 The trace of a square matrix is the sum of the entries on its main diagonal (from the top-left to the bottom-right corner). ■

For example, assume there is an n -order square matrix, the trace of it would be $\text{Tr}(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$. Also, the trace operator is linear, which satisfies additivity and homogeneity:

$$\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B), \quad (1)$$

$$\text{Tr}(kA) = k \times \text{Tr}(A). \quad (2)$$

6. Normal matrix

Definition 6 A complex square matrix X is called a normal matrix if it commutes with its conjugate transpose. ■

An equivalent formulation is:

$$XX^* = X^*X$$

It can also be shown using the *commutator* notation:

$$[X, X^*] = XX^* - X^*X = 0$$

Normal matrix is unitary diagonalizable. See the proof in Appendix. C

7. Schur's theorem

Lemma 1 Set $T \in M_n(\mathbb{C})$, which meets all the entries under the diagonal line is zero, then T is a upper triangular matrix. Set $A \in \mathbb{C}_{n \times n}$, there must be a unitary matrix U that meets $U^*AU = T$. ■

8. Symmetric Matrix

Definition 7 A square matrix Y is symmetric if $Y = Y^\top$, which implies $Y_{ij} = Y_{ji}$ for all $i \neq j$. Real symmetric matrices are orthogonally diagonalizable, and all eigenvalues are real numbers. ■

9. Basic Operational Properties

1. Transpose identity: $(AB)^\top = B^\top A^\top$.
2. Complex conjugate identity: $\overline{AB} = \overline{A} \overline{B}$.
3. Each real symmetric matrix admits a diagonal representation, with diagonal entries being its eigenvalues.

B. Intertia index

For an $n \times n$ Hermitian matrix $H \in M_n$ such that $H = H^* = \overline{H}^\top$, all eigenvalues of H are real. Counted with algebraic multiplicities, all its eigenvalues can be split into three subsets according to their signs: the total number of negative eigenvalues is denoted ν_- ; the total number of zero eigenvalues (same as the dimension of the null space of H) is denoted ν_0 , and the total number of positive eigenvalues is denoted ν_+ . The triple $\nu(H) = (\nu_-, \nu_0, \nu_+)$ is the inertia (inertia index) of the Hermitian matrix, and the three quantities always satisfy $\nu_- + \nu_0 + \nu_+ = n$, and ν_-, ν_0 and ν_+ are all non-negative integers.

C. Abelian group

Definition 8 An Abelian group is a group in which the binary operation is commutative, and an Abelian group satisfies

$$\text{closure, } \forall a, b \in G, a + b \in G, \quad (16)$$

$$\text{associativity, } \forall a, b, c \in G, (a + b) + c = a + (b + c), \quad (17)$$

$$\text{identity, } \exists 0 \in G, \forall a \in G, a + 0 = 0 + a = a, \quad (18)$$

$$\text{inverse, } \forall a \in G, \exists -a \in G, a + (-a) = (-a) + a = 0, \quad (19)$$

$$\text{and commutativity, } \forall u, v \in G, u + v = v + u. \quad (20)$$

■

Matrices of the same dimensions form an Abelian group under addition.

Consider a non-empty set H which is a subset of an Abelian group $(G, +)$. In order for H to be a subgroup of G , H must satisfy the following three conditions: H should be closed under the operation of the group ($\forall a, b \in H, a + b \in H$), H must contain the identity element of G ($0 \in H$) and H must be closed under inverse elements ($\forall a \in H, -a \in H$).

We set $\mathbb{F}$ as a number field ($\mathbb{R}/\mathbb{C}$). Then order- n general linear group $GL_n(\mathbb{F}) = \{A \in M_{n \times n}(\mathbb{F}) \mid \det(A) \neq 0\}$. We show that this is indeed a group.

(i) If $A, B \in GL_n$, then $\det(AB) = \det A \cdot \det B \neq 0$, so $AB \in GL_n$.

(ii) Matrix multiplication naturally satisfies $(AB)C = A(BC)$.

(iii) $AI_n = I_n A = A$.

(iv) For any $A \in GL_n$, A^{-1} is also reversible, $A^{-1} \in GL_n$.

D. Ring

Definition 9 A triple $(R, +, \cdot)$ is a ring if $(R, +)$ is an Abelian additive group. Multiplication $(\cdot)$ satisfies multiplicative closure and associativity.

$$\forall a, b \in R, ab \in R, \quad (21)$$

$$(ab)c = a(bc). \quad (22)$$

There is also distributive laws for a ring, which are left distributive and right distributive.

$$a(b + c) = ab + ac, \quad (23)$$

$$(b + c)a = ba + ca. \quad (24)$$

■

For a matrix ring M_n , it is a non-Abelian ring and it has multiplicative identity.

$$I_n \cdot A = A \cdot I_n = A, \quad \forall A \in M_n. \quad (25)$$

E. Linear space (vector space)

Definition 10 Let $\mathbb{F}$ be a field, can be the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A non-empty set V , along with two binary operations: vector addition and scalar multiplication, is said to be a linear space over $\mathbb{F}$ if it satisfies the following axioms:

axioms for vector addition,

$$\text{closure, } \forall u, v \in V, u + v \in V, \quad (26)$$

$$\text{associativity, } \forall u, v, w \in V, (u + v) + w = u + (v + w), \quad (27)$$

$$\text{identity, } \exists \mathbf{0} \in V, \forall u \in V, u + \mathbf{0} = \mathbf{0} + u = u, \quad (28)$$

$$\text{inverse, } \forall u \in V, \exists -u \in V, u + (-u) = (-u) + u = \mathbf{0}, \quad (29)$$

$$\text{commutativity, } \forall u, v \in V, u + v = v + u, \quad (30)$$

axioms for scalar multiplication,

$$\text{closure, } \forall k \in \mathbb{F}, u \in V, ku \in V, \quad (31)$$

$$\text{associativity, } \forall k, l \in \mathbb{F}, u \in V, k(lu) = (kl)u, \quad (32)$$

$$\text{identity, } \forall u \in V, 1 \cdot u = u, \quad (33)$$

$$\text{distributivity over vector addition, } \forall k \in \mathbb{F}, u, v \in V, k(u + v) = ku + kv, \quad (34)$$

$$\text{and distributivity over scalar addition, } \forall k, l \in \mathbb{F}, u \in V, (k + l)u = ku + lu. \quad (35)$$

■

Linearity contains additivity and homogeneity. For $F : X \rightarrow Y$, additivity is

$$f(x + y) = f(x) + f(y), \quad \forall x, y \in X, \quad (36)$$

and homogeneity is

$$k \cdot f(x) = f(kx), \quad \forall x \in X, k \in \mathbb{F}. \quad (37)$$

III. RESULT

Suppose W is an $m \times n$ entanglement witness (EW). Therefore, W is an Hermitian matrix in $M_m \otimes M_n \cong M_{mn}$ with $m, n > 1$, and there is at least one vector $x \in \mathbb{C}^{mn}$ such that $x^* W x < 0$. Further for each product vector $a \otimes b \in \mathbb{C}^m \otimes \mathbb{C}^n$ ($a \in \mathbb{C}^m, b \in \mathbb{C}^n$), we have $(a \otimes b)^* W (a \otimes b) \geq 0$.

A. properties for EW

Lemma 2 If W_2 is a 2×2 EW and $P, Q \in GL_2$ are invertible, then

$$W_3 := (P \otimes Q)^* W_2 (P \otimes Q)$$

is a 2×2 EW.

Proof 1

$$\begin{aligned} W_3^* &= [(P \otimes Q)^* W_2 (P \otimes Q)]^* \\ &= (P \otimes Q)^* \cdot W_2^* \cdot ((P \otimes Q)^*)^* \\ &= (P \otimes Q)^* W_2 (P \otimes Q) \\ &= W_3 \end{aligned}$$

W_3 is a hermitian matrix

$$\begin{aligned} (a \otimes b)^* W_3 (a \otimes b) &= (a \otimes b)^* [(P \otimes Q)^* W_2 (P \otimes Q)] (a \otimes b) \\ &= (Pa \otimes Qb)^* W_2 (Pa \otimes Qb) \geq 0, \quad \forall a, b \end{aligned}$$

$W_3 \not\geq 0$. Therefore, W_3 has at least one negative eigenvalue. □

Lemma 3 If $P \in GL_m$ and $Q \in GL_n$, then $P \otimes Q \in GL_{mn} \subset M_{mn}$.

Proof 2

$$\begin{aligned} P \otimes Q &= \left(U \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} \cdot U^* \right) \otimes \left(V \cdot \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \cdot V^* \right) \\ &= (U \otimes V) \left[\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} \otimes \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \right] (U^* \otimes V^*). \end{aligned}$$

Let R denote the matrix in the brackets. Then

$$\det \left[(U \otimes V) \cdot R \cdot (U \otimes V)^* \right] = \det(U \otimes V) \cdot \det R \cdot \det((U \otimes V)^*) = \det R \neq 0,$$

hence $P \otimes Q \in \text{GL}_{mn}$. □

Lemma 4 Sylvester's law of inertia If P and P^{-1} are in M_n (i.e. $P \in \text{GL}_n$), then

$$\nu(PHP^*) = \nu(H).$$

Here $\nu(\cdot)$ denotes the inertia index.

Lemma 5 Let $A, B \in M_n$. Then A and B are congruent if and only if there exists $P \in \text{GL}_n$ such that $A = PBP^*$.
Notation: $A \simeq B \implies \nu(A) = \nu(B)$.

Lemma 6 If, in addition, $A = A^*$ and $B = B^*$, then

$$\nu(A) = \nu(B) \implies A \simeq B.$$

Lemma 7 If W is a 2×2 EW, then

$$(i) \nu(W) = (1, 0, 3);$$

(ii) W has exactly one negative eigenvalue.

Proof 3 We prove the statement in two parts: at least one negative eigenvalue, and at most one negative eigenvalue.

Part 1: At least one negative eigenvalue.

Let W be a 2×2 entanglement witness. By definition, W is Hermitian, and there exists some vector $x \in \mathbb{C}^2 \otimes \mathbb{C}^2$ such that $x^* W x < 0$.

Since a Hermitian matrix is positive semi-definite if and only if all its eigenvalues are 0 or positive, which is equivalent to $x^* W x \geq 0$ for all vectors x .

Because there exists x with $x^* W x < 0$, W is not positive semi-definite. Therefore, W cannot have all eigenvalues non-negative, i.e., W has at least one negative eigenvalue.

Part 2: At most one negative eigenvalue.

We prove this by contradiction. Assume that W has at least two negative eigenvalues.

Since W is Hermitian, it is unitarily diagonalizable:

$$W = U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix} U^* = \sum_{j=1}^4 \lambda_j u_j u_j^*$$

where $U = (u_1, u_2, u_3, u_4)$ is unitary, and $\{u_j\}$ are orthonormal eigenvectors with real eigenvalues λ_j .

Sort the eigenvalues in decreasing order. By our assumption that there are at least two negative eigenvalues, we have:

$$\lambda_1 \geq \lambda_2 \geq 0 > \lambda_3 \geq \lambda_4.$$

Let $V = \text{span}\{u_3, u_4\} \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ be the subspace spanned by the eigenvectors corresponding to negative eigenvalues. Then $\dim V = 2$.

Claim: V contains a non-zero product vector $a \otimes b$.

Proof of claim: There is a natural isomorphism between $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $M_2(\mathbb{C})$ (the space of 2×2 complex matrices), given by:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \longleftrightarrow \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}.$$

Under this correspondence, a non-zero vector is a product vector $a \otimes b$ if and only if the corresponding matrix has rank 1, which is equivalent to its determinant being zero:

$$\det \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} = 0 \iff s_1 s_4 = s_2 s_3.$$

Let the two basis vectors u_3, u_4 correspond to matrices $A, B \in M_2(\mathbb{C})$. Any vector in V is a linear combination $xu_3 + yu_4$, which corresponds to the matrix $xA + yB$.

Consider the equation:

$$\det(xA + yB) = 0.$$

This is a homogeneous quadratic equation in x and y . Over the complex numbers, a homogeneous quadratic equation always has a non-trivial solution $(x_0, y_0) \neq (0, 0)$.

For this solution, the matrix $x_0 A + y_0 B$ has determinant zero, hence $\text{rank} \leq 1$. Since x_0, y_0 are not both zero, the matrix is non-zero, so its rank is exactly 1. Therefore, the corresponding vector $x_0 u_3 + y_0 u_4$ is a non-zero product vector $a \otimes b \in V$.

This proves the claim.

Now we derive the contradiction. Since $a \otimes b \in V = \text{span}\{u_3, u_4\}$, we can write:

$$a \otimes b = xu_3 + yu_4$$

for some scalars $x, y \in \mathbb{C}$, not both zero.

Compute the quadratic form:

$$(a \otimes b)^* W (a \otimes b) = (xu_3 + yu_4)^* W (xu_3 + yu_4).$$

Since u_3, u_4 are orthonormal eigenvectors with eigenvalues $\lambda_3, \lambda_4 < 0$, the cross terms vanish and we get:

$$(a \otimes b)^* W (a \otimes b) = |x|^2 \lambda_3 \|u_3\|^2 + |y|^2 \lambda_4 \|u_4\|^2 = |x|^2 \lambda_3 + |y|^2 \lambda_4.$$

Since $\lambda_3 < 0$, $\lambda_4 < 0$, and x, y are not both zero, we have:

$$(a \otimes b)^* W (a \otimes b) < 0.$$

On the other hand, since W is an entanglement witness, by definition the quadratic form is non-negative on all product states:

$$(a \otimes b)^* W (a \otimes b) \geq 0.$$

This is a contradiction. Therefore, our assumption that W has at least two negative eigenvalues is false. Hence, W has at most one negative eigenvalue.

Looking at both Part 1 and Part 2, we find that every 2×2 entanglement witness has exactly one negative eigenvalue. ■

B. Proof of $2 \times n$ EW matrix have at most $n - 1$ negative eigenvalue

Suppose the Hermitian matrix $H \in \mathbb{C}^{2n \times 2n}$ has k negative eigenvalues $\lambda_1, \dots, \lambda_k < 0$, with corresponding orthonormal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_k$. Define

$$U = [\mathbf{u}_1 \cdots \mathbf{u}_k],$$

which satisfies

$$U^* U = I_k, \quad U^* H U = \text{diag}(\lambda_1, \dots, \lambda_k) \prec 0.$$

We proceed by contradiction. Assume $k \geq N$. Take the first N eigenvectors and construct

$$V = [\mathbf{u}_1 \cdots \mathbf{u}_N] \in \mathbb{C}^{2N \times N}.$$

It follows that

$$V^*V = I_N, \quad V^*HV = \Lambda_N \prec 0,$$

where $\Lambda_N = \text{diag}(\lambda_1, \dots, \lambda_N)$. Partition V as

$$V = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad X, Y \in \mathbb{C}^{N \times N}.$$

We suggest that a nonzero vector $\mathbf{a} \in \mathbb{C}^N$ exists such that

$$V\mathbf{a} = \begin{pmatrix} X\mathbf{a} \\ Y\mathbf{a} \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ \mu\mathbf{y} \end{pmatrix}$$

Case 1: X is singular. Since $\det X = 0$, the homogeneous equation $X\mathbf{a} = \mathbf{0}$ has a nontrivial solution. Choose a nonzero vector $\mathbf{a}$ such that $X\mathbf{a} = \mathbf{0}$. Then

$$V\mathbf{a} = \begin{pmatrix} \mathbf{0} \\ Y\mathbf{a} \end{pmatrix}.$$

As V has full column rank, $V\mathbf{a} \neq \mathbf{0}$, which implies $Y\mathbf{a} \neq \mathbf{0}$. Consequently, $X\mathbf{a} = \mathbf{0} \cdot Y\mathbf{a}$.

Case 2: Y is singular. By symmetry, there exists a nonzero vector $\mathbf{a}$ satisfying $Y\mathbf{a} = \mathbf{0}$ and $X\mathbf{a} \neq \mathbf{0}$, so $Y\mathbf{a} = \mathbf{0} \cdot X\mathbf{a}$.

Case 3: Both X and Y are invertible. Set $T = YX^{-1} \in \mathbb{C}^{N \times N}$. Then T is invertible. Every complex square matrix possesses eigenvalues, so there exist a nonzero vector $\mathbf{w} \in \mathbb{C}^N$ and a nonzero complex scalar μ such that $T\mathbf{w} = \mu\mathbf{w}$. Let $\mathbf{a} = X^{-1}\mathbf{w}$; clearly $\mathbf{a} \neq \mathbf{0}$. Substitution yields

$$X\mathbf{a} = \mathbf{w}, \quad Y\mathbf{a} = T\mathbf{w} = \mu\mathbf{w}.$$

Hence $Y\mathbf{a} = \mu X\mathbf{a}$.

for some nonzero $\mathbf{y} \in \mathbb{C}^N$ and $\mu \in \mathbb{C}$. Set $\mathbf{z} = V\mathbf{a}$. On one hand,

$$\mathbf{z}^*H\mathbf{z} = \mathbf{a}^*V^*HV\mathbf{a} = \mathbf{a}^*\Lambda_N\mathbf{a} < 0.$$

Moreover, $\mathbf{z}$ is a separable vector of the form $\begin{pmatrix} \mathbf{y} \\ \mu\mathbf{y} \end{pmatrix}$. According to the definition of entanglement witness, $\mathbf{z}^*H\mathbf{z} \geq 0$.

This yields a contradiction.

Therefore, the assumption $k \geq N$ fails, and we conclude $k \leq n - 1$.

C. Proof of $2 \times N$ space without product vector

Let H be a $2N \times 2N$ Hermitian entanglement witness matrix. Then the number of linearly independent eigenvectors associated with the negative eigenvalues of H is at most $N-1$. Assume $\dim S \geq N$, i.e. the total dimension of the negative eigenspace is at least N . From this negative eigenspace S , pick N linearly independent eigenvectors and form the matrix

$$V = [\mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_N] \in \mathbb{C}^{2N \times N}.$$

Since H is Hermitian, eigenvectors belonging to distinct eigenvalues are orthogonal, and for a repeated eigenvalue we may choose an orthonormal basis. Thus we can take V such that

$$V^*V = I_N, \quad V^*HV \prec 0.$$

Partition V into upper and lower $N \times N$ blocks:

$$V = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad X, Y \in \mathbb{C}^{N \times N}.$$

We look for a nonzero vector $\mathbf{a} \in \mathbb{C}^{N \times 1}$ such that

$$V\mathbf{a} = \begin{pmatrix} X\mathbf{a} \\ Y\mathbf{a} \end{pmatrix}$$

is a product vector. This means that there exists a scalar $\mu \in \mathbb{C}$ for which the two blocks are proportional; more precisely, a product vector has the form $(\alpha \mathbf{y}; \beta \mathbf{y})$, which covers the cases $X\mathbf{a} = \mathbf{0}, Y\mathbf{a} \neq \mathbf{0}$ (take $\alpha = 0, \beta = 1$) and $X\mathbf{a} \neq \mathbf{0}, Y\mathbf{a} = \mathbf{0}$ (take $\alpha = 1, \beta = 0$), as well as the generic proportional case $Y\mathbf{a} = \mu X\mathbf{a}$.

We distinguish three cases.

Case 1: X is singular. Then $\det X = 0$, so the homogeneous system $X\mathbf{a} = \mathbf{0}$ has a nonzero solution $\mathbf{a}$. Because the columns of V are linearly independent, $V\mathbf{a} \neq \mathbf{0}$; hence $Y\mathbf{a} \neq \mathbf{0}$. Thus $V\mathbf{a} = (0; Y\mathbf{a})$ is a nonzero product vector (with $\alpha = 0, \beta = 1$).

Case 2: Y is singular. Similarly, a nonzero $\mathbf{a}$ with $Y\mathbf{a} = \mathbf{0}$ exists. Again $V\mathbf{a} \neq \mathbf{0}$, so $X\mathbf{a} \neq \mathbf{0}$. Now $V\mathbf{a} = (X\mathbf{a}; 0)$ is a nonzero product vector (with $\alpha = 1, \beta = 0$).

Case 3: X and Y are both invertible. Let μ be one of the eigenvalue of the matrix YX^{-1} and $\mathbf{b} \neq \mathbf{0}$ a corresponding eigenvector:

$$YX^{-1}\mathbf{b} = \mu\mathbf{b}.$$

Set $\mathbf{a} = X^{-1}\mathbf{b} \neq \mathbf{0}$. Then

$$X\mathbf{a} = \mathbf{b}, \quad Y\mathbf{a} = YX^{-1}\mathbf{b} = \mu\mathbf{b},$$

so $Y\mathbf{a} = \mu X\mathbf{a}$. Hence $V\mathbf{a} = (\mathbf{b}; \mu\mathbf{b})$ is a nonzero product vector.

In every case we obtain a nonzero product vector $w = V\mathbf{a}$. For this vector,

$$w^*Hw = \mathbf{a}^*(V^*HV)\mathbf{a} < 0,$$

because V^*HV is negative definite. On the other hand, an entanglement witness matrix H satisfies $w^*Hw \geq 0$ for every product vector w . The assumption $\dim S \geq N$ is false is shown from this contradiction. Therefore

$$\dim S \leq N - 1,$$

i.e. the number of linearly independent eigenvectors that are associated with negative eigenvalues is at most $N - 1$.

D. Construction of entanglement witnesses on $\mathbb{C}^2 \otimes \mathbb{C}^n$ with k negative eigenvalues, $1 \leq k \leq n - 1$

We construct $\psi_j = e_1 \otimes f_j + c_{j+1} e_2 \otimes f_{j+1} \in \mathbb{C}^2 \otimes \mathbb{C}^n$, and $0 < c_1 < \dots < c_{n-2} < c_{n-1}$. $\mathcal{I}_2 = \{e_1, e_2\}$, $\mathcal{I}_n = \{f_1, f_2, \dots, f_n\}$.

$$P = \sum_{j=1}^{n-1} \psi_j \psi_j^*, \quad W = P^\Gamma, \quad (T_A \otimes I_B) \circ P := P^\Gamma.$$

We expand on P .

$$\psi_j \psi_j^* = (e_1 \otimes f_j + c_{j+1} e_2 \otimes f_{j+1})(e_1^\top \otimes f_j^* + c_{j+1} e_2^\top \otimes f_{j+1}^*) \quad (38)$$

$$= (e_1 \otimes f_j)(e_1^\top \otimes f_j^*) + (e_1 \otimes f_j)(c_{j+1} e_2^\top \otimes f_{j+1}^*) + (c_{j+1} e_2 \otimes f_{j+1})(e_1^\top \otimes f_j^*) + (c_{j+1} e_2 \otimes f_{j+1})(c_{j+1} e_2^\top \otimes f_{j+1}^*) \quad (39)$$

$$= (e_1 e_1^\top) \otimes (f_j f_j^*) + c_{j+1} (e_1 e_2^\top) \otimes (f_j f_{j+1}^*) + c_{j+1} (e_2 e_1^\top) \otimes (f_{j+1} f_j^*) + c_{j+1}^2 (e_2 e_2^\top) \otimes (f_{j+1} f_{j+1}^*). \quad (40)$$

And we know $(\psi_j \psi_j^*)^\Gamma = (T_A \otimes I_B) \circ (\psi_j \psi_j^*)$.

$$W = (e_1 e_1^\top) \otimes (f_j f_j^*) + c_{j+1} (e_1 e_2^\top) \otimes (f_j f_{j+1}^*) + c_{j+1} (e_2 e_1^\top) \otimes (f_{j+1} f_j^*) + c_{j+1}^2 (e_2 e_2^\top) \otimes (f_{j+1} f_{j+1}^*).$$

W must be an EW and have $n - 1$ negative eigenvalues. We prove it via the following (i), (ii) and (iii).

(i) The first thing is prove that W is Hermitian.

Recall that $\psi_j \in \mathbb{R}^{m \times n}$. $P^* = \psi_j \psi_j^* = P$. So P is Hermitian.

$$W^* = (P^\Gamma)^* = (P^*)^\Gamma = P^\Gamma = W.$$

So W is also Hermitian.

(ii) We need to prove $(a \otimes b)^* W (a \otimes b) \geq 0$. (product vector positive semi-definite).

We introduce a lemma.

Lemma 8 $\text{Tr}(AB) = \text{Tr}(BA)$.

For $\text{Tr}(AB)$, $(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$.

$$\text{Tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{ki}.$$

For $\text{Tr}(BA)$, $(BA)_{kk} = \sum_{i=1}^n B_{ki} A_{ik}$.

$$\text{Tr}(BA) = \sum_{k=1}^n (BA)_{kk} = \sum_{k=1}^n \sum_{i=1}^n B_{ki} A_{ik}.$$

And we find that $\sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{ki} = \sum_{k=1}^n \sum_{i=1}^n B_{ki} A_{ik}$. So $\text{Tr}(AB) = \text{Tr}(BA)$. ■

Now we return to W . Let $(a \otimes b)^* = A$, $W(a \otimes b) = B$.

$$\text{Tr}(AB) = \text{Tr}[(a \otimes b)^* W(a \otimes b)] = (a \otimes b)^* \cdot W \cdot (a \otimes b) \quad (41)$$

$$= \text{Tr}(BA) = \text{Tr}[W(a \otimes b)(a \otimes b)^*]. \quad (42)$$

Using $W = P^\Gamma$, we have

$$\begin{aligned} \text{Tr}[W \cdot (a \otimes b)(a \otimes b)^*] &= \text{Tr}[P^\Gamma \cdot (a \otimes b)(a \otimes b)^*] \\ &= \text{Tr}\left\{[(T_A \otimes I_B) \circ P] \cdot (a \otimes b)(a \otimes b)^*\right\} \\ &= \text{Tr}\left\{\left[\sum_{j,k} (e_j e_k^\top \otimes I) \cdot P \cdot (e_j e_k^\top \otimes I)\right] \cdot (a \otimes b)(a \otimes b)^*\right\} \\ &= \text{Tr}\left\{P \cdot \sum_{j,k} (e_j e_k^\top \otimes I)(a \otimes b)(a \otimes b)^* (e_j e_k^\top \otimes I)\right\} \\ &= \text{Tr}\{P \cdot (\bar{a} a^\top) \otimes b b^*\} \\ &= \text{Tr}\{P[\bar{a} \otimes b] (a^\top \otimes b^*)\}. \\ &= (a^\top \otimes b^*) \cdot P \cdot (\bar{a} \otimes b), \quad \text{let } v = \bar{a} \otimes b. \\ &= v^* P v \geq 0. \end{aligned}$$

(iii) We prove that W has exactly $n-1$ negative eigenvalues.

Take a set of bases for the full space of $\mathbb{C}^2 \otimes \mathbb{C}^n$,

$$\mathcal{B} = \{e_1 \otimes f_1, e_2 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_2, e_1 \otimes f_3, \dots, e_2 \otimes f_{n-1}, e_1 \otimes f_n, e_2 \otimes f_n\}.$$

The entries of W have four categories by means of (38).

1. $(e_1 \otimes f_j)(e_1^* \otimes f_j^*)$
2. $c_{j+1}(e_1 \otimes f_j)(e_2^* \otimes f_{j+1}^*)$
3. $c_{j+1}(e_1^* \otimes f_j^*)(e_2 \otimes f_{j+1})$
4. $c_{j+1}^2(e_2 \otimes f_{j+1})(e_2^* \otimes f_{j+1}^*)$

If we expand W , we can find

$$W = \begin{pmatrix} 1 & & & & 0 \\ & B_1 & & & \\ & & B_2 & & \\ & & & \ddots & \\ & & & & B_{n-1} \\ 0 & & & & & c_n^2 \end{pmatrix}_{2n \times 2n}, \quad B_j \in M_{2 \times 2}.$$

For B_1 , basis $(e_2 \otimes f_1, e_1 \otimes f_2)$

$$B_1 = \begin{pmatrix} 0 & c_2 \\ c_2 & 1 \end{pmatrix}, \quad \det B_1 = 0 \cdot 1 - c_2^2 = -c_2^2 < 0.$$

The determinant of Hermitian 2×2 matrix is negative, which means one positive eigenvalue and one negative eigenvalue exists.

For B_j , $2 \leq j \leq n-2$, basis $(e_2 \otimes f_j, e_1 \otimes f_{j+1})$.

$$B_j = \begin{pmatrix} c_j^2 & c_{j+1} \\ c_{j+1} & 1 \end{pmatrix}, \quad \det B_j = c_j^2 \cdot 1 - c_{j+1}^2 = c_j^2 - c_{j+1}^2 < 0.$$

We know $0 < c_1 < c_2 < \dots < c_{n-1}$, so $c_{j+1}^2 > c_j^2$.

$\det B_j = c_j^2 - c_{j+1}^2 < 0$. As same as B_1 . The eigenvalues of B_j are one positive and one negative.

For B_{n-1} , basis $(e_2 \otimes f_{n-1}, e_1 \otimes f_n)$.

$$B_{n-1} = \begin{pmatrix} c_{n-1}^2 & c_n \\ c_n & 0 \end{pmatrix}, \quad \det B_{n-1} = c_{n-1}^2 \cdot 0 - c_n^2 = -c_n^2 < 0.$$

So the eigenvalues of B_{n-1} have one positive and one negative.

There are $n-1$ B_j matrices and each matrix have one negative eigenvalue. And other n eigenvalues are all non-negative. So W have $n-1$ negative eigenvalues.

So W is entanglement witness and has $n-1$ negative eigenvalues.

In the following, we construct an $2 \times n$ EW of exactly $k \in [1, n-1]$ negative eigenvalues.

For a $2 \times n$ EW $W = \sum_{j=1}^{2n} \lambda_j x_j x_j^*$, we construct a $2 \times n$ EW

$$W_1 = W + \sum_{i=2}^k |\lambda_{n+i}| x_{n+i} x_{n+i}^*, \quad (k \in \{1, \dots, n\})$$

which can have $[1, n-1]$ negative eigenvalues.

First we need to prove that W_1 is still entanglement witness.

1. W is Hermitian matrix, $|\lambda| x x^*$ is still Hermitian matrix, and the addition of two Hermitian matrices still results in a Hermitian matrix.
2. W_1 have negative eigenvalue so W_1 is not a positive semi-definite matrix.
- 3.

$$(a \otimes b)^* W_1 (a \otimes b) = (a \otimes b)^* W (a \otimes b) + |\lambda_i| (a \otimes b)^* x x^* (a \otimes b).$$

Because W is EW so $(a \otimes b)^* W (a \otimes b) \geq 0$.

$$|\lambda_i| \cdot (a \otimes b)^* x x^* (a \otimes b) = |\lambda_i| \cdot |x^* (a \otimes b)|^2 \geq 0.$$

So $(a \otimes b)^* W_1 (a \otimes b) \geq 0$.

And then we need to prove that W_1 have $[1, n-1]$ negative eigenvalues.

When $k=2$, $W_1 = W + |\lambda_{n+2}| x_{n+2} x_{n+2}^*$.

We have $|\lambda_{n+2}| = -\lambda_{n+2}$, $x_{n+2}^* x_{n+2} = 1$.

$$\begin{aligned} W_1 x_{n+2} &= W x_{n+2} + |\lambda_{n+2}| x_{n+2} x_{n+2}^* x_{n+2} \\ &= \lambda_{n+2} x_{n+2} + |\lambda_{n+2}| x_{n+2} \cdot 1 \\ &= (\lambda_{n+2} - \lambda_{n+2}) x_{n+2} = 0. \end{aligned}$$

So original negative eigenvalue λ has now become zero.

For other eigenvectors y of W .

$$W_1 y = W y + |\lambda_{n+2}| x_{n+2} x_{n+2}^* y = W y + 0 = \mu y.$$

So among the original $n-1$ negative eigenvalues, one has been “raised” to 0, while the remaining $n-2$ negative eigenvalues remain unchanged.

As k increase, it is easy to see that more and more negative eigenvalues become 0, but the remaining negative eigenvalues remain unchanged.

So W_1 have $[1, n-1]$ negative eigenvalues.

E. An example of a $2 \times N$ subspace of dimension $N - 1$ containing no product vector

In this subsection, we construct a $2N \times 2N$ diagonal unitary matrix:

Let $\mathbb{C}^2 \otimes \mathbb{C}^N$ be the bipartite complex tensor product space of total dimension $2N$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote the standard basis of $\mathbb{C}^2$, and let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ denote the standard basis of $\mathbb{C}^N$. Define the linearly independent vectors

$$\mathbf{v}_k = \mathbf{e}_1 \otimes \mathbf{f}_k + \mathbf{e}_2 \otimes \mathbf{f}_{k+1}, \quad k = 1, 2, \dots, N-1, \quad (43)$$

and let $\mathcal{S} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{N-1}\}$. Then

1. $\dim \mathcal{S} = N - 1$;
2. $\mathcal{S}$ contains no nonzero product vectors. That is, there exists no nonzero $\mathbf{x} \in \mathcal{S}$ that can be written as $\mathbf{u} \otimes \mathbf{v}$ for some $\mathbf{u} \in \mathbb{C}^2$ and $\mathbf{v} \in \mathbb{C}^N$.

Step 1. Dimension. For each index $k \in \{1, 2, \dots, N-1\}$, the vector $\mathbf{v}_k$ has a unique nonzero entry at the position corresponding to $\mathbf{e}_1 \otimes \mathbf{f}_k$, which does not exist in any other $\mathbf{v}_j$ with $j \neq k$. Hence the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{N-1}\}$ is linearly independent, and therefore $\dim \mathcal{S} = N - 1$.

Step 2. No nonzero product vectors. We proceed by contradiction. Assume there exists a nonzero vector $\mathbf{x} \in \mathcal{S}$ such that $\mathbf{x} = \mathbf{u} \otimes \mathbf{v}$ for some $\mathbf{u} \in \mathbb{C}^2$ and $\mathbf{v} \in \mathbb{C}^N$. Write

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

By definition of the tensor product,

$$\mathbf{u} \otimes \mathbf{v} = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ \vdots \\ u_1 v_n \\ u_2 v_1 \\ u_2 v_2 \\ \vdots \\ u_2 v_n \end{pmatrix}.$$

Let $\mathbf{x}_1 \in \mathbb{C}^N$ be the first N components of $\mathbf{x}$, and $\mathbf{x}_2 \in \mathbb{C}^N$ be the last N components. The product structure implies

$$\mathbf{x}_2 = \lambda \mathbf{x}_1, \quad \lambda = \frac{u_2}{u_1},$$

i.e. the two blocks are linearly proportional.

On the other hand, since $\mathbf{x} \in \mathcal{S}$, there exist coefficients $c_1, c_2, \dots, c_{n-1} \in \mathbb{C}$ such that

$$\mathbf{x} = \sum_{k=1}^{n-1} c_k \mathbf{v}_k.$$

Expanding component-wise gives

$$\mathbf{x}_1 = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix}.$$

Substituting into the proportionality condition $\mathbf{x}_2 = \lambda \mathbf{x}_1$ yields the system

$$\begin{cases} 0 = \lambda c_1, \\ c_1 = \lambda c_2, \\ c_2 = \lambda c_3, \\ \vdots \\ c_{n-1} = \lambda \cdot 0. \end{cases}$$

From the last equation we obtain $c_{n-1} = 0$. Substituting backwards, we inductively get $c_{n-2} = 0, \dots, c_1 = 0$. Thus $\mathbf{x} = \mathbf{0}$, contradicting the assumption that $\mathbf{x}$ is nonzero.

Therefore, $\mathcal{S}$ contains no nonzero product vectors.

F. proof of 2×2 EW have a (1,0,3) inertia index

Let W be a 2×2 entanglement witness. Then $\text{In}(W) = (1, 0, 3)$.

Let W be a 2×2 EW of inertia

$$\text{In}(W) = (1, a, 3 - a), \quad a \in [0, 3].$$

(i) $a = 3$

$$W = \sum_{j=1}^4 \lambda_j x_j x_j^*, \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 > \lambda_4.$$

Thus $W = \lambda_4 x_4 x_4^*$. Since $x_4 \neq 0$, not every product vector is orthogonal to x_4 (otherwise vectorization forces $x_4 = 0$). Hence $\exists a \otimes b := c$ such that

$$c^* W c = c^* (\lambda_4 x_4 x_4^*) c = \lambda_4 \cdot (c^* x_4) \cdot (x_4^* c) = \lambda_4 \cdot |c^* x_4|^2.$$

Since $\lambda_4 < 0$ and $|c^* x_4|^2 \geq 0$, we have $c^* W c < 0$, a contradiction.

(ii) $a = 2$

$$W = \sum_{j=1}^4 \lambda_j x_j x_j^*, \quad \lambda_4 < 0 < \lambda_1.$$

Lemma 9 Let $V \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ have dimension three. Then V is spanned by product vectors.

Since $\dim V = 3$, we have $\dim V^\perp = 4 - 3 = 1$, so $V^\perp = \text{span}\{\psi\}$.

Case 1: ψ is a product vector. By a local unitary transformation, assume $\psi = e_1 \otimes e_1$. Take the standard product basis

$$e_1 \otimes e_1, \quad e_1 \otimes e_2, \quad e_2 \otimes e_1, \quad e_2 \otimes e_2.$$

Among them, $e_1 \otimes e_2$, $e_2 \otimes e_1$, and $e_2 \otimes e_2$ are all orthogonal to $\psi = e_1 \otimes e_1$, and they are linearly independent. So these three product vectors all lie in V , and since $\dim V = 3$, they span V .

Case 2: ψ is an entangled vector. Then ψ can be written as

$$\psi = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2.$$

A product vector $u \otimes v$, with $u = (x, y)^T$ and $v = (s, t)^T$, is orthogonal to ψ iff

$$\lambda_1 x s + \lambda_2 y t = 0.$$

We find three groups $(x, y; s, t)$ satisfying this equation:

1. $x = 1, y = 0; s = 0, t = 1 \Rightarrow e_1 \otimes e_2$, since $\lambda_1 \cdot 1 \cdot 0 + \lambda_2 \cdot 0 \cdot 1 = 0$;
2. $x = 0, y = 1; s = 1, t = 0 \Rightarrow e_2 \otimes e_1$, since $\lambda_1 \cdot 0 \cdot 1 + \lambda_2 \cdot 1 \cdot 0 = 0$;
3. $x = 1, y = 1; s = 1, t = -\lambda_1/\lambda_2$, since $\lambda_1 \cdot 1 \cdot 1 + \lambda_2 \cdot 1 \cdot (-\lambda_1/\lambda_2) = 0$.

All three product vectors $e_1 \otimes e_2$, $e_2 \otimes e_1$, and $(1, 1)^T \otimes (1, -\lambda_1/\lambda_2)^T$ are linearly independent. Since $\dim V = 3$, these three product vectors span V . ■

Now take a three-dimensional subspace $V = \text{span}\{x_4, x_3, x_2\}$. Then V is orthogonal to x_1 , and by the Lemma V is spanned by product vectors. Hence there exists a product vector

$$c_1 = \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4$$

with $\mu_4 \neq 0$ (otherwise all product vectors in V would lie in $\text{span}\{x_2, x_3\}$, which is only two-dimensional). Because $c_1 \perp x_1$,

$$c_1^* W c_1 = |\mu_4|^2 \lambda_4 < 0,$$

a contradiction.

(iii) $a = 1$

$$W = \lambda_4 x_4 x_4^* + \lambda_2 x_2 x_2^* + \lambda_1 x_1 x_1^*, \quad \lambda_4 < 0 < \lambda_2 \leq \lambda_1.$$

Take a two-dimensional space $U_1 = \text{span}\{x_4, x_3\}$. This space has only the negative eigen-direction and the zero eigen-direction. If we can find $c = a \otimes b$ in U_1 with $\mu_4 \neq 0$, then

$$c^* W c = |\mu_4|^2 \lambda_4 < 0.$$

So the problem transforms to: is there a product vector with x_4 -component in $U_1 = \text{span}\{x_4, x_3\}$?

Situation A: U_1 can be spanned by two product vectors. Then those two product vectors must have one with non-zero x_4 -component; otherwise they are linearly correlated and cannot span two dimensions. Hence we are done.

Situation B: U_1 cannot be spanned by two product vectors.

Lemma 10 Any two-dimensional subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ not spanned by product vectors is SLOCC-equivalent to

$$V_2 = \text{span}\{e_1 \otimes e_1 + e_2 \otimes e_2, e_1 \otimes e_2\}.$$

■

Under the vectorization isomorphism:

$$e_1 \otimes e_1 + e_2 \otimes e_2 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \otimes e_2 \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Take $v \in V_2$:

$$v = x(e_1 \otimes e_1 + e_2 \otimes e_2) + y(e_1 \otimes e_2).$$

Then

$$M_v = x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}.$$

When $\det(M_v) = 0$ and M_v is not the zero matrix, the rank of M_v is 1, i.e. v is a product vector. Now

$$\det \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = x \cdot x - y \cdot 0 = x^2 = 0 \implies x = 0.$$

So $M_v = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = y \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, i.e. $y \cdot (e_1 \otimes e_2)$. Thus V_2 cannot be spanned by product vectors.

In V_2 , the only product vector direction is $e_1 \otimes e_2$.

Under the V_2 type, through a local unitary transformation, assume $x_3 = e_1 \otimes e_2$. Since $x_4 \perp x_3$, the matrix of x_4 is $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. A general element $px_4 + qx_3 \in U_1$ corresponds to $\begin{pmatrix} pa & q \\ pc & pd \end{pmatrix}$, with determinant $p(pad - qc)$. Since U_1 is of V_2 -type, this vanishes for only one direction ($p = 0$), so $c = 0$. Thus x_4 is diagonal; after rescaling,

$$x_3 = e_1 \otimes e_2, \quad x_4 = \lambda e_1 \otimes e_1 + e_2 \otimes e_2, \quad \lambda > 0.$$

The positive eigenvectors are orthogonal to x_3 and x_4 , so

$$x_j = p_j(e_1 \otimes e_1 - \lambda e_2 \otimes e_2) + r_j e_2 \otimes e_1, \quad j = 1, 2.$$

In components:

$$x_4 = (\lambda, 0, 0, 1), \quad x_3 = (0, 1, 0, 0), \quad x_j = (p_j, 0, r_j, -\lambda p_j).$$

Take $v = (1, p) \otimes (1, q)$, and assume $pq = \frac{1}{\lambda}$, so $q = \frac{1}{\lambda p}$. Then

$$v = \left(1, \frac{1}{\lambda p}, p, \frac{1}{\lambda}\right).$$

Negative eigen-direction x_4 :

$$v^* x_4 = 1 \cdot \lambda + \frac{1}{\lambda} \cdot 1 = \lambda + \frac{1}{\lambda}, \quad |v^* x_4|^2 = \left(\lambda + \frac{1}{\lambda}\right)^2 > 0.$$

Zero eigen-direction x_3 :

$$v^* x_3 = \frac{1}{\lambda p},$$

but $\lambda_3 = 0$, so there is no contribution.

Positive eigen-direction x_j :

$$v^* x_j = 1 \cdot p_j + p \cdot r_j + \frac{1}{\lambda} \cdot (-\lambda p_j) = p_j + r_j p - p_j = r_j p, \quad |v^* x_j|^2 = |r_j|^2 |p|^2.$$

Therefore

$$v^* W v = \lambda_4 |v^* x_4|^2 + \lambda_1 |v^* x_1|^2 + \lambda_2 |v^* x_2|^2 = \lambda_4 \left(\lambda + \frac{1}{\lambda}\right)^2 + (\lambda_1 |r_1|^2 + \lambda_2 |r_2|^2) |p|^2.$$

Since $\lambda_4(\lambda + \frac{1}{\lambda})^2 < 0$ is a fixed negative number and the second term is non-negative and tends to 0 as $|p| \rightarrow 0$, for sufficiently small $|p| > 0$ (note $p \neq 0$ since $q = 1/(\lambda p)$) we have $v^* W v < 0$.

Since v is a product vector, this contradicts the definition of an entanglement witness.

Both Situation A and Situation B lead to a contradiction, so $a = 1$ is impossible.

Conclusion:

The cases $a = 3, 2, 1$ are all impossible. Therefore $a = 0$, and

$$\text{In}(W) = (1, 0, 3).$$

IV. CONCLUSION

We showed that for a $2 \times n$ entanglement witness, it has at most $n - 1$ negative eigenvalues. We proceeded this proof by first establishing the 2×2 case via a geometric rank-one argument. We then extended to the general $2 \times n$ case through a dimensional analysis of the negative eigenspace. Moreover, the method we used for construction demonstrates that every integer between 1 and $n - 1$ can be realized as the number of negative eigenvalues of some $2 \times n$ entanglement witness. These results completely resolve the negative eigenvalue counting problem for the $2 \times n$ bipartite system. We also proved that the inertia index of 2×2 entanglement witness is $(1, 0, 3)$.

Our results suggest several directions for future investigation. A most immediate problem is the generalization to general $m \times n$ bipartite systems. In this setting, the question reduces to determining the maximal dimension of a subspace of $M_{m \times n}(\mathbb{C})$ containing no rank-one matrices. It remains to be seen whether this bound is always attainable by entanglement witnesses, and whether sharper bounds exist for $m \neq n$. Another challenging direction is the extension to multipartite systems $\mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_k}$. Here the notion of separability is leading to different types of entanglement witnesses and correspondingly different constraints on negative eigenvalue counts. The geometric problem now involves subspaces avoiding matrices of tensor rank one, or more generally, avoiding states that are separable across certain partitions. A complete characterization in the multipartite setting appears to be a difficult but worthwhile and meaningful open problem.

V. APPENDIX

Appendix A: Proof of Properties of Hermitian Matrices

Proof for real eigenvalues:

Let A be a $n \times n$ Hermitian matrix, $A = A^*$. Let λ be an eigenvalue of A , and let v be a corresponding eigenvector, $v \neq 0$.

$$Av = \lambda v$$

$$(Av)^* = (\lambda v)^*$$

$$v^* A^* = \bar{\lambda} v^*$$

Since $A = A^*$,

$$v^* A = \bar{\lambda} v^*$$

Now, multiplying v^* on the left of $Av = \lambda v$:

$$v^* Av = \lambda v^* v$$

Similarly, multiplying v on the right of $v^* A = \bar{\lambda} v^*$:

$$v^* Av = \bar{\lambda} v^* v$$

Therefore,

$$\lambda v^* v = \bar{\lambda} v^* v$$

$$(\lambda - \bar{\lambda}) v^* v = 0$$

Since $v \neq 0$,

$$\lambda = \bar{\lambda}$$

Thus λ is equal to its own complex conjugate, which means λ is real.

Proof for orthogonal eigenvectors:

Let A be a Hermitian matrix. Let $\lambda_1 \neq \lambda_2$ be two distinct eigenvalues of A , and let v_1, v_2 be corresponding eigenvectors, $v_1, v_2 \neq 0$.

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2$$

Taking conjugate transpose of the second equation:

$$v_2^* A^* = \bar{\lambda}_2 v_2^*$$

Since $A = A^*$ and λ_2 is real (from the previous result),

$$v_2^* A = \lambda_2 v_2^*$$

Now, multiplying v_2^* on the left of $Av_1 = \lambda_1 v_1$:

$$v_2^* Av_1 = \lambda_1 v_2^* v_1$$

Similarly, multiplying v_1 on the right of $v_2^*A = \lambda_2 v_2^*$:

$$v_2^*Av_1 = \lambda_2 v_2^*v_1$$

Therefore,

$$\lambda_1 v_2^*v_1 = \lambda_2 v_2^*v_1$$

$$(\lambda_1 - \lambda_2)v_2^*v_1 = 0$$

Since $\lambda_1 \neq \lambda_2$,

$$v_2^*v_1 = 0$$

Thus v_1 and v_2 are orthogonal.

Proof for unitary diagonalization:

Let A be an $n \times n$ Hermitian matrix. We prove by induction on n that there exists a unitary matrix U and a real diagonal matrix D such that $A = UDU^*$. For $n = 1$, trivial. Assume true for $n - 1$. Let $\lambda_1 \in \mathbb{R}$ be an eigenvalue of A , and let v_1 be a corresponding eigenvector with $\|v_1\| = 1$. Extend v_1 to an orthonormal basis $\{v_1, w_2, \dots, w_n\}$ of $\mathbb{C}^n$. Let

$$U_1 = [v_1 \ w_2 \ \cdots \ w_n]$$

Then U_1 is unitary. Since $Av_1 = \lambda_1 v_1$ and $A = A^*$, we have

$$U_1^*AU_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix}$$

where B is an $(n - 1) \times (n - 1)$ Hermitian matrix. By induction hypothesis, there exist unitary V and real diagonal D' such that $B = VD'V^*$. Define

$$U = U_1 \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & D' \end{pmatrix}$$

Then U is unitary, D is real diagonal, and

$$A = UDU^*$$

Thus every Hermitian matrix is unitarily diagonalizable.

Appendix B: Proof of $U^*U = I_2 = UU^*$

Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$, $U^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$.

Since $UU^* = I_2$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & a\bar{c} + b\bar{d} \\ c\bar{a} + d\bar{b} & |c|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So,

$$\begin{cases} |a|^2 + |b|^2 = 1 \\ a\bar{c} + b\bar{d} = 0 \\ c\bar{a} + d\bar{b} = 0 \\ |c|^2 + |d|^2 = 1 \end{cases}$$

For U^*U ,

$$U^*U = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |c|^2 & \bar{a}b + \bar{c}d \\ \bar{b}a + \bar{d}c & |b|^2 + |d|^2 \end{pmatrix}.$$

In order to prove $U^*U = I_2$, $|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1$ and $\bar{a}b + \bar{c}d = \bar{b}a + \bar{d}c = 0$ should be shown.

$$\begin{aligned} a\bar{c} &= -b\bar{d} \\ |a\bar{c}|^2 &= |-b\bar{d}|^2 \\ |a|^2 \cdot |c|^2 &= |b|^2 \cdot |d|^2 \\ |a|^2|c|^2 &= (1 - |a|^2)(1 - |c|^2) \\ |a|^2|c|^2 &= 1 - |a|^2 - |c|^2 + |a|^2|c|^2 \\ |a|^2 + |c|^2 &= 1. \end{aligned}$$

Similarly, $|b|^2 + |d|^2 = 1$.

Let $S = \bar{a}b + \bar{c}d$.

$$\begin{aligned} |S|^2 &= (\bar{a}b + \bar{c}d)(a\bar{b} + c\bar{d}) \\ |S|^2 &= |a|^2|b|^2 + \bar{a}bcd\bar{c} + \bar{c}da\bar{b} + |c|^2|d|^2 \\ |S|^2 &= |a|^2|b|^2 + c\bar{d}\bar{a} + ab\bar{c}\bar{d} + |c|^2|d|^2 \\ |S|^2 &= |a|^2|b|^2 + |c|^2|d|^2 + (\bar{a}c)(b\bar{d}) + (\bar{c}a)(d\bar{b}) \\ |S|^2 &= 0. \end{aligned}$$

Since $|a|^2 + |b|^2 = 1$, $|c|^2 + |d|^2 = 1$, $|a|^2|c|^2 = |b|^2|d|^2$,

$$|b|^2 = |c|^2, \quad |a|^2 = |d|^2.$$

Then,

$$\begin{aligned} |S|^2 &= |a|^2|b|^2 + |b|^2|d|^2 - 2|b|^2|d|^2 \\ |S|^2 &= |a|^2|b|^2 - |b|^2|d|^2 \\ |S|^2 &= |b|^2(|a|^2 - |d|^2) \\ |S|^2 &= 0. \end{aligned}$$

Therefore $S = \bar{a}b + \bar{c}d = 0$. Similarly $\bar{b}a + \bar{d}c = 0$.

Thus $U^*U = I_2$.

Appendix C: Proof for normal matrix

set A as a normal matrix, B as a diagonal matrix, then

$$U^*AU = B$$

$$A = UBU^*$$

$$A^* = UB^*U^*$$

$$AA^* = (UBU^*)(UB^*U^*) = UBB^*U^*$$

$$A^*A = (UB^*U^*)(UBU^*) = UB^*BU^*$$

$$BB^* = B^*B$$

$$AA^* = A^*A$$

Appendix D: Proof for EW

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in M_2 \otimes M_2$$

$$W^T = W^* = W$$

$$(a \otimes b)^* W (a \otimes b) = a_{11} b_{11} \cdot a_{11} b_{11} + \overline{a_{12}} b_{11} \cdot a_{11} b_{12} + \overline{a_{11}} b_{12} \cdot a_{12} b_{11} + \overline{a_{12}} b_{12} \cdot a_{12} b_{12}$$

$$\overline{a_{11}} b_{11} \cdot a_{11} b_{11} = \overline{a_{11}} b_{11} a_{11} b_{11} = (\overline{a_{11}} a_{11})(\overline{b_{11}} b_{11}) = |a_{11}|^2 |b_{11}|^2$$

$$\overline{a_{12}} b_{11} \cdot a_{11} b_{12} = \overline{a_{12}} \overline{b_{11}} a_{11} b_{12} = (\overline{a_{12}} a_{11})(\overline{b_{11}} b_{12})$$

$$\overline{a_{11}} b_{12} \cdot a_{12} b_{11} = \overline{a_{11}} \overline{b_{12}} a_{12} b_{11} = (\overline{a_{11}} a_{12})(\overline{b_{12}} b_{11})$$

$$\overline{a_{12}} b_{12} \cdot a_{12} b_{12} = \overline{a_{12}} \overline{b_{12}} a_{12} b_{12} = (\overline{a_{12}} a_{12})(\overline{b_{12}} b_{12}) = |a_{12}|^2 |b_{12}|^2$$

$$(a \otimes b)^* W (a \otimes b) = |a_{11}|^2 |b_{11}|^2 + (\overline{a_{12}} a_{11})(\overline{b_{11}} b_{12}) + (\overline{a_{11}} a_{12})(\overline{b_{12}} b_{11}) + |a_{12}|^2 |b_{12}|^2$$

$$|\langle a, b \rangle|^2 = |\overline{a_{11}} b_{11} + \overline{a_{12}} b_{12}|^2$$

$$= (\overline{a_{11}} b_{11} + \overline{a_{12}} b_{12})(a_{11} \overline{b_{11}} + a_{12} \overline{b_{12}})$$

$$= |a_{11}|^2 |b_{11}|^2 + \overline{a_{11}} b_{11} a_{12} \overline{b_{12}} + \overline{a_{12}} b_{12} a_{11} \overline{b_{11}} + |a_{12}|^2 |b_{12}|^2$$

$$= |a_{11}|^2 |b_{11}|^2 + (\overline{a_{11}} a_{12})(\overline{b_{12}} b_{11}) + (\overline{a_{12}} a_{11})(\overline{b_{11}} b_{12}) + |a_{12}|^2 |b_{12}|^2$$

$$(a \otimes b)^* W (a \otimes b) = |\langle a, b \rangle|^2 = |\overline{a_{11}} b_{11} + \overline{a_{12}} b_{12}|^2$$

$$(a \otimes b)^* W (a \otimes b) \geq 0$$

Appendix E: Proof for Sylvester's law of inertia

Let A and B be Hermitian matrices of the same size. Assume $B = P^*AP$ for some invertible P . For any nonzero vector $x \in \mathbb{C}^n$, we have

$$x^*Bx = x^*P^*APx = (Px)^*A(Px).$$

Since P is invertible, the subspaces on which $x^*Bx > 0$, < 0 , or $= 0$ have the same dimensions as the corresponding subspaces for A . Hence

- The dimension of the subspace on which $x^*Bx > 0$ equals the dimension of the subspace on which $y^*Ay > 0$, where $y = Px$;
- The same holds for the cases less than 0 and equals 0.

Hence A and B have the same positive, negative, and zero inertia indices:

$$(\pi_A, \nu_A, \delta_A) = (\pi_B, \nu_B, \delta_B).$$

Suppose A and B have the same inertia. By the spectral theorem, there are unitary matrices U_A, U_B such that

$$A = U_A D_A U_A^*, \quad B = U_B D_B U_B^*,$$

where

$$D_A = D_B = \text{diag}(I_\pi, -I_\nu, 0_\delta) = D.$$

Thus

$$A = U_A D U_A^*, \quad B = U_B D U_B^*.$$

From the first equation,

$$D = U_A^* A U_A.$$

From the second,

$$D = U_B^* B U_B.$$

Hence

$$U_A^* A U_A = U_B^* B U_B.$$

Multiplying on the left by U_B and on the right by U_B^* , we obtain

$$B = (U_B U_A^*) A (U_A U_B^*).$$

Let $P = U_A U_B^*$. Since unitary matrices are invertible, P is invertible, and

$$B = P^* A P.$$

Therefore A and B are congruent.

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