

## S.-T. Yau High School Science Award

Mathematics Award

### Quantitative Local Convergence of Random Regular Graphs

A Configuration-Model Collision Bound

Student author(s): Jiayi Henry Sheng Qianyu Zhou  
School: Ningbo Xiaoshi High School  
State/Province: Zhejiang  
Country: China  
Instructor(s): DALLE LUCHE LEONARDO  
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## ABSTRACT

A random regular graph is locally similar to the infinite regular tree. We give a detailed proof of a quantitative convergence rate for a random  $d$  regular graph to the  $d$  regular tree in the Benjamini Schramm sense. For fixed integers  $d \geq 3$  and  $r \geq 1$ , we show that the probability that the induced radius  $r$  neighborhood of a uniformly chosen vertex in a uniformly random simple  $d$  regular graph on  $n$  vertices is not rooted isomorphic to the corresponding tree neighborhood is  $O_{d,r}(1/n)$ . We use the configuration model, explore the neighborhood by breadth first search, and estimate the probability of a collision during the exploration.

**Keywords:** random regular graph, configuration model, half-edge pairing, breadth-first search, local convergence, tree-like neighborhood, collision bound.

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## 1. INTRODUCTION

Random regular graphs are graphs in which every vertex has the same degree. They are useful models of sparse networks: the number of edges is only linear in the number of vertices, but the graph is otherwise random. When  $n$  is large, it is difficult to study the entire graph at once. A natural idea is therefore to choose a typical vertex and inspect only a fixed-radius neighborhood around it. For a random regular graph, this neighborhood is usually a tree, because a fixed exploration has only a bounded number of opportunities to close a cycle while there are  $nd$  possible half-edges in total.

The limiting object is the infinite  $d$ -regular tree  $(T_d, o)$ . The main statement of this paper makes the preceding idea quantitative.

**Theorem 1.1** (Main theorem). *Fix integers  $d \geq 3$  and  $r \geq 1$ . For every  $n > d$  such that  $nd$  is even, let  $G_{n,d}$  be a uniformly random simple  $d$ -regular graph on  $[n]$ , and let  $U_n$  be uniform on  $[n]$ , independently of the graph. Then, as  $n \rightarrow \infty$ ,*

$$\mathbb{P}_{G_{n,d}}(B_r(G_{n,d}, U_n) \not\cong B_r(T_d, o)) = O_{d,r}\left(\frac{1}{n}\right).$$

Here  $[n] = \{1, \dots, n\}$ , and  $B_r(G, u)$  is the rooted induced radius- $r$  ball defined in Section 2.

The proof has three main parts. First, we explore the neighborhood of the root in the configuration model. Second, we bound the probability that the exploration closes a cycle, including the possibility of an edge joining two boundary vertices. Finally, we condition on the graph being simple. The probability of simplicity stays bounded away from zero, so this conditioning preserves the order  $1/n$ .

## 2. BACKGROUND: GRAPHS AND THE CONFIGURATION MODEL

**2.1. Rooted balls and local convergence.** The graphs considered in this paper are  $d$  regular graphs. Every vertex in a  $d$  regular graph has degree  $d$ .

**Example 2.1.** Figure 1 illustrates an example of a 2-regular graph.

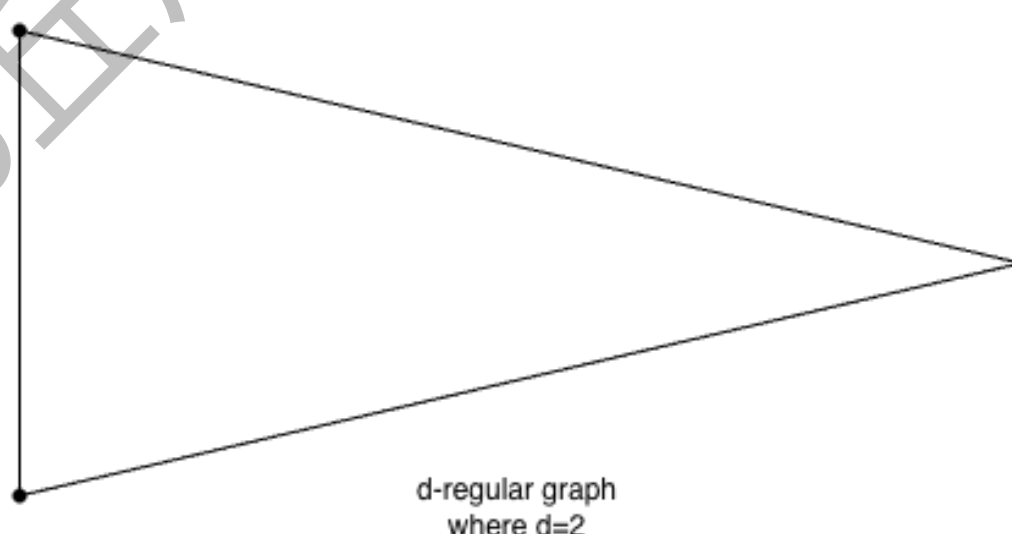


FIGURE 1. Example of a  $d$ -regular graph

However, a  $d$ -regular graph could be extended to an extremely large scale, and exploring it uniformly could be extremely difficult. Thus, instead of studying the entire graph, we are going to discuss the neighborhood around a typical vertex. If this local neighborhood has a predictable structure, we can gain insight into the trend of the whole graph.

By choosing a ‘typical vertex’, here we refer to the process of fixing a root, which means choosing one vertex uniformly at random from all the vertices in the graph. Fix the vertex and call it a root. This means that for a graph containing  $n$  vertices, each vertex has the possibility of  $\frac{1}{n}$  to be chosen.

In the following, we provide the formal definition for rooted locally finite graphs.

**Definition 2.2.** (Rooted locally finite graph)

$$(G, o)$$

is a rooted locally finite graph if  $G$  has a finite degree at each vertex and  $o \in V$ .

Before comparing two graphs, we must clarify what it means for them to have the same structure. This is the role of graph isomorphism.

**Definition 2.3** (Graph isomorphism). Let  $G$  and  $H$  be two graphs. An isomorphism between  $G$  and  $H$  is a bijection  $f : V(G) \rightarrow V(H)$  that preserves adjacency relations. That is,

$$\{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E(H)$$

If there exists such a bijection, we say that  $G$  is isomorphic to  $H$ .

Moreover, rooted isomorphism refers to the specific case in which the root is preserved. That is, for the two rooted graph  $(G, o_1)$  and  $(H, o_2)$

$$f(o_1) = o_2$$

**Example 2.4.** Figure 2 shows two rooted-isomorphic graphs, with the red dot indicating the root of each graph.

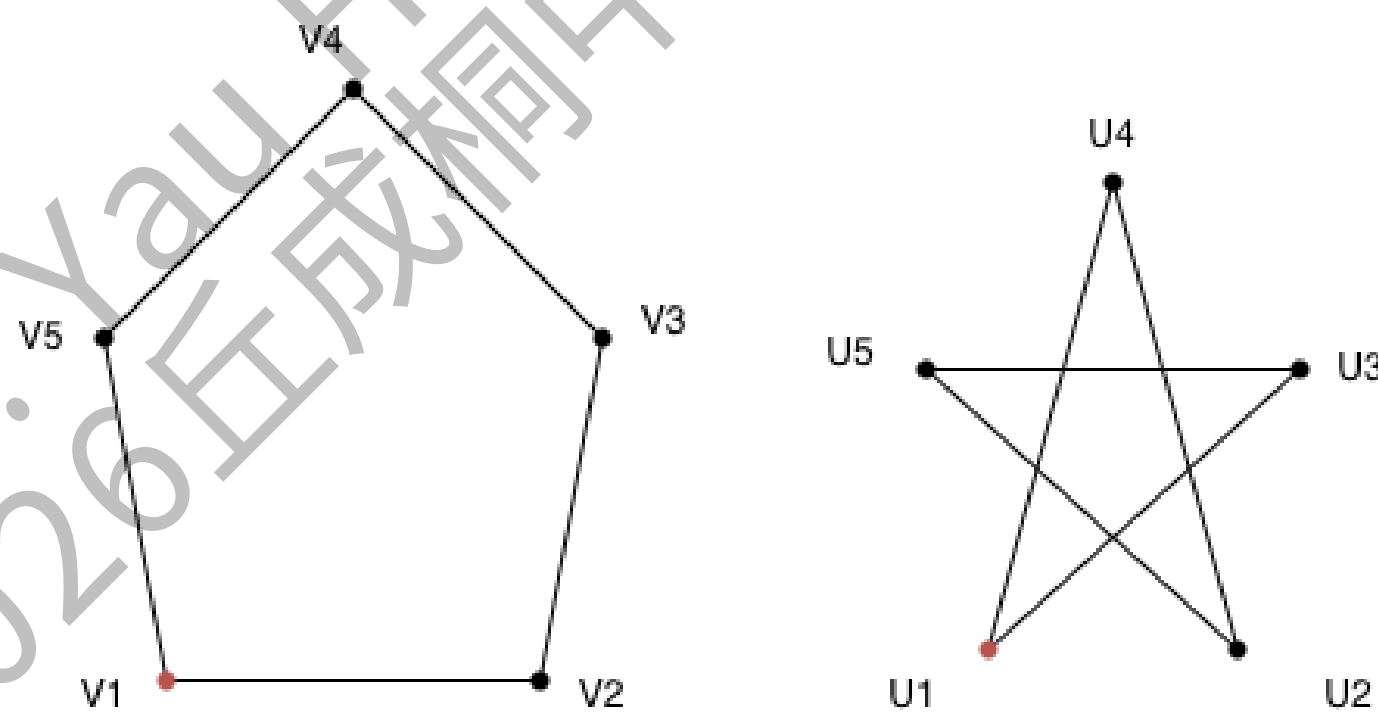
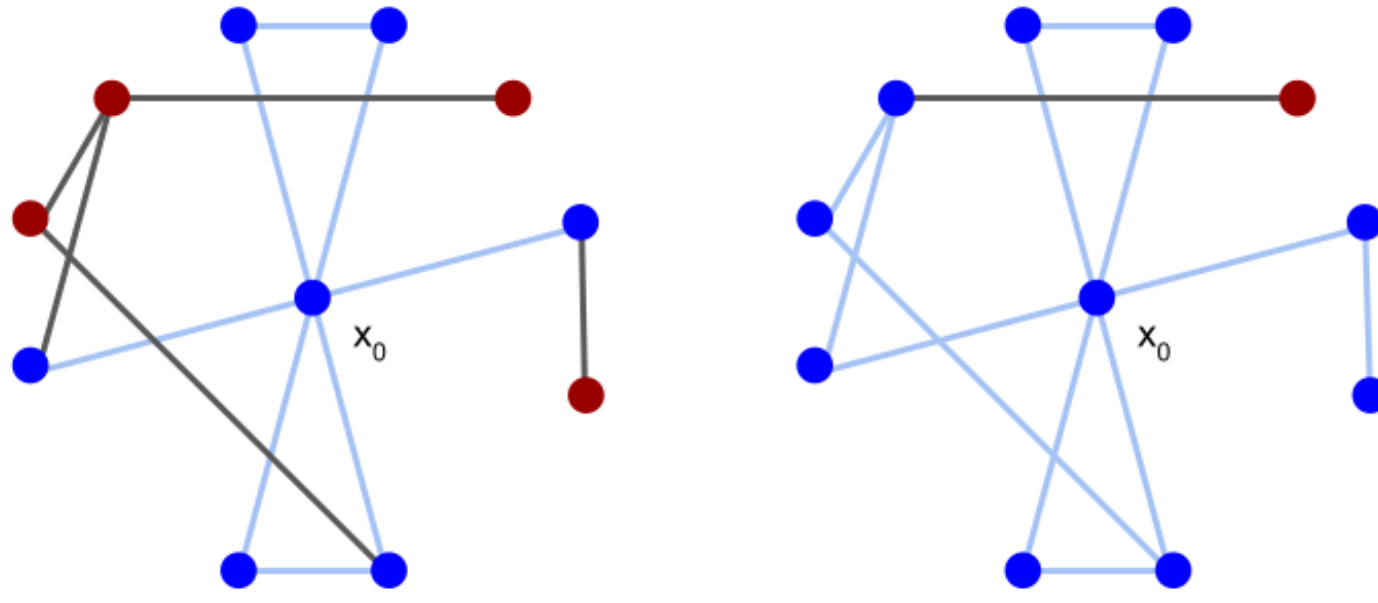


FIGURE 2. Example for rooted isomorphism. Specifically,  $f(v_1) = u_1$ ,  $f(v_2) = u_3$ ,  $f(v_3) = u_5$ ,  $f(v_4) = u_2$ , and  $f(v_5) = u_4$ . Adjacency relationships are preserved [3].

For the root of a graph, we say that the induced subgraph including exactly all the vertices within at most distance  $r$  is its induced  $r$ -ball. Here, we are exploring the properties of the neighborhood by gradually expanding the radius of the ball  $r$ .

**Example 2.5.** In the following figure,  $x_0$  is the root of the graph. The blue dots on the left indicate a 1-ball, while those on the right indicate a 2-ball.

FIGURE 3. Example for an  $r$ -ball

For two locally finite rooted graphs  $G(v)$  and  $(H, w)$ , define

$$R(G(v), (H, w)) := \sup\{s \geq 0 : B_s(G, v) \cong B_s(H, w)\}.$$

Subsequently, we have

$$d((G, v), (H, w)) := 2^{-R}$$

with the convention that  $2^{-\infty} := 0$ . A sequence of finite random graphs  $(G_n)$  converges locally in the annealed sense to a random rooted graph  $(G_\infty, o)$  if

$$(G_n, U_n) \Rightarrow (G_\infty, o)$$

in this metric, where conditional on  $G_n$ , the root  $U_n$  is uniform on  $V(G_n)$ .

The following type of graphs is probably the most important type that we will consider

**Definition 2.6** (Tree). A forest is a graph with no cycles. A tree is a connected forest.

The infinite tree is the natural limiting object for these local explorations.

**Definition 2.7.** (Infinite  $d$ -regular tree) The infinite  $d$  regular tree is an infinite connected graph in which every vertex has degree  $d$  and no cycles.

Local convergence refers to the possibility that the neighborhood around a root of a finite  $d$  regular graph becomes increasingly similar to a neighborhood in the infinite  $d$  regular tree. The pairing construction below provides the probabilistic mechanism for analyzing this convergence.

**2.2. The pairing construction.** When we choose a random  $d$ -regular graph, we mean a graph chosen uniformly from the labeled simple  $d$ -regular graphs on vertex set  $[n]$ . Directly sampling from this set is inconvenient because the number of such graphs is very large. The configuration model gives a convenient equivalent sampling method after conditioning on simplicity.

For fixed  $n$  and  $d$  with  $nd$  even, the configuration model is constructed as follows:

- (1) Give each vertex  $d$  distinguishable half-edges.
- (2) Pair all  $nd$  half-edges in a uniformly random perfect matching.

There are  $(nd - 1)!!$  such perfect matchings, and they are all equally likely. Each matched pair becomes an edge between the vertices incident to its two half-edges. A half-edge may be paired to another half-edge at the same vertex, producing a self-loop, and two vertices may be joined by more than one edge. Therefore the resulting object is generally a multigraph. We denote it by  $\text{CM}_{n,d}$ . Probabilities concerning this matching will be written  $\mathbb{P}_{\text{CM}_{n,d}}$  when it is useful to distinguish them from probabilities under the simple-graph law  $G_{n,d}$ ; the latter

will be written  $\mathbb{P}_{G_{n,d}}$ . The root  $U_n$  is sampled uniformly from  $[n]$ , independently of this matching.

**Example 2.8.** Figure 4 illustrates an example of the pairing, where the blue lines are the given vertices and the red lines are the pairing construction.

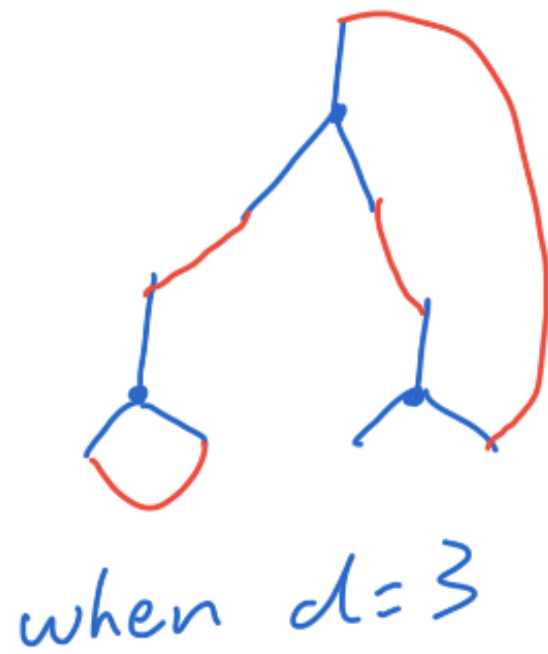


FIGURE 4. Example for the configuration model when  $d=3$

The next lemma records the symmetry that makes sequential exploration possible. Even after part of the matching has been revealed, the remaining matching is still uniform.

**Lemma 2.9.** *Suppose that some pairs in a uniform perfect matching have been revealed. Choose any remaining half-edge  $h$  by a rule measurable with respect to the revealed history. Conditional on that history, the partner of  $h$  is uniform among all other unmatched half-edges.*

*Proof.* After the revealed pairs are fixed, every perfect matching of the remaining half-edges is still equally likely: each such matching extends the revealed partial matching in exactly one way, and all complete matchings had the same initial probability. If  $N$  half-edges remain, symmetry therefore gives

$$\mathbb{P}(h \text{ is paired with } h' \mid \text{revealed history}) = \frac{1}{N-1}$$

for each of the other  $N-1$  half-edges  $h'$ .

For example, with 6 half-edges the first partner has probability  $1/5$ . After one pair is revealed, the partner of a specified remaining half-edge has probability  $1/3$ . No independence of successive pairings is being assumed.  $\square$

Let's proceed. The configuration model may contain self-loops or multiple edges. The resulting object is therefore a multigraph rather than a simple graph. Multiple edges and self-loops are called collisions.

Because a cycle occurs when collisions happen during BFS, while a tree has no cycles. Those collisions are considered as unwanted situations that we want to avoid.

**Example 2.10.** Figure 5 illustrates two collision events, with a multiple edge on the right and a self-loop on the left. These examples show why cycles may appear during the exploration.

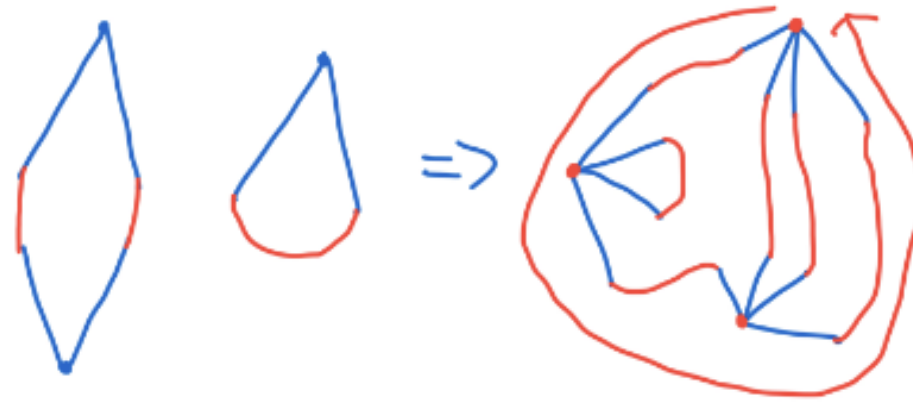


FIGURE 5. Avoiding collisions and cycles

**Definition 2.11** (Simple event). Let  $S_n$  be the event that the multigraph produced by the configuration model has no self-loops and no multiple edges. Equivalently, the outcome graph is a simple  $d$ -regular graph on  $[n]$ .

We conclude this section with the conditioning fact that connects the two models.

**Lemma 2.12** (Conditioning on simplicity). *Conditional on  $S_n$ , the projected configuration model has the uniform distribution on simple labeled  $d$ -regular graphs on  $[n]$ .*

*Proof.* Every configuration projects to a  $d$ -regular multigraph. On  $S_n$  there are no loops or multiple edges, so the projection is a simple  $d$ -regular graph. Conversely, every simple  $d$ -regular graph has exactly  $(d!)^n$  configurations: at each vertex, its  $d$  incident edges can be assigned to the  $d$  distinguishable half-edges in  $d!$  ways. Since all configurations were initially equally likely, all simple graphs have the same conditional probability. This is the standard configuration-model argument (see, for example, [4]).  $\square$

### 3. NEIGHBORHOOD EXPLORATION

We now describe an exploration that reveals the pairings around the root. We first count the vertices and half-edges in the tree neighborhood that we hope to obtain.

**3.1. Tree enumeration.** For a rooted  $d$ -regular tree  $T_d$ , the root is the only vertex at distance 0 (to the root). There are  $d$  vertices at distance 1, and each vertex at a positive distance has one parent and  $d - 1$  children. Therefore the number of vertices at distance exactly  $k \geq 1$  is  $d(d - 1)^{k-1}$ . Then the number of vertices other than the root within distance  $r$  is

$$M_{d,r} = d + d(d - 1) + d(d - 1)^2 + \cdots + d(d - 1)^{r-1} = \frac{d((d - 1)^r - 1)}{d - 2}.$$

Thus the radius  $r$ -ball contains  $M_{d,r} + 1$  vertices and  $M_{d,r}$  tree edges. The number of vertices on the boundary is  $d(d - 1)^{r-1}$ . Each boundary vertex has  $d - 1$  half-edges left after its edge to its parent is revealed, so the number of unmatched boundary half-edge is

$$b_{d,r} = d(d - 1)^r.$$

**3.2. Exploration algorithm.** We perform a first-depth exploration starting from a uniformly chosen root  $U_n$ . Starting with the discovered vertex set  $\{U_n\}$ , we iteratively reveal vertices at increasing distances from the root. After reaching level  $k$ , we extend to level  $k + 1$  by pairing the unmatched half-edges incident to the vertices at level  $k$ . We repeat this process until we have discovered the vertices of the ball of radius  $r$ . During exploration, we reveal half-edge pairings sequentially.

More precisely, give the half-edges a fixed order to break ties, and use the following procedure.

- Step 1 Mark  $U_n$  as discovered and give it level 0.
- Step 2 Choose an unmatched half-edge at a discovered vertex of level less than  $r$ , processing smaller levels first, and reveal its partner.
- Step 3 If the partner belongs to an undiscovered vertex, mark that vertex as discovered and give it level one greater than the current vertex. The new edge becomes its edge to its parent. On the contrary, if the partner belongs to an already discovered vertex, record a collision and stop the procedure.
- Step 4 If no collision has occurred, continue until all half-edges at levels  $0, \dots, r-1$  have been paired.

If the procedure finishes without a collision, it has revealed exactly  $M_{d,r}$  pairings and discovered  $M_{d,r} + 1$  vertices. The revealed edges form the desired rooted tree. There are still  $b_{d,r}$  unmatched half-edges at level  $r$ .

Our goal is for the induced ball to be a tree. As section 2 mentioned, if a collision occurs, the explored ball contains a cycle and cannot be a tree. There is one more point to check: two of the remaining boundary half-edges might be paired to each other. Such an edge is part of the induced ball and also creates a cycle. We therefore control both a collision during the exploration and a boundary-boundary pairing in the remaining matching. Edges from the boundary to vertices outside the discovered set do not belong to the induced radius- $r$  ball.

Then we bound the probability of both types of bad events.

#### 4. THE COLLISION ESTIMATE AND THE SIMPLE MODEL

In the previous section, we described the breadth first search exploration process and defined the collisions that may occur during this exploration. In this section, we prove that for fixed  $d$  and  $r$ , the probability that a collision occurs during the exploration up to depth is  $\frac{1}{n}$ . A standard approach is to first establish a collision bound conditional on the simple event  $S_n$ , then transfer it to the unconditional setting. It is known [1] that

$$\mathbb{P}(S_n) \rightarrow \exp\left(\frac{1-d^2}{4}\right) =: s_d > 0.$$

Thus, for sufficiently large  $n$ ,  $\mathbb{P}(S_n) \geq \frac{s_d}{2} > 0$ . Consequently, an  $\frac{1}{n}$  bound on the conditional probability implies an  $\frac{1}{n}$  bound on the unconditional probability.

**Proposition 4.1** (Configuration-model bound). *For fixed  $d \geq 3$  and  $r \geq 1$ . If  $nd$  is sufficiently large, then*

$$\mathbb{P}(B_r(\text{CM}_{n,d}, U_n) \not\cong B_r(T_d, o)) = O_d\left(\frac{(d-1)^{2r}}{n}\right).$$

For fixed  $d$  and  $r$ , this is  $O_{d,r}(1/n)$ .

*Proof.* If  $n < M_{d,r} + 1$ , then  $M_{d,r}(M_{d,r} + 1)/n > 1$ , so the displayed bound is trivial. We may therefore assume  $n \geq M_{d,r} + 1$  and proceed with the collision-free exploration.

Reveal the half-edge pairings sequentially, layer by layer, beginning at the uniformly chosen root  $U_n$ . We control two possible sources of collision: a collision among the first  $M_{d,r}$  pairings that discover new vertices, and a pairing between two boundary half-edges after those vertices have been discovered.

Suppose that the first  $j$  pairings have succeeded without a collision, where  $0 \leq j \leq M_{d,r}-1$ . Then  $j+1$  vertices have been discovered. They contribute  $d(j+1)$  incident half-edges, of which  $2j$  have already been paired. Excluding the current half-edge, at most

$$d(j+1) - 2j - 1$$

unmatched half-edges are incident to discovered vertices and hence can cause a collision. The current half-edge has

$$N - 2j - 1$$

possible unmatched partners. By [theorem 2.9](#),

$$\mathbb{P}_{\text{CM}_{n,d}}(\text{the next pairing creates a collision} \mid \text{that collision-free history}) \leq \frac{d(j+1)}{N - 2j - 1}.$$

For sufficiently large  $n$ , for example when  $n \geq (4M_{d,r} + 2)/d$ , every denominator above is at least  $N/2$ . Therefore, writing  $E_j$  for the event that the first  $j$  pairings are collision-free, the conditional bound above gives  $\mathbb{P}_{\text{CM}_{n,d}}(E_j \cap \{\text{a collision at step } j+1\}) \leq 2(j+1)/n$ . Summing these bounds (a union bound) gives

$$\begin{aligned} \mathbb{P}_{\text{CM}_{n,d}}(\text{a collision in the first } M_{d,r} \text{ pairings}) &\leq \sum_{j=0}^{M_{d,r}-1} \frac{d(j+1)}{N - 2j - 1} \\ &\leq \sum_{j=0}^{M_{d,r}-1} \frac{2(j+1)}{n} = \frac{M_{d,r}(M_{d,r} + 1)}{n}. \end{aligned}$$

Condition further on any particular collision-free history of these first  $M_{d,r}$  pairings. The remaining matching is uniform by [theorem 2.9](#), and the  $b_{d,r}$  boundary half-edges are all still unmatched. For any fixed pair of boundary half-edges, the probability that they are matched to each other is  $1/(N - 2M_{d,r} - 1)$ . A union bound over the  $\binom{b_{d,r}}{2}$  pairs gives, uniformly over all such histories,

$$\begin{aligned} &\mathbb{P}_{\text{CM}_{n,d}}(\text{a boundary-boundary pairing} \mid \text{that history}) \\ &\leq \binom{b_{d,r}}{2} \frac{1}{N - 2M_{d,r} - 1} \leq \frac{b_{d,r}(b_{d,r} - 1)}{dn}, \end{aligned}$$

where the last inequality uses the same sufficiently-large- $n$  condition. Averaging over the collision-free histories gives the same bound conditional on no early collision.

If neither type of collision occurs, every newly exposed vertex is new. All half-edges incident to levels  $0, \dots, r-1$  have already been exposed, so the only edges between vertices at distance at most  $r$  are the tree edges. Any remaining boundary half-edge paired to a half-edge outside the explored set cannot put that new vertex within distance  $r$ : every path of length at most  $r$  would have entered it through a half-edge incident to a level  $< r$  vertex, and all such half-edges were already exposed. Thus the new vertex is at distance at least  $r+1$  and does not belong to the induced radius- $r$  ball. Hence the rooted induced ball is  $B_r(T_d, o)$ . Combining the two bounds proves the displayed inequality. Finally,  $M_{d,r} = O_d((d-1)^r)$  and  $b_{d,r} = O_d((d-1)^r)$ , which gives the stated order of magnitude.  $\square$

## 5. THE PROBABILITY OF SIMPLICITY

The previous sections established that a fixed neighborhood in the configuration model agrees with the corresponding neighborhood of the infinite  $d$ -regular tree with probability

$1 - O_{d,r}(1/n)$ . This section goes a step further. First, we are going to show the general convergence trend via Poisson distribution, then condition on its simplicity.

**5.1. Exploring the convergence rate.** We index each possible self-loop or parallel edges by  $i \in I_n$ . Recall that a self-loop is two half-edges at one vertex pair together and a double-edge is two distinct vertices have two pairs of half-edges paired. Let  $B_i$  be the  $i$ -th bad event, i.e. self-loop or parallel edges, happen. Its complement  $\overline{B_i}$  is the event that the bad event does not happen.

We would like to introduce the indicator variable

$$\chi_i = \begin{cases} 1, & B_i \text{ occurs,} \\ 0, & B_i \text{ does not occur.} \end{cases}$$

Each indicator records whether one self-loop or double-edge appears. Moreover,

$$\chi_n = \sum_{i \in I_n} \chi_i = L_n + D_n$$

where  $L_n$  is the number of self-loops and  $D_n$  is the number of unordered pairs of parallel edges between distinct vertices. If  $m$  edges join the same two distinct vertices, they contribute  $\binom{m}{2}$  to  $D_n$ . For example, three parallel edges contribute three, not one. Thus

$$S_n = \{L_n = 0, D_n = 0\} = \{\chi_n = 0\} = \bigcap_{i \in I_n} \overline{B_i}.$$

The events  $B_i$  are not independent, so the probability of this intersection cannot in general be replaced by a product of probabilities.

We expect each of the two bad events to be rare. A Poisson distribution often describes the total number of rare events when their dependence is sufficiently weak. We establish this using counting and inclusion exclusion, a method known as Brun's sieve. See [5, Section 8.3, Theorem 8.3.1]. A random variable  $Z$  has the Poisson distribution with parameter  $\mu > 0$  if

$$\mathbb{P}(Z = t) = e^{-\mu} \frac{\mu^t}{t!}, \quad t = 0, 1, 2, \dots$$

The first step is calculating the limiting mean. Let's first compute the mean number of self-loops. Each vertex has  $\binom{d}{2}$  possible pairs of half-edges, so there are  $n\binom{d}{2}$  possible pairs in total. Recalling Lemma 2.9

$$\mathbb{P}_{\text{CM}_{n,d}}(\text{one particular pair is matched}) = \frac{1}{nd - 1}.$$

Thus by linearity of expectation,

$$\mathbb{E}_{\text{CM}_{n,d}}(L_n) = n \binom{d}{2} \frac{1}{nd - 1} = \frac{nd(d-1)}{2(nd-1)} \rightarrow \frac{d-1}{2} =: \theta_1.$$

Similarly, let's compute the mean number of double-edge. We choose two distinct vertices  $u$  and  $v$  from all  $n$  vertices. There are  $\binom{n}{2}$  choices. We then choose two half-edges at each vertex which there are  $\binom{d}{2}^2$  choices. Say the half-edges chosen from  $u$  are  $a, b$  and those chosen from  $v$  are  $c, e$ . There are exactly two ways to pair them across the vertices:  $(a, c), (b, e)$  or  $(a, e), (b, c)$ . For either pairing the probability that both pairs occur is

$$\frac{1}{(nd-1)(nd-3)}.$$

Indeed the first specified pair has probability  $1/(nd - 1)$  and conditional on it we have the second has probability  $1/(nd - 3)$ . So

$$\mathbb{E}_{\text{CM}_{n,d}}(D_n) = \binom{n}{2} \binom{d}{2}^2 \frac{2}{(nd - 1)(nd - 3)} \rightarrow \frac{(d - 1)^2}{4} =: \theta_2.$$

By adding  $\theta_1$  and  $\theta_2$  we can see that

$$\mathbb{E}_{\text{CM}_{n,d}}(\chi_n) = \sum_{i \in I_n} \mathbb{P}_{\text{CM}_{n,d}}(B_i) \rightarrow \theta_1 + \theta_2 = \frac{d^2 - 1}{4}$$

So we have

**Theorem 5.1.** *For fixed  $d \geq 3$ , as  $n \rightarrow \infty$  through integers with  $nd$  even,*

$$\mathbb{P}_{\text{CM}_{n,d}}(\chi_n = t) \rightarrow e^{-\mu} \frac{\mu^t}{t!} \quad \text{for every fixed } t \geq 0, \quad \mu = \frac{d^2 - 1}{4}.$$

*In particular,*

$$\mathbb{P}_{\text{CM}_{n,d}}(S_n) \rightarrow e^{-\mu} = \exp\left(-\frac{d^2 - 1}{4}\right) > 0.$$

*Remark 5.2.* This simplicity limit is classical. It follows from the enumeration of Bender and Canfield [1]. We give the counting the inclusion-exclusion proof here so that the dependence between the bad events is addressed explicitly. The proof of this theorem will be shown later.

**Lemma 5.3.** *Suppose  $q$  disjoint pairs of half-edges are prescribed, with no half-edge appearing in more than one pair. In a uniform perfect matching of  $N = nd \geq 2q$  half-edges, the probability that all of these pairs occur is*

$$p_q(N) := \prod_{j=0}^{q-1} \frac{1}{N - 2j - 1}.$$

*We set  $p_0(N) = 1$ . If two prescriptions require the same half-edge to have different partner, their joint probability is zero.*

*Proof.* Reveal the prescribed pairs one in a time. By Lemma 2.9, after  $j$  pairs have been revealed, the next specified pair occurs with conditional probability  $1/(N - 2j - 1)$ . Multiplying these conditional probabilities proves the formula. A half-edge has exactly one partner, which proves the last statement.  $\square$

For the later of this section, for a nonnegative integer  $x$  and  $k \geq 1$ , write

$$(x)_k = x(x - 1) \cdots (x - k + 1), \quad (x)_0 = 1.$$

And thus  $(x)_k = k! \binom{x}{k}$ .

**Lemma 5.4.** *For every fixed integer  $k \geq 0$ ,*

$$\mathbb{E}_{\text{CM}_{n,d}}(\chi_n)_k \rightarrow \mu^k.$$

*Proof.*  $k = 0$  is trivial. For  $k \geq 1$ ,  $(\chi_n)_k$  counts the ordered lists of  $k$  distinct bad event/collision, again recall that to be self-loop or parallel-edges, that occur. Therefore

$$\mathbb{E}_{\text{CM}_{n,d}}(\chi_n)_k = \sum_{\text{distinct } i_1, \dots, i_k \in I_n} \mathbb{P}_{\text{CM}_{n,d}}(B_{i_1} \cap \cdots \cap B_{i_k}).$$

We divide this sum according to whether the patterns use disjoint sets of original vertices.

**Patterns on disjoint vertex set.** Fix an order of  $\ell$  loop patterns and  $m$  double-edge patterns, with  $\ell + m = k$ . Together they use  $q = \ell + 2m$  vertices and prescribe  $q$  pairs of half-edges.

There are  $(n)_q/2^m$  ways to choose the distinct vertex labels for these ordered patterns: each loop uses one vertex, and the two vertices of each double-edge pattern are unordered. At every chosen vertex there are  $\binom{d}{2}$  ways to choose its two half-edges. Each double-edge pattern also has two possible pairings. Consequently the number of such lists is exactly

$$\binom{(n)_q}{2^m} \binom{d}{2}^q 2^m = (n)_q \binom{d}{2}^q.$$

By Lemma 5.3 their contribution to the sum is

$$(n)_q \binom{d}{2}^q p_q(nd) \rightarrow \left( \binom{d}{2} / d \right)^q = \theta_1^\ell \theta_2^m.$$

The limit follows because  $q$  is fixed,  $(n)_q/n^q \rightarrow 1$  and  $(nd)^q p_q(nd) \rightarrow 1$ . There are  $\binom{k}{\ell}$  ways to place the loop patterns among the  $k$  positions. Summing over  $\ell$  shows that all lists on disjoint vertex sets contribute a quantity tending to

$$\sum_{\ell=0}^k \binom{k}{\ell} \theta_1^\ell \theta_2^{k-\ell} = (\theta_1 + \theta_2)^k = \mu^k.$$

**Patterns sharing a vertex.** A list in which some half-edge is required to have two different partners contributes zero. For a list with no such conflict, let  $H$  be the multigraph formed by the union of its prescribed pairs. Repeated prescriptions of the same pair are counted only once. Write  $v$  for its number of vertices and  $e$  for its number of edges.

Every vertex of  $H$  has degree at least two, since it belongs to a loop or a double-edge pattern. A loop contributes two to the degree. Moreover two distinct patterns in such a list that share a vertex must use at least three half-edges there. To see this, suppose they use the same two half-edges at that vertex. If these are paired together, both patterns must be that same loop. Otherwise their partners are fixed by the prescribed pairings, and both patterns must be that same double-edge patterns. Both cases contradict distinctness. Therefore at least one vertex has degree greater than two, and the degree-sum formula gives

$$2e = \sum_{x \in V(H)} \deg_H(x) > 2v, \quad \text{so } e \geq v + 1.$$

For fixed  $d$  and  $k$ , there are only finitely many possible arrangements of vertices and half-edges in such a list: at most  $2k$  vertices and  $2k$  edges can occur. Choosing  $v$  vertex labels gives at most  $n^v$  possibilities, and the remaining choices are bounded by a constant depending only on  $d$  and  $k$ . Thus there are  $O_{d,k}(n^v)$  lists of each arrangement. Each has probability  $p_e(nd) = O_{d,k}(n^{-e})$ , so its arrangement contributes

$$O_{d,k}(n^{v-e}) = O_{d,k}(1/n).$$

Summing over the finitely many arrangements shows that all overlapping lists have total contribution  $O_{d,k}(1/n)$ . Combining the two parts proves the lemma.  $\square$

These moments limits give the claimed probabilities. Define

$$S_{n,k} := \sum_{J \in \binom{[n]}{k}} \mathbb{P}_{\text{CM}_{n,d}} \left( \bigcap_{i \in J} B_i \right) = \mathbb{E}_{\text{CM}_{n,d}} \left( \binom{\chi_n}{k} \right), \quad S_{n,0} = 1.$$

Here each unordered set of  $k$  distinct patterns appears once. By Lemma 5.4,

$$S_{n,k} \rightarrow \frac{\mu^k}{k!} \quad \text{for every fixed } k.$$

The Bonferroni inequalities (a special union bound, or called inclusion-exclusion bound) give, for each integer  $s \geq 0$ ,

$$\sum_{k=0}^{2s+1} (-1)^k S_{n,k} \leq \mathbb{P}_{\text{CM}_{n,d}}(\chi_n = 0) \leq \sum_{k=0}^{2s} (-1)^k S_{n,k}.$$

These are the finite truncations used in Brun's sieve [5, Section 8.3]

*proof of Theorem 5.1.* First consider  $t = 0$ . Fix  $s$  and let  $n \rightarrow \infty$  in the Bonferroni bounds. We obtain

$$\begin{aligned} \sum_{k=0}^{2s+1} \frac{(-\mu)^k}{k!} &\leq \liminf_{n \rightarrow \infty} \mathbb{P}_{\text{CM}_{n,d}}(\chi_n = 0) \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{P}_{\text{CM}_{n,d}}(\chi_n = 0) \leq \sum_{k=1}^{2s} \frac{(-\mu)^k}{k!}. \end{aligned}$$

As  $s \rightarrow \infty$ , both finite sums converges to  $e^{-\mu}$ . Hence

$$\mathbb{P}_{\text{CM}_{n,d}}(\chi_n = 0) \rightarrow e^{-\mu}.$$

The same argument proves the limit for every fixed  $t \geq 0$ . The generalized Bonferroni bounds are

$$\sum_{j=0}^{2s+1} (-1)^j \binom{t+j}{t} S_{n,t+j} \leq \mathbb{P}_{\text{CM}_{n,d}}(\chi_n = t) \leq \sum_{j=0}^{2s} (-1)^j \binom{t+j}{t} S_{n,t+j}.$$

Alternatively, we can see that for any integer  $x > t$ , the truncated binomial sum satisfies

$$\sum_{j=0}^K (-1)^j \binom{x-j}{j} = (-1)^K \binom{x-t-1}{K}.$$

It is nonnegative for even  $K$  and nonpositive for odd  $K$ . When  $x = t$ , the sum equals one. Multiply by  $\binom{x}{t}$ . When  $x < t$ , take the resulting expression to be zero. This bounds the indicator of  $\{x = t\}$  from above for even  $K$  and below for odd  $K$ . The identity

$$\binom{x}{t} \binom{x-t}{j} = \binom{t+j}{t} \binom{x}{t+j}$$

then gives the displayed probability bounds after setting  $x = \chi_n$  and taking expectations.

For fixed  $s$ , the terms in these bounds converge to

$$\binom{t+j}{t} \frac{\mu^{t+j}}{(t+j)!} = \frac{\mu^t \mu^j}{t! j!}.$$

Letting first  $n \rightarrow \infty$  and then  $s \rightarrow \infty$ , the lower and upper bounds both tend to

$$\frac{\mu^t}{t!} \sum_{j=0}^{\infty} \frac{(-\mu)^j}{j!} = e^{-\mu} \frac{\mu^t}{t!}.$$

□

**5.2. Conditioning on the simplicity of the graph.** Throughout the proofs, we have worked in the configuration model because it is much easier to analyze. However, the configuration model can produce multigraphs containing loops and multiple edges. Theorem 5.1 tells us that the probability of having none of these defects tends to the positive constant  $s_d$ . After conditioning on this event, the distribution becomes exactly the uniform simple  $d$ -regular graph by Theorem 2.12.

*Proof of Theorem 1.1.* Recall from Definition 2.11 that  $S_n$  is the event that the configuration model is simple. Define the event

$$A_n = \{B_r(\text{CM}_{n,d}, U_n) \cong B_r(T_d, o)\}.$$

We are looking for  $\mathbb{P}_{\text{CM}_{n,d}}(A_n \mid S_n)$ , or equivalently an upper bound on  $\mathbb{P}_{\text{CM}_{n,d}}(A_n^c \mid S_n)$ . The root is independent of the matching, so conditioning on  $S_n$  leaves it uniform and independent of the resulting simple graph. Therefore,

$$\begin{aligned} \mathbb{P}_{G_{n,d}}(B_r(G_{n,d}, U_n) \not\cong B_r(T_d, o)) &= \mathbb{P}_{\text{CM}_{n,d}}(A_n^c \mid S_n) \\ &= \frac{\mathbb{P}_{\text{CM}_{n,d}}(A_n^c \cap S_n)}{\mathbb{P}_{\text{CM}_{n,d}}(S_n)} \\ &\leq \frac{\mathbb{P}_{\text{CM}_{n,d}}(A_n^c)}{\mathbb{P}_{\text{CM}_{n,d}}(S_n)}. \end{aligned}$$

By Proposition 4.1, the numerator is at most

$$\frac{C_{d,r}}{n}, \quad C_{d,r} := M_{d,r}(M_{d,r} + 1) + \frac{b_{d,r}(b_{d,r} - 1)}{d},$$

for sufficiently large  $n$ . By Theorem 5.1, the denominator is at least  $s_d/2$  for sufficiently large  $n$ . Consequently,

$$\mathbb{P}_{G_{n,d}}(B_r(G_{n,d}, U_n) \not\cong B_r(T_d, o)) \leq \frac{2C_{d,r}}{s_d n} = O_{d,r}(1/n),$$

which proves the theorem. □

## 6. DISCUSSION

The results of this paper provide a quantitative explanation for the local tree-like structure of large random regular graphs. When the degree  $d$  and the radius  $r$  are fixed, the breadth first exploration exposes only a finite number of half-edges. At each step, however, the partner half-edge is selected from a set whose size is approximately  $nd$ . As a result, the probability of encountering a previously discovered vertex becomes small as  $n$  increases.

In case of collisions, a collision may create a self-loop, a multiple edge, or an additional connection within the explored neighborhood. The number of potentially dangerous half-edges grows with the size of the explored region. Since the number of vertices within distance

$r$  is of order  $(d-1)^r$ , the total collision probability is of order

$$\frac{(d-1)^{2r}}{n}.$$

This explains why the neighborhood becomes increasingly similar to the corresponding neighborhood of the infinite  $d$  regular tree.

The configuration model also makes it possible to study the difference between random multigraphs and random simple graphs. The probability that a configuration produces a simple graph converges to a positive constant. Therefore, conditioning on simplicity changes the probability estimates only by a constant factor. The local tree like estimate obtained in the configuration model consequently remains valid for uniformly random simple  $d$  regular graphs.

The proof relies on the assumption that  $d$  and  $r$  remain fixed as  $n$  tends to infinity. If the radius grows too quickly with  $n$ , the explored neighborhood may contain a number of vertices comparable to the total number of vertices in the graph. In that regime, collisions are no longer sufficiently rare for the present estimate to guarantee a tree like neighborhood. A more refined analysis would be needed to study larger radii.

In conclusion, the configuration model provides a direct and intuitive method for proving local convergence. By counting the number of exposed half-edges and applying a union bound, we obtain

$$\mathbb{P}(B_r(G_{n,d}, U_n) \not\cong B_r(T_d, o)) = O_{d,r}\left(\frac{1}{n}\right).$$

Thus, from the perspective of a uniformly chosen vertex, a large random regular graph is very likely to resemble the infinite regular tree within any fixed radius. This local approximation is useful because it allows properties of infinite trees to serve as accurate models for the local behavior of large finite random regular graphs.

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Qianyu Zhou contributed to the study of the probability of simplicity in the configuration model. Qianyu worked on the counting of self loops and multiple edges, the calculation of the limiting expectations, and the Poisson approximation argument. Qianyu also contributed to the discussion of boundary pairings and the conditioning argument that transfers the result from the configuration model to the uniform simple regular graph.

Both student authors participated in reviewing the mathematical literature, checking the definitions and calculations, discussing the proof strategy, preparing the figures, revising the written report, and verifying the final version of the manuscript. Both students approved the final report and accept responsibility for its contents.

**External Assistance.** The project was completed under the guidance of the instructor identified above. Any additional assistance should be described below:

No person or organization outside the student authors and the identified instructor contributed to the mathematical ideas, proofs, calculations, or research results presented in this report.

**Disclosure of Artificial Intelligence Use.** The student authors independently developed the research question, mathematical arguments, proofs, calculations, and conclusions presented in this report.

Artificial intelligence tools were not used to generate or replace the students' original mathematical work. Any assistance used during preparation was limited to grammar checking, spelling correction, sentence level editing, and improving the clarity of the English prose. All mathematical content and references were independently reviewed and verified by the student authors.

All mathematical statements, formulas, citations, and proofs were reviewed and verified by the student authors.

## ACADEMIC INTEGRITY DECLARATION

We declare that this report presents our own research and that all sources used in preparing it have been properly cited. We have not fabricated mathematical results, references, or data. We understand that plagiarism, undisclosed assistance, and other forms of academic misconduct may result in disqualification.

Jiaqi Honey Sheng signature: Jiaqi Honey Sheng

Date: 2026. 9. 14

Qianyu Zhou signature: Qianyu Zhou

Date: 2026. 9. 14

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## REFERENCES

- [1] Edward A. Bender and E. Rodney Canfield. *The asymptotic number of labeled graphs with given degree sequences*. Journal of Combinatorial Theory, Series A **24**(3) (1978), 296–307. doi:10.1016/0097-3165(78)90059-6.
- [2] Justin Salez. *Sparse Random Graphs: Structure & Dynamics*. Habilitation thesis, 2016. Sections 1.1–1.2, pp. 11–14. <https://www.ceremade.dauphine.fr/~salez/hdr.pdf>.
- [3] Gabriel Spessato Agostini. *Benjamini-Schramm Convergence of Random Rooted Graphs*. Topology course project, Fall 2021. <https://www.uvm.edu/~smillere/TProjects/GAgostini21f.pdf>.
- [4] Alan M. Frieze. *On Random Regular Graphs with Non-Constant Degree*. Research Report No. 88-2, Department of Mathematics, Carnegie Mellon University, February 1988. Section 2. <https://www.math.cmu.edu/~af1p/Texfiles/REGNONCONSTANT.pdf>.
- [5] Noga Alon and Joel H. Spencer. *The Probabilistic Method*, 4th edition. John Wiley & Sons, 2016. Section 8.3, especially Theorem 8.3.1, p. 133.

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