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Paper Title: Asymptotic Enumeration of Pants Decomposition
Types with a Fixed Number of Separating Curves

Asymptotic Enumeration of Pants Decomposition Types with a Fixed Number of Separating Curves

Chang Gao

Abstract

Let Σ_g be a closed orientable surface of genus $g \geq 2$. Write $\mathcal{P}(g)$ for its pants decomposition types, where a type is a mapping class group orbit of ambient isotopy classes of pants decompositions. Let $\mathcal{P}_k(g) \subseteq \mathcal{P}(g)$ consist of the types with exactly k separating curves. For each fixed integer $k \geq 0$, we prove that

$$|\mathcal{P}_k(g)| \sim \frac{2^{k+1}}{3k! \sqrt{2\pi} g^{3/2}} \left(\frac{3g}{2e} \right)^g \quad (g \rightarrow \infty).$$

If a type is chosen uniformly from $\mathcal{P}(g)$, its number of separating curves converges in distribution to Poisson(2). For fixed k , the proportion of types in $\mathcal{P}_k(g)$ for which every separating curve cuts off a one-holed torus tends to one. The proof uses dual cubic multigraphs, where separating curves correspond to bridges. A degree-sequence count gives the main term. Separate estimates handle connectivity and automorphism groups.

Keywords: pants decomposition; mapping class group; separating curve; cubic multigraph.

1 Introduction

A pants decomposition cuts a closed orientable surface into pairs of pants. Once a decomposition is fixed, the lengths and twists of its curves give Fenchel–Nielsen coordinates on Teichmüller space [1, 2]. Wolpert used these coordinates to express the Weil–Petersson symplectic form [2].

Hatcher defines the pants complex using pants decompositions as vertices and elementary local moves as edges, and proves that it is simply connected [3]. Brock proved that the pants graph is quasi-isometric to Teichmüller space with the Weil–Petersson metric and related this geometry to the convex-core volumes of quasi-Fuchsian three-manifolds [5, Theorems 1.1 and 1.2]. Mirzakhani’s generalized McShane identity gives recursive formulas for Weil–Petersson volumes. These formulas sum over simple closed geodesics that, together with boundary components, bound pairs of pants [6, Theorem 1.3 and Section 5].

For fixed genus, the mapping class group has only finitely many orbits of pants decompositions. The associated dual cubic multigraph is obtained by cutting the surface, assigning a vertex to each resulting pair of pants, and assigning an edge to each decomposition curve; loops and multiple edges are permitted. This construction identifies mapping class group orbits with isomorphism classes of unlabeled cubic multigraphs [8, Sections 2.3 and 6].

Wormald gave the asymptotic count for vertex-labeled cubic multigraphs with loops and multiple edges [12]. Parlier and Petri reproved the corresponding unlabeled count and applied it to the graph genus of the modular pants graph [8, Theorems 3.5 and 6.2]. Sultan used the correspondence between separating curves and bridges in the dual graph [10, Section 3.3]. Chae obtained asymptotic counts and connectivity estimates for vertex-labeled cubic multigraphs with a fixed number of loops [13, Equation (4.2) and Theorem 4].

Our count fixes the number of bridges in the dual graph. Loops force the pendant bridges described in [definition 6.1](#); these remain in the leading term, while graphs with a non-pendant bridge make a lower-order contribution. Chae proved the corresponding estimate for loop-free vertex-labeled graphs [[13](#), pp. 1199–1201, Proposition 4]. Here loops are allowed and the graphs are unlabeled. This gives the main asymptotic formula. It also shows that, for fixed k , the proportion of types in $\mathcal{P}_k(g)$ for which every separating curve cuts off a one-holed torus tends to one.

We also find that the limiting proportion of graphs with at least one bridge is $1 - e^{-2}$, not the value $1 - e^{-1}$ reported by Grave de Peralta and Kolpakov [[9](#), Section 3.2, p. 1830]; see [section 9](#) for the correction.

2 Main Results

For $g \geq 2$, let $\mathcal{P}(g)$ be the set of mapping class group orbits of ambient isotopy classes of pants decompositions of Σ_g , and set $P(g) := |\mathcal{P}(g)|$; see [definitions 4.2](#) and [4.3](#).

Definition 2.1. The number of separating curves is invariant under homeomorphisms, so it is well defined for each type $[\Gamma]$. For $g \geq 2$ and $k \geq 0$, define

$$\mathcal{P}_k(g) := \{[\Gamma] \in \mathcal{P}(g) : \Gamma \text{ has } k \text{ separating curves}\}, \quad P_k(g) := |\mathcal{P}_k(g)|.$$

Theorem 2.2 (Main Theorem). For each fixed integer $k \geq 0$,

$$P_k(g) \sim \frac{2^k}{k!} \frac{(6g-7)!!}{6^{2g-2}(2g-2)!} \quad (g \rightarrow \infty). \quad (2.1)$$

Equivalently,

$$P_k(g) \sim \frac{2^{k+1}}{3k! \sqrt{2\pi} g^{3/2}} \left(\frac{3g}{2e}\right)^g \quad (g \rightarrow \infty). \quad (2.2)$$

Combining the total count for connected unlabeled cubic multigraphs in [equation \(5.7\)](#) with the dual-graph bijection in [proposition 4.5](#) yields

$$P(g) \sim \frac{2e^2}{3\sqrt{2\pi} g^{3/2}} \left(\frac{3g}{2e}\right)^g \quad (g \rightarrow \infty). \quad (2.3)$$

The same asymptotic formula was obtained by Parlier and Petri for all unlabeled cubic multigraphs, without imposing connectivity [[8](#), Theorem 3.5]. The argument for [proposition 5.1](#) supplies the connectivity estimate and justifies the removal of labels. Comparison with [theorem 2.2](#) then yields the following limit law.

Corollary 2.3. Let X_g be uniformly distributed on the finite set $\mathcal{P}(g)$, and let K_g be the number of separating curves in the type X_g . For every fixed $k \geq 0$,

$$\mathbb{P}(K_g = k) \longrightarrow e^{-2} \frac{2^k}{k!} \quad (g \rightarrow \infty).$$

Thus, K_g converges in distribution to Poisson(2) as $g \rightarrow \infty$.

Corollary 2.4 (Typical subsurfaces cut off by separating curves). Fix $k \geq 0$. Let $\mathcal{E}_k(g) \subseteq \mathcal{P}_k(g)$ be the set of types for which at least one separating curve does not cut off a one-holed torus. Then

$$\frac{|\mathcal{E}_k(g)|}{P_k(g)} = O_k(g^{-1}).$$

Thus, as $g \rightarrow \infty$, the proportion of types in $\mathcal{P}_k(g)$ for which every separating curve cuts off a one-holed torus tends to 1.

3 Proof Structure

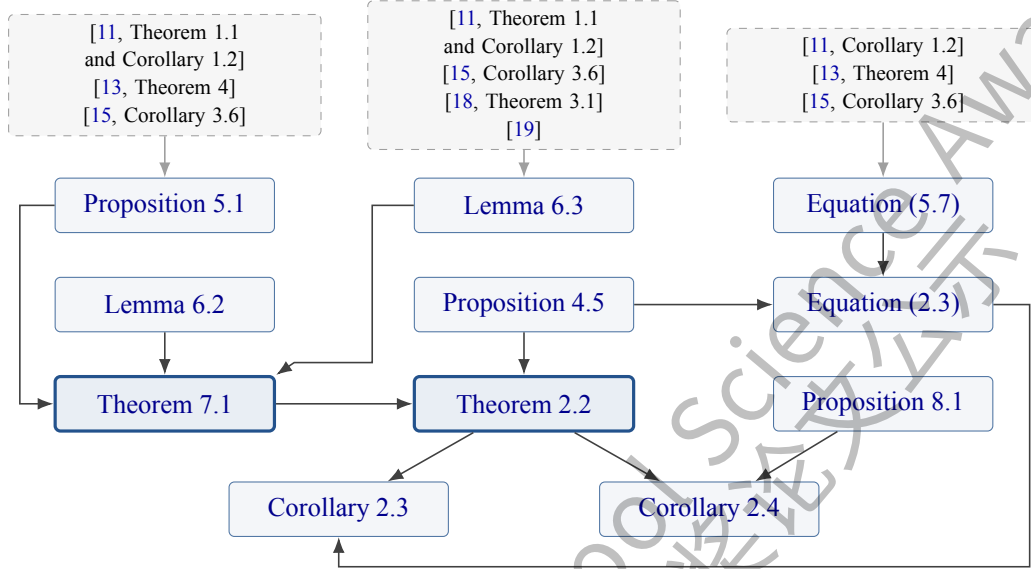


Figure 1: Structure of the proof. Each gray dashed box lists external results used directly by the blue box below it. Solid arrows show direct dependencies among results proved in this paper.

4 Preliminaries

This section fixes the surface and graph notation used throughout the paper. The map Φ_g links pants decomposition types with the corresponding graph classes, allowing graph estimates to be transferred directly to surface counts.

4.1 Group Actions, Surfaces, and Pants Decompositions

Definition 4.1 (Groups, group actions, and orbits). A set H with a binary operation is a *group* if the operation is associative, there is an identity element, and every element has an inverse. A subset $N \subseteq H$ is a *subgroup* of H if it is itself a group under the same operation. If, in addition, $hNh^{-1} = N$ for every $h \in H$, then N is a *normal subgroup*, written $N \trianglelefteq H$. In this case, the set of left cosets

$$H/N := \{hN : h \in H\}$$

is the *quotient group* under the multiplication $(h_1N)(h_2N) := (h_1h_2)N$.

Let X be a set. A map $H \times X \rightarrow X$, $(h, x) \mapsto h \cdot x$, is a *left action* of H on X if $e \cdot x = x$ and $(h_1h_2) \cdot x = h_1 \cdot (h_2 \cdot x)$. For $x \in X$, the set

$$H \cdot x := \{h \cdot x : h \in H\}$$

is the *orbit* of x , and the subgroup

$$H_x := \{h \in H : h \cdot x = x\}$$

is the *stabilizer* of x . The set of all orbits is the *orbit space*, written $H \backslash X$.

The orbit–stabilizer theorem [20, Chapter 1, Section 7] says that if a finite group H acts on a set X , then

$$|H \cdot x| = \frac{|H|}{|H_x|}$$

for every $x \in X$.

Every connected closed orientable surface is homeomorphic to a connected sum of g tori for a unique integer $g \geq 0$; see, for example, Munkres [7, Chapter 12]. We denote this surface by Σ_g and call g its *genus*. Unless stated otherwise, $g \geq 2$ throughout. An orientable surface of genus 1 with exactly one boundary component is called a *one-holed torus*.

Definition 4.2 (Pants decomposition). A *pair of pants*, denoted by Y , is a sphere with three pairwise disjoint open disks removed. A simple closed curve on Σ_g is *essential* if it is not contractible. A *pants decomposition* is a finite set $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of pairwise disjoint, pairwise non-isotopic essential simple closed curves such that every connected component obtained by cutting Σ_g along Γ is a pair of pants. A curve $\gamma \in \Gamma$ is a *separating curve* if $\Sigma_g \setminus \gamma$ is disconnected. A concrete example on Σ_3 is shown in figure 2.

This is equivalent to the standard definition in terms of a maximal multicurve [3, 4]. We determine both $|\Gamma|$ and the number of complementary pairs of pants in proposition 4.5.

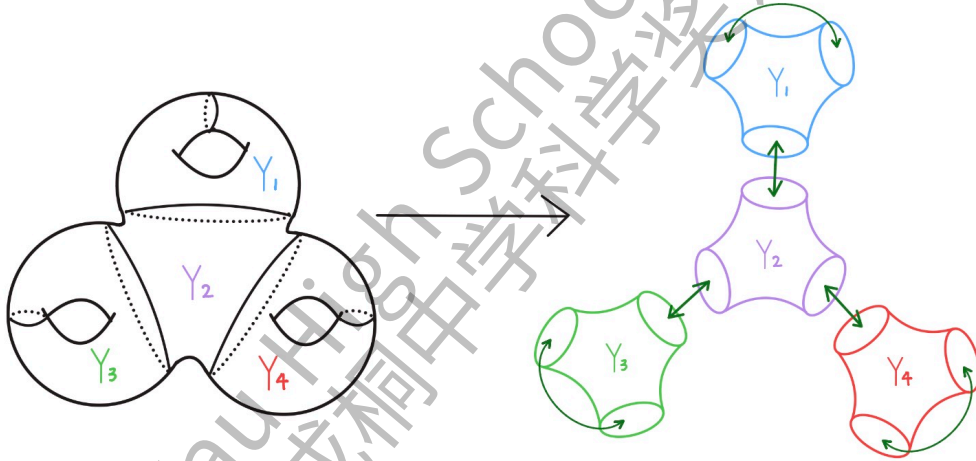


Figure 2: A pants decomposition of a closed surface of genus 3.

Definition 4.3 (Equivalence of pants decompositions). Two pants decompositions Γ_1 and Γ_2 are *ambient isotopic* if there is a family of surface homeomorphisms F_t , depending continuously on $t \in [0, 1]$, such that $F_0 = \text{id}_{\Sigma_g}$ and $F_1(\Gamma_1) = \Gamma_2$. Let $\text{PD}(\Sigma_g)$ be the set of *ambient isotopy classes* of pants decompositions.

The *homeomorphism group* $\text{Homeo}(\Sigma_g)$ consists of all homeomorphisms of Σ_g under composition, and its orientation-preserving elements form the subgroup $\text{Homeo}^+(\Sigma_g)$. Let $\text{Homeo}_0(\Sigma_g)$ consist of the homeomorphisms in $\text{Homeo}^+(\Sigma_g)$ that are isotopic to the identity. Since this subgroup is normal in $\text{Homeo}^+(\Sigma_g)$, the quotient-group definition in definition 4.1 gives the *mapping class group*

$$\text{Mod}(\Sigma_g) := \text{Homeo}^+(\Sigma_g) / \text{Homeo}_0(\Sigma_g).$$

This quotient identifies orientation-preserving homeomorphisms that are isotopic. Its natural action on $\text{PD}(\Sigma_g)$ is

$$[f] \cdot [\Gamma] := [f(\Gamma)].$$

The set of *pants decomposition types* is the orbit space

$$\mathcal{P}(g) := \text{Mod}(\Sigma_g) \backslash \text{PD}(\Sigma_g).$$

Two levels of quotient are involved. In the action formula, $[\Gamma]$ denotes an ambient isotopy class; in an expression such as $[\Gamma] \in \mathcal{P}(g)$, it denotes the corresponding mapping class group orbit. Two ambient isotopy classes are *equivalent* precisely when they lie in the same orbit, or, equivalently, when an orientation-preserving homeomorphism sends a representative of one to a representative of the other.

4.2 Cubic Multigraphs

Definition 4.4 (Multigraph terminology). A finite undirected multigraph G may have several edges between two vertices. Such edges are called *multiple edges*, or *parallel edges*. An edge whose two endpoints are the same is called a *loop*. A *half-edge* is one incidence of an edge with a vertex. If an edge e has endpoints u, v , where $u = v$ is allowed, then e has two half-edges, one incident to u and one incident to v . If $u = v$, both half-edges are incident to the same vertex. For each vertex v of G , its *degree* $\deg_G(v)$ is the number of half-edges incident to v . The graph G is *cubic*, or *3-regular*, if every vertex has degree 3. An edge is a *bridge* if deleting it increases the number of connected components.

An isomorphism between two multigraphs is a bijection between their vertex sets that preserves the number of edges between each pair of distinct vertices and the number of loops at each vertex. An isomorphism from G to itself is a *graph automorphism*. These automorphisms form the *automorphism group* $\text{Aut}(G)$ under composition. Let $\ell(G)$ and $b(G)$ denote the numbers of loops and bridges in G , respectively. For distinct vertices u, v , let $m_{uv}(G)$ be the number of edges between them. An unordered pair $\{u, v\}$ with $m_{uv}(G) = 2$ is called a *double-edge pair*. Define

$$d(G) := \#\{\{u, v\} : u \neq v, m_{uv}(G) = 2\}.$$

Thus, $d(G)$ counts double-edge pairs, not the two edges in each pair separately. An isomorphism class of graphs is called an *unlabeled graph*. To simplify notation, we use G for both a graph and its isomorphism class. For a positive integer V , let

$$\begin{aligned} \mathcal{G}(V) &:= \{G : G \text{ is an unlabeled cubic multigraph on } V \text{ vertices}\}, \\ \mathcal{G}^{\text{con}}(V) &:= \{G \in \mathcal{G}(V) : G \text{ is connected}\}. \end{aligned}$$

For $k \geq 0$, let

$$\mathcal{G}_{b=k}^{\text{con}}(V) := \{G \in \mathcal{G}^{\text{con}}(V) : b(G) = k\}.$$

Proposition 4.5 (Dual-graph correspondence and parameters). Every pants decomposition of Σ_g contains $2g - 2$ pairs of pants and $3g - 3$ decomposition curves. The construction that assigns a vertex to each pair of pants and an edge to each decomposition curve gives a bijection

$$\Phi_g : \mathcal{P}(g) \longrightarrow \mathcal{G}^{\text{con}}(2g - 2), \quad \Phi_g([\Gamma]) = G_\Gamma.$$

Figure 3 gives a schematic example of the map Φ_g when $g = 2$. Under this bijection,

$$\#\{\text{separating curves in } \Gamma\} = b(G_\Gamma). \quad (4.1)$$

Therefore, Φ_g induces a bijection between $\mathcal{P}_k(g)$ and $\mathcal{G}_{b=k}^{\text{con}}(2g - 2)$. Conversely, the same gluing construction associates every connected cubic multigraph with a closed connected orientable surface Σ_g . If the graph has V vertices, then $V = 2g - 2$, so V must be even. If $G \in \mathcal{G}(V)$ is disconnected, apply this gluing

construction to each connected component. The result is a finite disjoint union of closed orientable surfaces. Every component has an even number of vertices, so the total number V is still even.

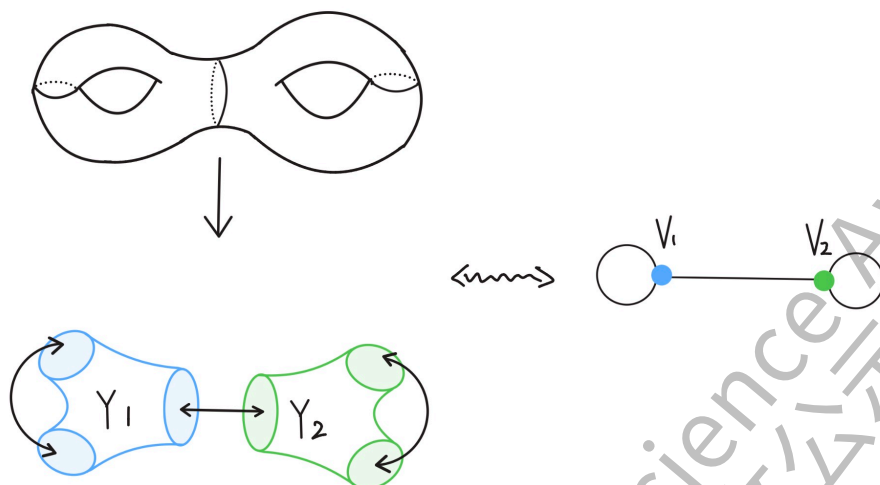


Figure 3: A schematic example of the map Φ_2 . The two pairs of pants correspond to the vertices v_1, v_2 of the dual cubic multigraph.

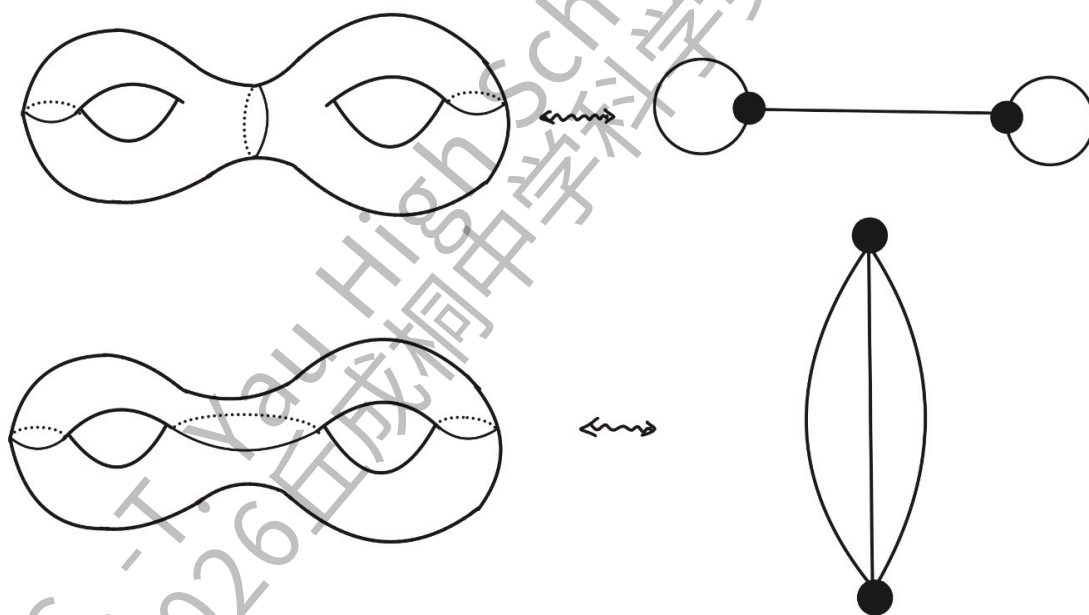


Figure 4: The two pants decomposition types on a closed surface of genus 2 and their corresponding cubic multigraphs.

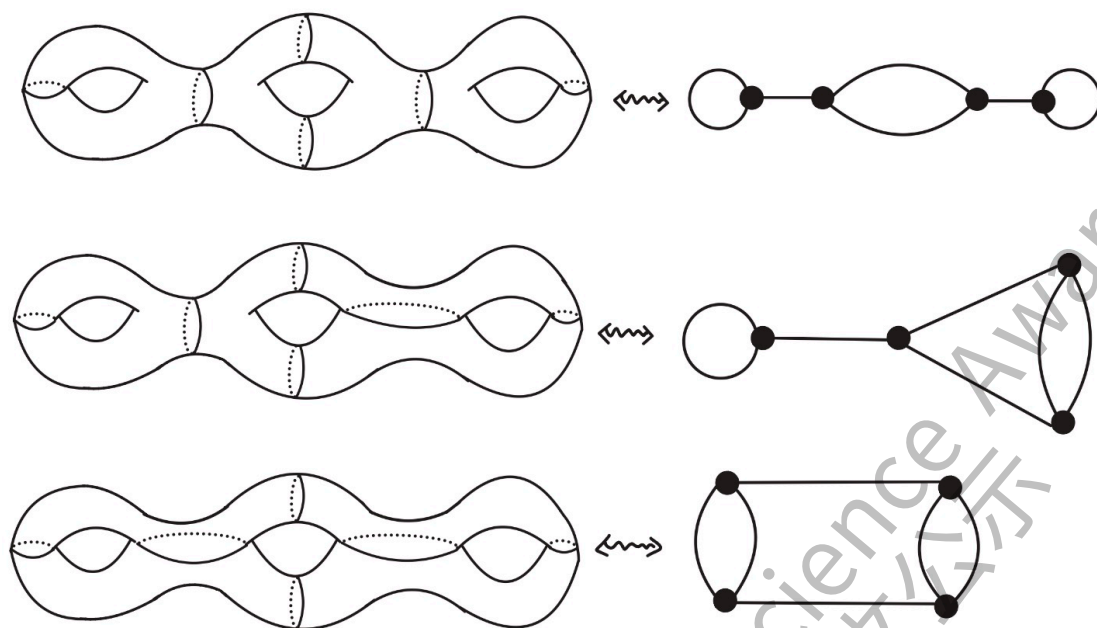


Figure 5: Pants decomposition types on a closed surface of genus 3 and their corresponding cubic multi-graphs, Part I: three of the five types.

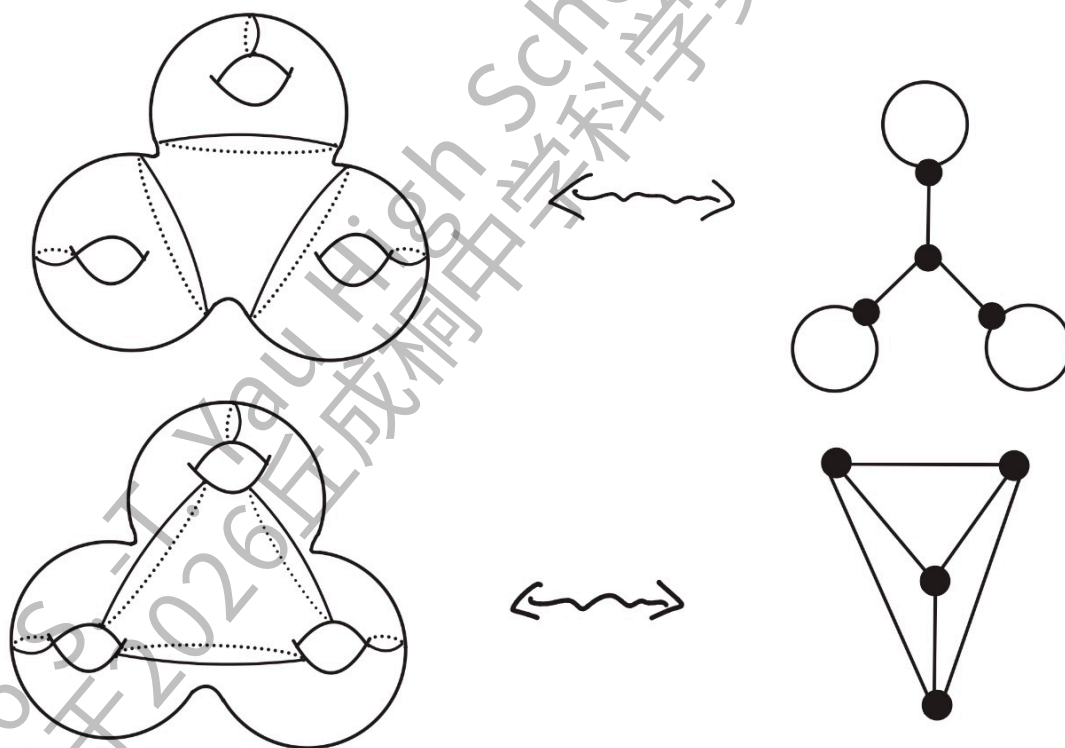


Figure 6: Pants decomposition types on a closed surface of genus 3 and their corresponding cubic multi-graphs, Part II: the remaining two types.

Proof. The parameters follow from Euler characteristic. For the pair of pants Y in [definition 4.2](#), one has $\chi(Y) = -1$. Let S^1 denote the circle. Since $\chi(S^1) = 0$, Euler characteristic is additive under gluing along boundary circles. If cutting along the pants decomposition gives V pairs of pants, then

$$2 - 2g = \chi(\Sigma_g) = V\chi(Y) = -V,$$

so $V = 2g - 2$. The V pairs of pants have $3V$ boundary circles altogether, and every decomposition curve pairs two of them. The number of decomposition curves is consequently $3V/2 = 3g - 3$.

The bijection is the standard correspondence described by Parlier and Petri [8, Sections 2.3 and 6]. Once the surface is cut along Γ , each pair of pants represents a vertex and its three boundary circles represent the incident half-edges. The pairing of two boundary circles along a decomposition curve joins their half-edges to form an edge. Two circles from the same pair of pants give a loop, while several curves joining the same two pairs of pants give multiple edges. Connectedness of Σ_g ensures that G_Γ is connected.

For the inverse construction, start with a connected cubic multigraph on V vertices and replace each vertex by a pair of pants. Along every edge, glue the corresponding boundary circles by an orientation-reversing homeomorphism. The resulting surface is closed, connected, and orientable, with Euler characteristic $-V$. The classification of closed connected orientable surfaces then provides an integer $g' \geq 2$ such that $-V = 2 - 2g'$, or $V = 2g' - 2$. In particular, a connected cubic multigraph has an even number of vertices. In the present inverse construction $V = 2g - 2$, hence $g' = g$, and the surface is homeomorphic to Σ_g .

The seams formed by the glued circles are essential and pairwise non-isotopic. Indeed, cutting along all of them recovers the pairs of pants. An innermost contractible seam would instead produce a disk component, and an adjacent pair of isotopic seams would produce an annulus component. Both possibilities contradict the description of every component as a pair of pants. Hence the seams form a pants decomposition.

The construction is independent of its auxiliary choices. Any two orientation-reversing gluing homeomorphisms between circles are isotopic, and the isotopy extends to a boundary collar; changing the gluing maps therefore leaves the pants decomposition type unchanged. Every permutation of the boundary components of a pair of pants is also realized by an orientation-preserving homeomorphism. Given a graph isomorphism, choose such a homeomorphism on each pair of pants to realize the induced permutation of boundary components, and adjust it in boundary collars to agree with the gluing maps. The resulting local homeomorphisms combine into an orientation-preserving homeomorphism between the resulting surfaces. Every graph isomorphism therefore lifts, and the construction just described is an inverse of Φ_g . In particular, Φ_g is well defined and bijective.

To identify separating curves, let γ correspond to the edge e . After cutting only along γ , the remaining pairs of pants are still glued according to the edges of $G_\Gamma - e$. The connected components of the cut surface therefore correspond exactly to those of $G_\Gamma - e$. Thus Σ_g cut along γ is disconnected if and only if deleting e disconnects the graph, which proves [equation \(4.1\)](#). $\square$

Remark 4.6 (Low-genus pants decomposition types). For $g = 2$, the two pants decomposition types and their dual cubic multigraphs appear in [figure 4](#). For $g = 3$, the five types and their corresponding graphs are divided between [figures 5](#) and [6](#). The genus-3 list is split only to keep each diagram readable.

4.3 Three Counting Models and Their Weights

By [proposition 4.5](#), the surface problem is a counting problem for unlabeled isomorphism classes of connected cubic multigraphs. The enumeration and automorphism estimates in Sections 5 and 6 are carried out initially for vertex-labeled graphs, while Section 9 requires a careful comparison with the probability

spaces used in earlier work. Since “uniform” refers to three different distributions here, we record the three finite sample spaces explicitly. In each case, the probability of an event is the size of the relevant subfamily divided by the size of the sample space. For a positive even integer V , fix the vertex-label set

$$\mathcal{V} := \{1, \dots, V\}.$$

For a vertex-labeled graph, its underlying abstract vertex set may be written as $V(G) = \{v_1, \dots, v_V\}$, distinct in principle from the label set $\mathcal{V}$. A label-preserving isomorphism fixes every vertex label, so each abstract vertex is uniquely identified with its label. We make this identification from this point onward and write $V(G) = \mathcal{V}$. The symbol G will also stand for the label-preserving isomorphism class of a vertex-labeled graph. Half-edge labels enter only in the pairing model below.

Definition 4.7 (Three finite counting models).

1. *Uniform unlabeled model*: With the notation of [definition 4.4](#), the sample space is $\mathcal{G}(V)$, and every unlabeled graph G has the same probability.
2. *Uniform vertex-labeled model*: Let $\mathcal{M}(\mathcal{V})$ be the set of label-preserving isomorphism classes of all vertex-labeled cubic multigraphs with vertex-label set $\mathcal{V}$, and let $M(V) := |\mathcal{M}(\mathcal{V})|$. The distribution on the finite set $\mathcal{M}(\mathcal{V})$ is uniform.
3. *Uniform vertex-labeled half-edge pairing model*: Place three distinct half-edges (v, h) at each labeled vertex $v \in \mathcal{V}$, where $h \in \{1, 2, 3\}$. Let $\Omega(\mathcal{V})$ be the set of all pairings of these $3V$ labeled half-edges, with the uniform distribution on its $|\Omega(\mathcal{V})| = (3V - 1)!!$ elements. Each pair $\{(u, h), (v, h')\}$ produces an edge between u and v . If $u = v$, this edge is a loop. Forgetting the half-edge labels gives a vertex-labeled cubic multigraph. Both the vertex label v and the half-edge label h are part of the sample data.

The two forgetful maps fit into the projection chain

$$\Omega(\mathcal{V}) \xrightarrow{\pi_1} \mathcal{M}(\mathcal{V}) \xrightarrow{\pi_2} \mathcal{G}(V). \quad (4.2)$$

The map π_1 forgets the half-edge labels, whereas π_2 forgets the vertex labels. This chain is illustrated in [Figure 7](#).

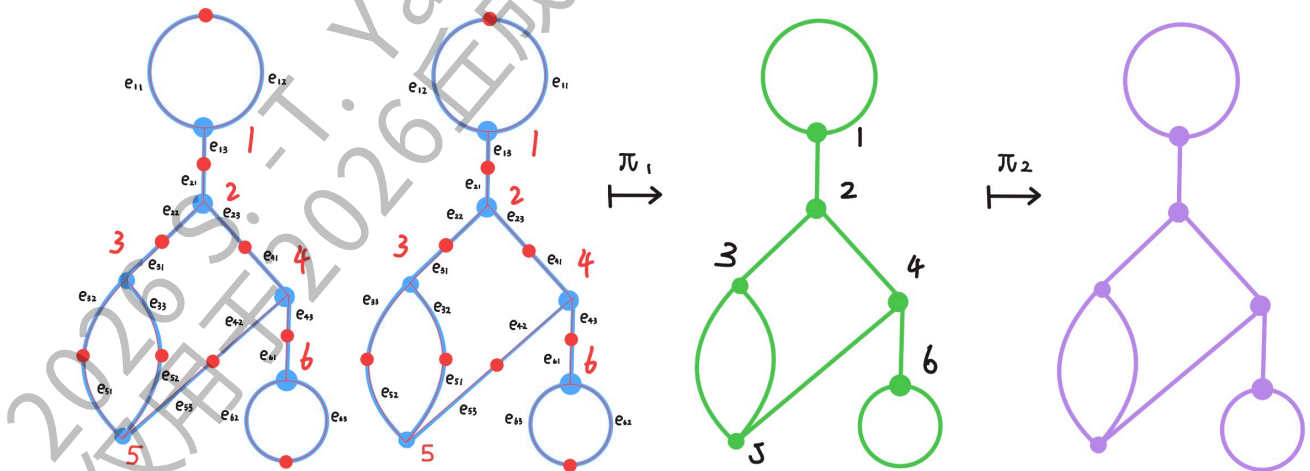


Figure 7: $\Omega(\mathcal{V}) \xrightarrow{\pi_1} \mathcal{M}(\mathcal{V}) \xrightarrow{\pi_2} \mathcal{G}(V)$

Weights in the uniform vertex-labeled half-edge pairing model. For $G \in \mathcal{M}(\mathcal{V})$, retain the notation $m_{uv}(G)$ from [definition 4.4](#), and let $m_{vv}(G)$ count the loops at v . The fiber of π_1 in [equation \(4.2\)](#) can be counted directly. At each vertex there are $3!$ bijections between the three labeled half-edges and the three half-edges of G incident to that vertex, giving $(3!)^V$ initial assignments. The final pairing is unchanged when the two half-edges of a loop are swapped, the $m_{vv}(G)$ loops at a vertex are permuted, or the $m_{uv}(G)$ parallel edges between u and v are permuted. It follows that

$$\begin{aligned} |\pi_1^{-1}(G)| &= \frac{(3!)^V}{\prod_{v \in \mathcal{V}} 2^{m_{vv}(G)} m_{vv}(G)! \prod_{\substack{u, v \in \mathcal{V} \\ u < v}} m_{uv}(G)!} \\ &= \frac{6^V}{2^{\ell(G)} \prod_{\substack{u, v \in \mathcal{V} \\ u < v}} m_{uv}(G)!}. \end{aligned} \quad (4.3)$$

For the second equality, the fact that G is cubic implies that each vertex supports at most one loop. Hence $m_{vv}(G) \in \{0, 1\}$, $m_{vv}(G)! = 1$, and $\sum_{v \in \mathcal{V}} m_{vv}(G) = \ell(G)$.

Remark 4.8 (Multiplicity of parallel edges in cubic graphs). Let G be a connected cubic multigraph with more than two vertices. For distinct vertices u, v , if $m_{uv}(G) > 0$, then $m_{uv}(G) \in \{1, 2\}$. If $m_{uv}(G) = 3$, the three parallel edges exhaust the degrees of u and v , making $\{u, v\}$ a connected component. Connectedness would then force G to have exactly two vertices. When G is connected and has more than two vertices, it follows that

$$\prod_{u < v} m_{uv}(G)! = 2^{d(G)}.$$

Connectivity is essential here: a disconnected cubic multigraph with more than two vertices may still contain a component consisting of two vertices joined by three parallel edges.

If G is connected and $V > 2$, [remark 4.8](#) simplifies the general fiber formula to

$$|\pi_1^{-1}(G)| = \frac{6^V}{2^{\ell(G)+d(G)}}.$$

Let W be uniformly distributed on $\Omega(\mathcal{V})$, and set $G_{\text{pair}} := \pi_1(W)$. Its distribution is

$$\mathbb{P}_{\text{pair}}(G) := \mathbb{P}(G_{\text{pair}} = G) = \frac{|\pi_1^{-1}(G)|}{|\Omega(\mathcal{V})|} = \frac{6^V}{(3V-1)!!} \frac{1}{2^{\ell(G)} \prod_{\substack{u, v \in \mathcal{V} \\ u < v}} m_{uv}(G)!}. \quad (4.4)$$

For connected G with $V > 2$, the same probability can be written as

$$\mathbb{P}_{\text{pair}}(G) = \frac{6^V}{(3V-1)!! 2^{\ell(G)+d(G)}}.$$

The uniform vertex-labeled model, by comparison, assigns probability $1/M(V)$ to every $G \in \mathcal{M}(\mathcal{V})$. Formula [equation \(4.4\)](#) shows that a loop contributes a factor $1/2$ to the pairing weight. For connected G with $V > 2$, each double-edge pair also contributes a factor $1/2$; in the general disconnected sample space, three parallel edges contribute $1/6$. The pairing and vertex-labeled models therefore differ as soon as loops or multiple edges are admitted.

Janson, Łuczak, and Ruciński work with the uniform vertex-labeled half-edge pairing model in Chapter 9 [17, Chapter 9]. Grave de Peralta and Kolpakov instead treated the probability of having no loops

in this model as if it were the corresponding probability in the uniform vertex-labeled graph model. This substitution produces an incorrect limiting constant for the proportion of graphs with a bridge. In [section 9](#), we locate the error and explain the difference between the two probabilities.

Transfer from the uniform vertex-labeled model to the uniform unlabeled model. Relabeling vertices defines an action of the symmetric group $\text{Sym}(\mathcal{V})$ on $\mathcal{M}(\mathcal{V})$. Fix $H \in \mathcal{G}(V)$ and choose a vertex-labeled representative $\tilde{G} \in \pi_2^{-1}(H)$. By [equation \(4.2\)](#), $\pi_2^{-1}(H)$ is precisely the relabeling orbit:

$$\pi_2^{-1}(H) = \text{Sym}(\mathcal{V}) \cdot \tilde{G}.$$

The stabilizer is naturally isomorphic to the automorphism group of the unlabeled graph H , so the orbit–stabilizer theorem gives

$$|\text{Sym}(\mathcal{V}) \cdot \tilde{G}| = \frac{|\text{Sym}(\mathcal{V})|}{|\text{Aut}(H)|} = \frac{V!}{|\text{Aut}(H)|}.$$

After restriction to connected graphs, pushing the uniform vertex-labeled model forward under π_2 assigns an unlabeled graph $H \in \mathcal{G}^{\text{con}}(V)$ a weight proportional to $1/|\text{Aut}(H)|$. The uniform unlabeled model assigns equal weight to every such graph instead. The counts in [Sections 5 and 6](#) are obtained in $\mathcal{M}(\mathcal{V})$; quantitative estimates of McKay and Wormald for automorphism group orders then control the passage from vertex-labeled to unlabeled graphs.

Definition 4.9 (Counting scales). Define the basic *unlabeled scale*

$$A(V) := \frac{(3V-1)!!}{6^V V!}, \quad (4.5)$$

and the corresponding *vertex-labeled scale*

$$Q(V) := V!A(V) = \frac{(3V-1)!!}{6^V}. \quad (4.6)$$

4.4 Asymptotic Conventions

Definition 4.10 (Asymptotic notation). Unless stated otherwise, all asymptotic statements involving the graph size V are taken as V tends to infinity through even integers. In the surface application, $V = 2g - 2$, so V is even. The integer k remains fixed. For functions f and h that are eventually positive, define

$$f(V) \sim h(V) \iff \frac{f(V)}{h(V)} \longrightarrow 1 \quad (V \rightarrow \infty, V \text{ even}).$$

We write $f(V) \asymp h(V)$ if there are constants $c, C > 0$ such that $ch(V) \leq f(V) \leq Ch(V)$ for every sufficiently large even V . We write $f(V) = O(h(V))$ if $|f(V)| \leq C|h(V)|$ eventually, and $f(V) = o(h(V))$ if $f(V)/h(V) \rightarrow 0$ as $V \rightarrow \infty$ through even integers. The constants c, C may change from one line to the next, but they are independent of the parameter that tends to infinity. They may depend on fixed parameters such as k . We write $O_k(\cdot)$ when this dependence should be shown explicitly.

Theorem (Markov’s inequality; see Mitzenmacher–Upfal [18], Theorem 3.1). If Y is a nonnegative random variable and $a > 0$, then

$$\mathbb{P}(Y \geq a) \leq \frac{\mathbb{E}Y}{a}. \quad (4.7)$$

Theorem (Robbins’ two-sided Stirling bounds [19]). For every positive integer a ,

$$\sqrt{2\pi} a^{a+1/2} e^{-a+1/(12a+1)} < a! < \sqrt{2\pi} a^{a+1/2} e^{-a+1/(12a)}.$$

In particular, there are absolute constants $c_0, C_0 > 0$ such that

$$c_0 \sqrt{a} \left(\frac{a}{e}\right)^a \leq a! \leq C_0 \sqrt{a} \left(\frac{a}{e}\right)^a \quad (a \geq 1). \quad (4.8)$$

5 Cubic Multigraphs with a Fixed Number of Loops

Direct enumeration by the number of bridges is awkward, since the bridge condition depends on the global connectivity of the graph. Loops, by contrast, are local: deleting them reduces the problem to one with a prescribed degree sequence. We begin with this simpler parameter and later show, in [section 6](#), that fixed loop counts supply the leading term for fixed bridge counts.

For an even integer V and an integer $k \geq 0$, define

$$\mathcal{G}_{\ell=k}^{\text{con}}(V) := \{G \in \mathcal{G}^{\text{con}}(V) : \ell(G) = k\}.$$

Let $\vec{\text{deg}}$ be an ordered degree vector indexed by a fixed set of vertex labels $\mathcal{V}$. We denote by $M(\vec{\text{deg}})$ the number of vertex-labeled multigraphs with this degree vector, allowing loops and multiple edges, and by $M_{\ell=0}(\vec{\text{deg}})$ the corresponding loop-free count. No cubicity condition is imposed: these graphs are cubic precisely when every entry of the degree vector is 3. For a fixed $k \geq 0$, define

$$\vec{\text{deg}}_k := (\underbrace{1, \dots, 1}_{k \text{ entries}}, \underbrace{3, \dots, 3}_{V-k \text{ entries}}). \quad (5.1)$$

The specified k labels have been placed first only for convenience; any other fixed set of k labels gives the same count.

Theorem (Greenhill–McKay [11], special case of Theorem 1.1). For every fixed k , as $V \rightarrow \infty$ through even values,

$$M_{\ell=0}(\vec{\text{deg}}_k) = (1 + O_k(V^{-1})) \frac{(3V - 2k - 1)!!}{6^{V-k}}. \quad (5.2)$$

Theorem (Greenhill–McKay [11], Corollary 1.2). As $V \rightarrow \infty$ through even values,

$$M(V) = e^{2+O(V^{-1})} Q(V) \sim e^2 Q(V). \quad (5.3)$$

Theorem (McKay–Wormald [15], Corollary 3.6). If G is sampled from the uniform vertex-labeled model, then

$$\mathbb{E}(|\text{Aut}(\pi_2(G))| - 1) = O(V^{-1}), \quad \sum_{G \in \mathcal{M}(\mathcal{V})} (|\text{Aut}(\pi_2(G))| - 1) = O\left(\frac{M(V)}{V}\right). \quad (5.4)$$

Let

$$\mathcal{M}^{\text{dis}}(\mathcal{V}) := \{G \in \mathcal{M}(\mathcal{V}) : G \text{ is disconnected}\}.$$

Theorem (Chae [13], Theorem 4). As $V \rightarrow \infty$ through even values,

$$|\mathcal{M}^{\text{dis}}(\mathcal{V})| = o(M(V)). \quad (5.5)$$

Together with $M(V)/Q(V) \rightarrow e^2$ from [equation \(5.3\)](#), this estimate yields

$$\frac{|\mathcal{M}^{\text{dis}}(\mathcal{V})|}{Q(V)} = \frac{|\mathcal{M}^{\text{dis}}(\mathcal{V})|}{M(V)} \frac{M(V)}{Q(V)} \longrightarrow 0 \cdot e^2 = 0.$$

Consequently, $|\mathcal{M}^{\text{dis}}(\mathcal{V})| = o(Q(V))$.

Proposition 5.1 (A fixed number of loops). For every fixed integer $k \geq 0$,

$$|\mathcal{G}_{\ell=k}^{\text{con}}(V)| \sim \frac{2^k}{k!} A(V) \quad (V \rightarrow \infty, V \text{ even}). \quad (5.6)$$

Moreover,

$$|\mathcal{G}^{\text{con}}(V)| \sim e^2 A(V) \quad (V \rightarrow \infty, V \text{ even}). \quad (5.7)$$

Proof. To determine the constant $2^k/k!$, let

$$\mathcal{M}_{\ell=k}(V) := \{G \in \mathcal{M}(\mathcal{V}) : \ell(G) = k\}, \quad \mathcal{M}_{\ell=k}^{\text{con}}(V) := \{G \in \mathcal{M}_{\ell=k}(V) : G \text{ is connected}\}.$$

A degree-3 vertex cannot support two loops, which would already contribute 4 to its degree. Fix a k -element subset $L \subseteq \mathcal{V}$. Deleting the loop at every vertex of L identifies graphs with loop-bearing set exactly L with loop-free multigraphs having degree 1 on L and degree 3 elsewhere. Relabeling shows that this family has size $M_{\ell=0}(\overrightarrow{\deg}_k)$, independently of the choice of L , and hence

$$|\mathcal{M}_{\ell=k}(V)| = \binom{V}{k} M_{\ell=0}(\overrightarrow{\deg}_k). \quad (5.8)$$

Applying [equation \(5.2\)](#) to [equation \(5.8\)](#) and normalizing by $Q(V)$ from [equation \(4.6\)](#) yields

$$\begin{aligned} \frac{|\mathcal{M}_{\ell=k}(V)|}{Q(V)} &= (1 + o(1)) \binom{V}{k} 6^k \frac{(3V - 2k - 1)!!}{(3V - 1)!!} \\ &= (1 + o(1)) \frac{1}{k!} \prod_{j=0}^{k-1} \frac{6(V - j)}{3V - (2j + 1)}. \end{aligned} \quad (5.9)$$

Here we have used

$$\binom{V}{k} = \frac{1}{k!} \prod_{j=0}^{k-1} (V - j), \quad \frac{(3V - 1)!!}{(3V - 2k - 1)!!} = \prod_{j=0}^{k-1} (3V - (2j + 1)).$$

For each fixed j ,

$$\frac{6(V - j)}{3V - (2j + 1)} = \frac{6(1 - j/V)}{3 - (2j + 1)/V} \longrightarrow 2 \quad (V \rightarrow \infty, V \text{ even}).$$

Since k is fixed, only finitely many factors occur in [equation \(5.9\)](#), so

$$|\mathcal{M}_{\ell=k}(V)| \sim \frac{2^k}{k!} Q(V). \quad (5.10)$$

For $k = 0$, the product is understood as the empty product 1.

It remains to impose connectivity and then forget the vertex labels. By [equations \(5.3\)](#) and [\(5.5\)](#), the total number of disconnected graphs is $o(Q(V))$, whereas the fixed- k family in [equation \(5.10\)](#) has order $Q(V)$. Discarding its disconnected members therefore changes the count by only a relative error of $o(1)$, and

$$|\mathcal{M}_{\ell=k}^{\text{con}}(V)| \sim \frac{2^k}{k!} Q(V). \quad (5.11)$$

Vertex relabeling defines an action of $\text{Sym}(\mathcal{V})$ on $\mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V})$. For $G \in \mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V})$, the orbit $\text{Sym}(\mathcal{V}) \cdot G$ has size $V! / |\text{Aut}(\pi_2(G))|$; see [section 4.3](#). For every $H \in \mathcal{G}_{\ell=k}^{\text{con}}(\mathcal{V})$, the fiber $\pi_2^{-1}(H)$ is one such orbit, and therefore $|\pi_2^{-1}(H)| \cdot |\text{Aut}(H)| = V!$. Summing over unlabeled graphs and then expanding the fibers yields

$$\begin{aligned} |\mathcal{G}_{\ell=k}^{\text{con}}(\mathcal{V})| &= \sum_{H \in \mathcal{G}_{\ell=k}^{\text{con}}(\mathcal{V})} \frac{|\pi_2^{-1}(H)| \cdot |\text{Aut}(H)|}{V!} \\ &= \frac{1}{V!} \sum_{G \in \mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V})} |\text{Aut}(\pi_2(G))| \\ &= \frac{|\mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V})|}{V!} + \frac{1}{V!} \sum_{G \in \mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V})} (|\text{Aut}(\pi_2(G))| - 1). \end{aligned} \quad (5.12)$$

Every term $|\text{Aut}(\pi_2(G))| - 1$ is nonnegative, while $\mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V}) \subseteq \mathcal{M}(\mathcal{V})$. It follows that

$$\begin{aligned} 0 &\leq \sum_{G \in \mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V})} (|\text{Aut}(\pi_2(G))| - 1) \\ &\leq \sum_{G \in \mathcal{M}(\mathcal{V})} (|\text{Aut}(\pi_2(G))| - 1) = O\left(\frac{M(V)}{V}\right) = O\left(\frac{Q(V)}{V}\right). \end{aligned}$$

The two big- O estimates come from [equation \(5.4\)](#) and [equation \(5.3\)](#), respectively. After division by $V!$, the identity $Q(V) = V!A(V)$ makes the second term in [equation \(5.12\)](#) $O(A(V)/V)$. Meanwhile, [equations \(4.6\)](#) and [\(5.11\)](#) gives

$$\frac{|\mathcal{M}_{\ell=k}^{\text{con}}(\mathcal{V})|}{V!} = \left(\frac{2^k}{k!} + o(1)\right) \frac{Q(V)}{V!} = \left(\frac{2^k}{k!} + o(1)\right) A(V) \sim \frac{2^k}{k!} A(V).$$

Together, these two estimates in [equation \(5.12\)](#) establish [equation \(5.6\)](#).

For the unrestricted count, [equations \(5.3\)](#) and [\(5.5\)](#) show that the number of connected vertex-labeled graphs is $(e^2 + o(1))Q(V)$. The orbit-stabilizer identity, together with [equation \(5.4\)](#), now yields

$$\begin{aligned} |\mathcal{G}^{\text{con}}(\mathcal{V})| &= \frac{1}{V!} \sum_{\substack{G \in \mathcal{M}(\mathcal{V}) \\ G \text{ is connected}}} |\text{Aut}(\pi_2(G))| \\ &= \frac{(e^2 + o(1))Q(V)}{V!} + O\left(\frac{M(V)}{V V!}\right) \sim e^2 A(V). \end{aligned}$$

This is [equation \(5.7\)](#). □

6 Negligibility of Non-Pendant Bridges

The fixed-loop enumeration must now be converted into one with a prescribed number of bridges. Loops and pendant bridges correspond bijectively by [lemma 6.2](#), so it suffices to show that non-pendant bridges occur only in a lower-order family.

For the loop-free subfamily of the uniform vertex-labeled model, Chae [13, pp. 1199–1201, Proposition 4] marks a bridge uv , deletes u and v , and, at each deleted vertex, joins the other endpoints of its two remaining incident edges. The resulting object is a disconnected cubic graph with two specified edges, an encoding Chae uses to show that graphs with bridges form a lower-order family. Here, rooting at a bridge means only marking it for enumeration, not introducing a new graph-theoretic object.

Our model includes loops, so the exceptional edges are specifically the non-pendant bridges, which cannot arise from loops. We bound the probability that a uniformly sampled vertex-labeled graph is connected and has such a bridge by $O(V^{-1})$, then transfer this estimate to the unlabeled setting through the automorphism moment and the orbit–stabilizer identity. The resulting exceptional set has size $O(A(V)/V)$. Section 7 then shows that the fixed bridge and loop counts differ by $o(A(V))$.

Definition 6.1 (Pendant and non-pendant bridges). Let G be a connected cubic multigraph with more than 2 vertices. A bridge is called a *pendant bridge* if removing it leaves a connected component with only one vertex. Every other bridge is called a *non-pendant bridge*. Let $r(G)$ be the number of non-pendant bridges.

Figure 8 compares a pendant bridge with a non-pendant bridge.

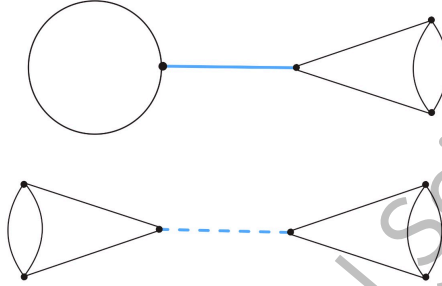


Figure 8: A pendant bridge and a non-pendant bridge. In the upper graph, removing the pendant bridge leaves exactly one vertex on one side. The lower graph contains a non-pendant bridge.

Lemma 6.2 (Loops and pendant bridges). Let G be a connected cubic multigraph with $V > 2$ vertices. There is a one-to-one correspondence between its loops and its pendant bridges. Therefore,

$$b(G) = \ell(G) + r(G). \quad (6.1)$$

If one side of a bridge contains s vertices, then s is odd. Thus each side of a non-pendant bridge contains at least three vertices.

Proof. Suppose that a vertex v carries a loop. The loop contributes 2 to the degree of v , and, because G is cubic, exactly one non-loop edge is incident to v . Deleting this edge isolates v , so the edge is a pendant bridge.

Conversely, suppose that deleting a bridge produces a component consisting only of a vertex v . The bridge accounts for 1 of the degree of v , while the remaining edges on the one-vertex side must contribute another 2 for v to have degree 3. Only a loop at v can do so. Since G has more than 2 vertices, the bridge cannot have a single vertex on both sides. The two constructions are therefore inverse, which proves equation (6.1). The excluded two-vertex case is different: a bridge may join two vertices that each carry one loop.

For the parity assertion, let S be the vertex set on one side of the bridge, and let $E(S)$ be the multiset of edges whose two endpoints both lie in S . Exactly one edge leaves S , so

$$3s = 2|E(S)| + 1. \quad (6.2)$$

This forces s to be odd. If $s \neq 1$, then $s \geq 3$, and the same argument applies on the other side of the bridge. $\square$

For even $V \geq 4$, define

$$\mathcal{G}_{r \neq 0}^{\text{con}}(V) := \{G \in \mathcal{G}^{\text{con}}(V) : r(G) \neq 0\}.$$

For an odd integer $m \geq 1$, let $\mathcal{V}_m := \{1, \dots, m\}$ be the vertex-label set, and define

$$\overrightarrow{\deg}_m^{\text{cut}} := (2, \underbrace{3, \dots, 3}_{m-1 \text{ entries}}).$$

Here $M(\overrightarrow{\deg}_m^{\text{cut}})$ counts multigraphs on the label set $\mathcal{V}_m$ for which vertex 1 has degree 2 and every other vertex has degree 3; loops and multiple edges are allowed. The initial entry is 2 because deleting a cut edge lowers the degree of its endpoint from 3 to 2. This auxiliary family is therefore not cubic, and its total degree $3m - 1$ requires m to be odd.

Theorem (Greenhill–McKay [11], special case of Theorem 1.1). As $m \rightarrow \infty$ through odd values,

$$M(\overrightarrow{\deg}_m^{\text{cut}}) = e^{2+O(m^{-1})} \frac{(3m-2)!!}{2 \cdot 6^{m-1}}. \quad (6.3)$$

The estimates in equations (5.3) and (6.3) give the following bounds for all sufficiently large even V and sufficiently large odd m . Enlarging the upper-bound constant to absorb the finitely many remaining odd values of m , there are absolute constants $c, C > 0$ such that, for every sufficiently large even V and every odd $m \geq 1$,

$$c \frac{(3V-1)!!}{6^V} \leq M(V) \leq C \frac{(3V-1)!!}{6^V}, \quad M(\overrightarrow{\deg}_m^{\text{cut}}) \leq C \frac{(3m-2)!!}{2 \cdot 6^{m-1}}. \quad (6.4)$$

Lemma 6.3 (A quantitative upper bound for non-pendant bridges).

$$|\mathcal{G}_{r \neq 0}^{\text{con}}(V)| = O\left(\frac{A(V)}{V}\right) \quad (V \rightarrow \infty, V \text{ even}). \quad (6.5)$$

Proof.

Step 1: Count graphs for a fixed cut (S, S^c) . Sample G uniformly from $\mathcal{M}(\mathcal{V})$, and fix a set of labels $S \subseteq \mathcal{V}$ with odd size s , where $1 \leq s \leq V-1$, and set $t := V-s$. For a graph G , let $E(S, S^c)$ be the multiset of edges joining S to its complement. The condition $|E(S, S^c)| = 1$ says precisely that (S, S^c) is a one-edge cut.

For a graph with exactly one crossing edge, its endpoints in S and S^c can be chosen in s and t ways, respectively. Deleting this edge lowers the degree of each endpoint from 3 to 2. Each endpoint consequently has degree 2 in $G[S]$ or $G[S^c]$, while all other vertices on the same side retain degree 3. After fixing endpoints $u \in S$ and $v \in S^c$, fix bijections $S \rightarrow \mathcal{V}_s$ and $S^c \rightarrow \mathcal{V}_t$ that send u and v to label 1, respectively. Under these relabelings, the possible induced multigraphs on the two sides are counted by $M(\overrightarrow{\deg}_s^{\text{cut}})$ and $M(\overrightarrow{\deg}_t^{\text{cut}})$. The choices on the two sides are independent, so each of the st pairs (u, v) admits $M(\overrightarrow{\deg}_s^{\text{cut}})M(\overrightarrow{\deg}_t^{\text{cut}})$ possibilities. Reattaching the edge uv reverses the construction uniquely. The exact number of graphs for which the fixed partition (S, S^c) has exactly one crossing edge is consequently

$$st M(\overrightarrow{\deg}_s^{\text{cut}}) M(\overrightarrow{\deg}_t^{\text{cut}}). \quad (6.6)$$

Division by the sample-space size $M(V)$, followed by [equation \(6.4\)](#), yields

$$\begin{aligned} \mathbb{P}(|E(S, S^c)| = 1) &= \frac{st M(\deg_s^{\rightarrow \text{cut}}) M(\deg_t^{\rightarrow \text{cut}})}{M(V)} \\ &\leq Cst \frac{(3s-2)!!}{2 \cdot 6^{s-1}} \frac{(3t-2)!!}{2 \cdot 6^{t-1}} \frac{6^V}{(3V-1)!!} \\ &= C 9st \frac{(3s-2)!!(3t-2)!!}{(3V-1)!!}. \end{aligned} \quad (6.7)$$

The numerical factor in the last line follows from $s + t = V$: $6^V / (4 \cdot 6^{s-1} 6^{t-1}) = 6^2 / 4 = 9$.

Step 2: Use the expected number of cuts to bound the probability of a non-pendant bridge. For odd s , let

$$\mathcal{S}_s := \{S \subseteq \mathcal{V} : |S| = s\}, \quad X_s(G) := \#\{S \in \mathcal{S}_s : |E(S, S^c)| = 1\}.$$

The value $X_s(G)$ records the label sets S of size s for which exactly one edge joins S to S^c —equivalently, one-edge cuts with a distinguished side S of size s . For a compact form of the right-hand side of [equation \(6.7\)](#), define

$$T(V, s) := \binom{V}{s} 9s(V-s) \frac{(3s-2)!!(3(V-s)-2)!!}{(3V-1)!!}. \quad (6.8)$$

Vertex permutations act transitively on the sets $S \in \mathcal{S}_s$. Linearity of expectation and [equation \(6.7\)](#) then yield

$$\begin{aligned} \mathbb{E}X_s &= \sum_{S \in \mathcal{S}_s} \mathbb{E}\mathbf{1}_{\{|E(S, S^c)|=1\}} \\ &= \sum_{S \in \mathcal{S}_s} \mathbb{P}(|E(S, S^c)| = 1) \\ &= \binom{V}{s} \mathbb{P}(|E(S, S^c)| = 1) \leq C T(V, s). \end{aligned} \quad (6.9)$$

Introduce the following auxiliary statistic on the full vertex-labeled sample space:

$$r'(G) := \sum_{\substack{3 \leq s \leq V/2 \\ s \text{ odd}}} X_s(G).$$

Although $r'(G)$ is defined on the full sample space, it is compared with $r(G)$ only for connected G . For such a graph G , both induced subgraphs of a one-edge cut are connected, and the crossing edge is a bridge. A non-pendant bridge whose two components have unequal sizes has exactly one side of size at most $V/2$, and therefore contributes once to $r'(G)$. If a bridge separates the vertex set into two components of size $V/2$, the parity statement in [lemma 6.2](#) forces $V/2$ to be odd; both sides then contribute to $X_{V/2}(G)$, and the bridge is counted twice.

A connected graph contains at most one equal-split bridge. To see this, let e be one such bridge and consider any other bridge f , which must lie inside a component of $G - e$. Removing f leaves a component of $G - f$ not containing e that is a proper subset of the original $V/2$ -vertex component. It has fewer than $V/2$ vertices, so f cannot also determine an equal split. For connected G , let $\mathbf{1}_{\text{eq}}(G)$ be the indicator of the event that an equal-split bridge exists. Then

$$r'(G) = r(G) + \mathbf{1}_{\text{eq}}(G), \quad r(G) \leq r'(G) \leq r(G) + 1, \quad r(G) = 0 \iff r'(G) = 0.$$

For connected G , the last equivalence identifies the events $r(G) \neq 0$ and $r'(G) \neq 0$. Markov's inequality in [equation \(4.7\)](#), together with linearity of expectation and [equation \(6.9\)](#), now gives

$$\begin{aligned}
 \mathbb{P}(G \text{ is connected and has a non-pendant bridge}) &= \mathbb{P}(G \text{ is connected and } r'(G) \neq 0) \\
 &\leq \mathbb{P}(r'(G) \geq 1) \\
 &\leq \mathbb{E}r'(G) = \sum_{\substack{3 \leq s \leq V/2 \\ s \text{ odd}}} \mathbb{E}X_s \\
 &\leq C \sum_{\substack{3 \leq s \leq V/2 \\ s \text{ odd}}} T(V, s). \tag{6.10}
 \end{aligned}$$

Step 3: Uniformly estimate the sum over cut sizes. The required double-factorial bounds follow from [equation \(4.8\)](#). Indeed, the identity $(2q-1)!! = (2q)!/(2^q q!)$ gives

$$\begin{aligned}
 \frac{c_0 \sqrt{2q} (2q/e)^{2q}}{2^q C_0 \sqrt{q} (q/e)^q} &\leq (2q-1)!! \\
 &\leq \frac{C_0 \sqrt{2q} (2q/e)^{2q}}{2^q c_0 \sqrt{q} (q/e)^q}.
 \end{aligned}$$

The lower and upper expressions are both constant multiples of $(2q/e)^q$. After adjusting the constants, this becomes

$$c_0 \left(\frac{2q}{e}\right)^q \leq (2q-1)!! \leq C_0 \left(\frac{2q}{e}\right)^q \quad (q \geq 1). \tag{6.11}$$

Applying [equation \(4.8\)](#) separately to the factorials in the binomial coefficient, and using $s+t=V$, also gives

$$\begin{aligned}
 \binom{V}{s} &= \frac{V!}{s! t!} \\
 &\leq \frac{C_0 \sqrt{V} (V/e)^V}{c_0^2 \sqrt{st} (s/e)^s (t/e)^t} \leq C \sqrt{\frac{V}{st}} \frac{V^V}{s^s t^t}. \tag{6.12}
 \end{aligned}$$

Since s and t are odd while V is even, the quantities

$$q_s := \frac{3s-1}{2}, \quad q_t := \frac{3t-1}{2}, \quad q_V := \frac{3V}{2},$$

are integers, and $q_s + q_t = q_V - 1$. Using the upper estimate in [equation \(6.11\)](#) for the numerator and the lower estimate for the denominator, we find

$$\begin{aligned}
 \frac{(3s-2)!!(3t-2)!!}{(3V-1)!!} &\leq C \frac{((3s-1)/e)^{q_s} ((3t-1)/e)^{q_t}}{(3V/e)^{q_V}} \\
 &\leq C \frac{s^{(3s-1)/2} t^{(3t-1)/2}}{V^{3V/2}}. \tag{6.13}
 \end{aligned}$$

For the second inequality, $3s-1 \leq 3s$ and $3t-1 \leq 3t$; because $q_V - q_s - q_t = 1$, cancellation of the powers of 3 and e leaves only the constant factor $e/3$.

Insert equations (6.12) and (6.13) into equation (6.8), initially keeping the powers separate before collecting like terms:

$$\begin{aligned}
 T(V, s) &\leq Cst \sqrt{\frac{V}{st}} \frac{V^V s^{3s/2-1/2} t^{3t/2-1/2}}{V^{3V/2}} \\
 &= C\sqrt{V} \frac{s^{s/2} t^{t/2}}{V^{V/2}} \\
 &= C\sqrt{V} \left(\frac{s}{V}\right)^{s/2} \left(\frac{t}{V}\right)^{t/2}.
 \end{aligned} \tag{6.14}$$

In the second line, $st\sqrt{V/(st)}s^{-1/2}t^{-1/2} = \sqrt{V}$; the last line uses $s + t = V$. The exclusion of pendant bridges is also visible at this scale. Direct evaluation shows that the term with $s = 1$ has order 1, whereas a non-pendant bridge has a smaller side of at least $s = 3$ vertices, where the bound is already of order V^{-1} .

It remains to split the sum in equation (6.10) into three ranges. When $s = 3$, we have $(t/V)^{t/2} \leq 1$, so

$$T(V, 3) \leq C\sqrt{V} \left(\frac{3}{V}\right)^{3/2} = O(V^{-1}). \tag{6.15}$$

When $5 \leq s \leq V^{1/3}$, we have $s/V \leq V^{-2/3}$, and hence $(s/V)^{s/2} \leq V^{-s/3}$. For this range, extend the summation to all integers $s \geq 5$ and bound $(t/V)^{t/2}$ by 1:

$$\begin{aligned}
 \sum_{\substack{5 \leq s \leq V^{1/3} \\ s \text{ odd}}} T(V, s) &\leq C\sqrt{V} \sum_{s \geq 5} V^{-s/3} \\
 &= C\sqrt{V} \frac{V^{-5/3}}{1 - V^{-1/3}} = O(V^{-7/6}).
 \end{aligned} \tag{6.16}$$

Since the original sum restricts to odd s and all terms are nonnegative, extending the range and dropping the parity restriction can only increase the bound.

In the remaining range $V^{1/3} < s \leq V/2$,

$$\left(\frac{s}{V}\right)^{s/2} \leq 2^{-s/2} \leq 2^{-V^{1/3}/2}.$$

There are at most V integers in this range, and hence

$$\sum_{\substack{V^{1/3} < s \leq V/2 \\ s \text{ odd}}} T(V, s) \leq CV^{3/2} 2^{-V^{1/3}/2} = o(V^{-1}). \tag{6.17}$$

The final comparison follows because

$$\log(V^{5/2} 2^{-V^{1/3}/2}) = \frac{5}{2} \log V - \frac{\log 2}{2} V^{1/3} \rightarrow -\infty.$$

The three ranges show that the right-hand side of equation (6.10) is $O(V^{-1})$.

Step 4: Pass from vertex-labeled graphs to unlabeled isomorphism classes. Define

$$\mathcal{M}_{r \neq 0}^{\text{con}}(\mathcal{V}) := \{G \in \mathcal{M}(\mathcal{V}) : G \text{ is connected and } r(G) \neq 0\}.$$

Since the uniform vertex-labeled model has sample-space size $M(V)$, the probability estimate from the first three steps is equivalent to

$$|\mathcal{M}_{r \neq 0}^{\text{con}}(\mathcal{V})| = O(M(V)/V). \tag{6.18}$$

An unlabeled graph H has $V!/|\text{Aut}(H)|$ vertex-labeled versions. The orbit–stabilizer identity shows that its isomorphism class contributes a total of $V!$ to the following sum:

$$|\mathcal{G}_{r \neq 0}^{\text{con}}(V)| = \frac{1}{V!} \sum_{G \in \mathcal{M}_{r \neq 0}^{\text{con}}(\mathcal{V})} |\text{Aut}(\pi_2(G))|. \quad (6.19)$$

Decompose each summand as $|\text{Aut}(\pi_2(G))| = 1 + (|\text{Aut}(\pi_2(G))| - 1)$. The second term is nonnegative, so its sum over the exceptional family is bounded by the corresponding sum over the full family. [Equations \(5.4\), \(6.18\) and \(6.19\)](#) now imply

$$\begin{aligned} |\mathcal{G}_{r \neq 0}^{\text{con}}(V)| &= \frac{1}{V!} \left(|\mathcal{M}_{r \neq 0}^{\text{con}}(\mathcal{V})| + \sum_{G \in \mathcal{M}_{r \neq 0}^{\text{con}}(\mathcal{V})} (|\text{Aut}(\pi_2(G))| - 1) \right) \\ &\leq \frac{1}{V!} \left(O(M(V)/V) + \sum_{G \in \mathcal{M}(\mathcal{V})} (|\text{Aut}(\pi_2(G))| - 1) \right) \\ &= O\left(\frac{M(V)}{V V!}\right). \end{aligned}$$

The normalization is supplied by [equations \(4.5\) and \(6.4\)](#):

$$\frac{M(V)}{V!} \asymp \frac{(3V-1)!!}{6^V V!} = A(V).$$

Together with the previous bound, this yields $|\mathcal{G}_{r \neq 0}^{\text{con}}(V)| = O(A(V)/V)$. $\square$

7 A Fixed Number of Bridges and the Main Theorem

No further graph enumeration is needed: the preceding two sections show that, outside an exceptional set of $O(A(V)/V)$ isomorphism classes, every bridge is pendant. On this complement, [lemma 6.2](#) identifies bridges with loops.

Theorem 7.1 (A fixed number of bridges). For every fixed integer $k \geq 0$,

$$|\mathcal{G}_{b=k}^{\text{con}}(V)| \sim \frac{2^k}{k!} A(V) \quad (V \rightarrow \infty, V \text{ even}). \quad (7.1)$$

Proof. We may restrict attention to sufficiently large even V , so $V \geq 4$ and [lemmas 6.2 and 6.3](#) apply. For any two sets A and B , their *symmetric difference* is $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Using $b(G) = \ell(G) + r(G)$ from [equation \(6.1\)](#), we have

$$\mathcal{G}_{b=k}^{\text{con}}(V) \triangle \mathcal{G}_{\ell=k}^{\text{con}}(V) = \{G \in \mathcal{G}^{\text{con}}(V) : r(G) \neq 0, b(G) = k \text{ or } \ell(G) = k\} \subseteq \mathcal{G}_{r \neq 0}^{\text{con}}(V).$$

The inclusion may be strict: $r(G) \neq 0$ ensures only that $b(G) \neq \ell(G)$, not that either $b(G)$ or $\ell(G)$ equals the prescribed value k . Consequently,

$$||\mathcal{G}_{b=k}^{\text{con}}(V)| - |\mathcal{G}_{\ell=k}^{\text{con}}(V)|| \leq |\mathcal{G}_{b=k}^{\text{con}}(V) \triangle \mathcal{G}_{\ell=k}^{\text{con}}(V)| \leq |\mathcal{G}_{r \neq 0}^{\text{con}}(V)| = O(A(V)/V) = o(A(V)).$$

By [proposition 5.1](#), $|\mathcal{G}_{\ell=k}^{\text{con}}(V)| \sim (2^k/k!)A(V)$, and a perturbation of size $o(A(V))$ does not affect this leading term. This establishes [equation \(7.1\)](#). $\square$

Returning to the surface problem, the duality map Φ_g translates [theorem 7.1](#) directly into [theorem 2.2](#).

Proof of Theorem 2.2. By [proposition 4.5](#), the restriction of Φ_g is a bijection from $\mathcal{P}_k(g)$ to $\mathcal{G}_{b=k}^{\text{con}}(2g-2)$, and hence

$$P_k(g) = |\mathcal{G}_{b=k}^{\text{con}}(2g-2)|.$$

Taking $V = 2g - 2$ in [equations \(4.5\)](#) and [\(7.1\)](#), we obtain

$$P_k(g) \sim \frac{2^k}{k!} A(2g-2) = \frac{2^k}{k!} \frac{(6g-7)!!}{6^{2g-2}(2g-2)!}.$$

This is [equation \(2.1\)](#).

For the Stirling form, let $n := g - 1$. The double-factorial identity gives

$$A(2n) = \frac{(6n-1)!!}{6^{2n}(2n)!} = \frac{(6n)!}{2^{3n}(3n)! 6^{2n}(2n)!}. \quad (7.2)$$

Applying

$$m! = \sqrt{2\pi m} \left(\frac{m}{e}\right)^m (1 + o(1)) \quad (m \rightarrow \infty)$$

to each of the three factorials in [equation \(7.2\)](#) yields

$$\begin{aligned} A(2n) &\sim \frac{\sqrt{12\pi n}}{\sqrt{6\pi n}\sqrt{4\pi n}} \frac{(6n)^{6n} e^{-6n}}{2^{3n} 6^{2n} (3n)^{3n} e^{-3n} (2n)^{2n} e^{-2n}} \\ &= \frac{1}{\sqrt{2\pi n}} n^n e^{-n} \left(\frac{6^6}{2^3 6^2 3^3 2^2}\right)^n \\ &= \frac{1}{\sqrt{2\pi n}} \left(\frac{3n}{2e}\right)^n. \end{aligned} \quad (7.3)$$

The simplifications use

$$\frac{\sqrt{12\pi n}}{\sqrt{6\pi n}\sqrt{4\pi n}} = \frac{1}{\sqrt{2\pi n}}, \quad \frac{6^6}{2^3 6^2 3^3 2^2} = \frac{3}{2}.$$

Returning to $n = g - 1$ and expressing the right-hand side entirely in terms of g gives

$$\begin{aligned} \frac{1}{\sqrt{2\pi(g-1)}} \left(\frac{3(g-1)}{2e}\right)^{g-1} &= \frac{1}{\sqrt{2\pi g}} \frac{2e}{3g} \left(1 - \frac{1}{g}\right)^{g-\frac{3}{2}} \left(\frac{3g}{2e}\right)^g \\ &\sim \frac{2}{3\sqrt{2\pi} g^{3/2}} \left(\frac{3g}{2e}\right)^g. \end{aligned} \quad (7.4)$$

The last asymptotic relation uses $(1 - 1/g)^{g-3/2} \rightarrow e^{-1}$. Multiplying [equation \(7.4\)](#) by $2^k/k!$ recovers [equation \(2.2\)](#). $\square$

8 Probabilistic and Topological Consequences

The main asymptotic formula has a probabilistic consequence after normalization by the total count. Its topological content comes from the fact that the size of a connected component on one side of a bridge determines the genus of the subsurface cut off by the corresponding separating curve.

Proof of Corollary 2.3. From [definition 2.1](#),

$$\mathbb{P}(K_g = k) = \frac{P_k(g)}{P(g)}.$$

Dividing the asymptotic in [equation \(2.2\)](#) by that in [equation \(2.3\)](#) cancels the common factors that depend on g , leaving

$$\mathbb{P}(K_g = k) = \frac{P_k(g)}{P(g)} \longrightarrow \frac{2^k/k!}{e^2} = e^{-2} \frac{2^k}{k!} \quad (g \rightarrow \infty).$$

For fixed $m \geq 0$, the sum below has only finitely many terms, and its limit may be taken term by term:

$$\mathbb{P}(K_g \leq m) = \sum_{j=0}^m \mathbb{P}(K_g = j) \longrightarrow \sum_{j=0}^m e^{-2} \frac{2^j}{j!} \quad (g \rightarrow \infty).$$

The resulting probability masses have total

$$\sum_{j \geq 0} e^{-2} \frac{2^j}{j!} = e^{-2} e^2 = 1.$$

The finite-sum limit above is therefore the distribution function of $\text{Poisson}(2)$, and convergence in distribution follows. $\square$

Proposition 8.1 (The genus on one side of a separating curve). Let γ be a separating curve in a pants decomposition. Suppose that one side of γ consists of s pairs of pants and is an orientable surface of genus $h \geq 1$ with one boundary component. Then

$$s = 2h - 1. \quad (8.1)$$

In particular, after the corresponding bridge is deleted, one connected component consists of a single vertex if and only if γ cuts off a one-holed torus.

Proof. The side assembled from s pairs of pants has Euler characteristic $-s$. Viewed as an orientable surface of genus h with one boundary component, the same side has Euler characteristic

$$2 - 2h - 1 = 1 - 2h.$$

Equating the two descriptions gives $-s = 1 - 2h$, or $s = 2h - 1$; in particular, $s = 1$ if and only if $h = 1$. $\square$

Proof of Corollary 2.4. Take g large enough that $V = 2g - 2 \geq 4$. Under [proposition 8.1](#), a separating curve that does not cut off a one-holed torus corresponds to a non-pendant bridge. Thus $\mathcal{E}_k(g)$, as defined in the corollary, consists exactly of the types whose dual graph contains at least one such bridge. Together, the dual graph bijection and [lemma 6.3](#) yield

$$0 \leq |\mathcal{E}_k(g)| \leq |\mathcal{G}_{r \neq 0}^{\text{con}}(2g - 2)| = O\left(\frac{A(2g - 2)}{g}\right).$$

For the denominator, [theorem 2.2](#) gives

$$P_k(g) \sim \frac{2^k}{k!} A(2g - 2),$$

where $2^k/k! > 0$. The quotient consequently satisfies

$$\frac{|\mathcal{E}_k(g)|}{P_k(g)} = O_k(g^{-1}) \quad (g \rightarrow \infty).$$

$\square$

Remark 8.2. The asymptotic formula for vertex-labeled graphs with a fixed number of loops can already be obtained from a formula of Chae [13]. Chae, Palmer, and Robinson [14] also gave exact recurrences refined by connectivity, loops, and multiple edges. Chae’s count [13, pp. 1199–1201, Proposition 4] of graphs with a specified bridge in the loop-free subfamily of the uniform vertex-labeled model already shows that the proportion of graphs with a bridge in this subfamily tends to zero. In our count, the pendant bridges forced by loops remain in the leading term, while the estimate controls the non-pendant bridges separately. For the connected subfamily of the uniform unlabeled model, we also obtain the quantitative bound [equation \(6.5\)](#). These two features produce the local asymptotic formula for a fixed number of bridges and allow the constant correction in [corollary 9.1](#).

A refinement could record the genera of the subsurfaces cut off by the separating curves.

9 Correction of the Bridge Probability in Grave de Peralta and Kolpakov

We identify below the source of an incorrect constant in [9]. We nevertheless believe that its qualitative conclusion that the outer conductance tends to zero is correct. The exact location of the error is in Section 3.2. At the end of p. 1829, the authors define $B_n \subset U_n = V(\Gamma_n)$, where U_n is the set of isomorphism classes of connected cubic multigraphs on $2n$ vertices, B_n is the subset with at least one bridge, and L_n is the corresponding set of vertex-labeled graphs. On p. 1830, they define the events Bridged, Looped $\subset L_n$. In the sentence beginning with “By applying [9, Theorem 9.5]”, they state, for a graph g chosen uniformly from L_n , that $\mathbb{P}(g \in \text{Looped}) = 1 - e^{-1} + o(1)$. They use this statement to conclude that $\mathbb{P}(g \in \text{Bridged}) = 1 - e^{-1} + o(1)$ and then obtain

$$\frac{|B_n|}{|U_n|} = 1 - e^{-1} + o(1), \quad (\text{GK})$$

for the uniform unlabeled model.

On p. 1829, they cite Bollobás [16] and use $|L_n| \sim (2n)!|U_n|$ to transfer the asymptotic probabilities of isomorphism-invariant events from the uniform vertex-labeled model conditioned on connectedness to the uniform unlabeled model conditioned on connectedness. Such a transfer would be valid if this asymptotic relation had been established for the precise class of multigraphs used here, which allows loops and multiple edges. The orbit–stabilizer identity reads

$$\frac{|L_n|}{(2n)!} = \sum_{H \in U_n} \frac{1}{|\text{Aut}(H)|}.$$

Under the asserted asymptotic relation, the proportion of graphs with a nontrivial automorphism in the uniform unlabeled connected model would be $o(1)$, and asymptotic probabilities could be transferred for any isomorphism-invariant event. The cited result of Bollobás, however, concerns simple regular graphs. The original paper does not establish its direct applicability to the present graph class, which allows loops and multiple edges. Combining our automorphism moment estimate [equation \(5.4\)](#) with the orbit–stabilizer calculation supplies the missing argument and gives an additive $O(n^{-1})$ error in the comparison of the two event probabilities.

The decisive error occurs on p. 1830, in the sentence beginning with “By applying [9, Theorem 9.5]”. The cited Theorem 9.5 of Janson, Łuczak, and Ruciński concerns the uniform vertex-labeled half-edge pairing model, in which the number of loops converges in distribution to Poisson(1). In that model, the probability of having no loops tends to e^{-1} [17, Theorem 9.5]. The model on L_n , however, is the uniform vertex-labeled model defined in [section 4.3](#), conditioned on connectedness. For $n > 1$, each graph G in

this model is connected and has $2n > 2$ vertices. Once the half-edge labels are forgotten, [equation \(4.3\)](#) and [remark 4.8](#) give

$$|\pi_1^{-1}(G)| = \frac{6^{2n}}{2^{\ell(G)+d(G)}}.$$

Each loop and each double-edge pair changes this fiber size. Hence uniform pairings do not induce the uniform distribution on vertex-labeled graphs: the two probability spaces are different. For the uniform vertex-labeled model conditioned on connectedness, the connected count with a fixed number of loops and the total connected count are given by [equations \(5.3\), \(5.5\) and \(5.11\)](#). Their ratio yields

$$\mathbb{P}(\ell(G) = k) \longrightarrow e^{-2} \frac{2^k}{k!}.$$

Accordingly, the number of loops converges in distribution to Poisson(2) rather than Poisson(1). The same limit holds in the uniform unlabeled connected model by [proposition 5.1](#). In this model, every loop forces a pendant bridge, whereas [lemma 6.3](#) bounds the proportion of graphs with a non-pendant bridge by $O(n^{-1})$. The resulting correction is recorded next.

Corollary 9.1 (Correction to the bridge-probability constant). For even V , let G_V be a uniformly random element of the finite set $\mathcal{G}^{\text{con}}(V)$. As V tends to infinity through the even integers,

$$\mathbb{P}(b(G_V) = 0) \longrightarrow e^{-2}, \quad \mathbb{P}(b(G_V) \geq 1) \longrightarrow 1 - e^{-2}.$$

Equivalently, if a pants decomposition type is chosen uniformly from $\mathcal{P}(g)$, the probability that it contains at least one separating curve tends to $1 - e^{-2}$. In particular, this changes the constant $1 - e^{-1}$ in Grave de Peralta and Kolpakov [[9](#), Section 3.2, p. 1830] to $1 - e^{-2}$.

Proof. Setting $k = 0$ in [proposition 5.1](#) and [theorem 7.1](#) yields

$$\mathbb{P}(b(G_V) = 0) = \frac{|\mathcal{G}_{b=0}^{\text{con}}(V)|}{|\mathcal{G}^{\text{con}}(V)|} \longrightarrow \frac{1}{e^2} = e^{-2}.$$

The complementary event is the presence of a bridge, whose probability tends to $1 - e^{-2}$. With $V = 2g - 2$, [proposition 4.5](#) translates this limit into the equivalent statement about pants decompositions. $\square$

The next clause of the same sentence on p. 1830 cites Theorem 9.23 only to estimate the loop-free non-Hamiltonian family, whose probability is $o(1)$. Even a valid model transfer for this family would not affect the leading-term error caused by the incorrect loop constant.

The correction also changes the leading constant displayed in the outer conductance estimate on p. 1832. The original constant $(e - 1)/(3n)$ should be replaced by $(e^2 - 1)/(3n)$. Since both expressions tend to 0, this change of constant alone does not invalidate the conclusion that the outer conductance tends to zero. We therefore believe that the qualitative conclusion of Grave de Peralta and Kolpakov remains correct; the required corrections concern the proportion of graphs with a bridge and the explicit constant derived from it.

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