

Additive energy and color forcing in directed L-grid Ramsey problems

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Abstract

Take two equally colored main-diagonal cells (u, u) and (v, v) in an integer grid, with $u < v$. The cell (v, u) must have a different color to avoid a monochromatic triple $(x, y), (x + d, y), (x + d, y + d)$ with $d > 0$. If many diagonal positions share a color, these restrictions act on the same set of cells. We study what happens when that set uses only two colors, by counting equal differences and examining small Boolean cubes.

For a finite mark set $A \subset \mathbb{Z}$, its strict source is $V(A) = \{(v, u) : u, v \in A, u < v\}$. Put $n = |A|$, $M = \binom{n}{2}$, and $L = \max A - \min A$, with empty span zero. Each corner in $V(A)$ comes from two occurrences of the same positive difference. If $T(A)$ counts these corners, then

$$E(A) = 2n^2 - n + 4T(A).$$

Thus an upper bound for the number of corners gives an upper bound for additive energy. Cauchy–Schwarz also turns $T(A) \leq CM$ into $M \leq (2C + 1)L$, relating repeated differences to the length occupied by the marks.

A Golomb set has no repeated positive difference. For disjoint Golomb sets B, C , write $b = |B|$, $c = |C|$, $n = b + c$, and $\rho = |\Delta B \cap \Delta C|$. By counting the two classes together, we prove

$$T(B \sqcup C) \leq \min\{2M, M + 5\rho\}, \quad E(B \sqcup C) \leq 6n^2 - 5n.$$

The separate-class estimate derived in the paper is $E \leq 8n^2$. Counting the classes together gives the smaller bound above and a further loss when their sizes differ. Summing these losses over the fibers refines the cube-moment estimates. Two explicit examples disprove stronger proposed counts. The two-Golomb partition occurs within each fiber of a properly colored source; it is not assumed for the whole mark set.

We next determine the largest attainable dimension of an injective Boolean cube whose strict source has a proper two-coloring. It is four: every five-dimensional cube is excluded, and a complete coloring works for all separated four-dimensional cubes. The obstruction uses the known binary Sidon number $a(4) = 7$, for which we include a pair-counting proof adapted to weighted cubes. The value seven itself is not a new extremal result. For every injective positive-integer three-cube, a proper two-coloring has at least nine corners whose parallel endpoints have different colors. A construction attains nine when the generators are 2-dissociated, including every separated three-cube.

To track corners that share a completion, we place the actual source cells in a forcing graph Γ . For a proper sign coloring it satisfies

$$2T(A) = M + 2|E(\Gamma)| - \sum_{d>0} s_d^2,$$

where s_d is the color imbalance in the fiber of offset d . The identity counts the effect of corners sharing one cell. We also prove bounds for progression products, direct sums, translation matchings, and sources with two-distance representations. The family $\{\pm 5^i : 0 \leq i < r\}$ gives arbitrarily large proper sources with $T/M \rightarrow 1/2$. The constants in the structured bounds need not be optimal.

The energy estimates, cube colorings, and structured-source bounds are proved without the open assumptions in the final section. That section gives conditional small-color Ramsey bounds and a thirteen-mark counterexample to one proposed rainbow bound. A general source estimate and the other local rainbow premises remain open. An appendix applies the external corners theorem of Jaber et al. to obtain a general quasipolynomial bound. Full coloring tables and companion programs give separate checks of the finite counts.

Keywords: Ramsey theory; lattice colorings; additive energy; Golomb rulers; Boolean cubes.

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1 Introduction

Consider two cells (u, u) and (v, v) of the same color on the main diagonal, with $u < v$. They form a right isosceles triangle with (v, u) , so this third cell must have another color. We study how these restrictions interact when many diagonal positions have the same color.

The directed L configuration. Write $[n] = \{1, \dots, n\}$ and set

$$\mathcal{L}(x, y; d) = \{(x, y), (x + d, y), (x + d, y + d)\}, \quad d \in \mathbb{Z}_{>0}.$$

A coloring of a domain is *proper* when it has no monochromatic copy of this configuration, meaning no copy whose three cells all share one color. The directed L-grid Ramsey number $R_k(\mathcal{L})$ is the least n for which every map $[n]^2 \rightarrow [k]$ fails to be proper. The two legs are equal and positive by definition; Figure 1 shows how this convention differs from two close relatives.

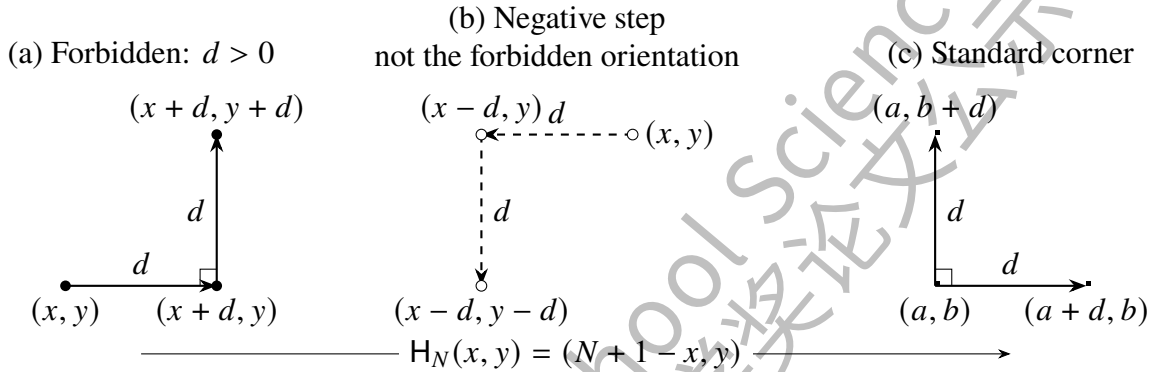


Figure 1: Direction and reflection. Solid legs show the forbidden positive L and its standard-corner image; dashed legs show the negative-step comparison. In (c), $a = N + 1 - x - d$ and $b = y$. Reflection H_N sends the right-angle vertex of (a) to (a, b) . Notice that this reflection differs from the coordinate transposition proved in Lemma 2.1.

By the pigeonhole principle, some color appears at least $\lceil n/k \rceil$ times on the main diagonal of a k -colored $n \times n$ grid. Let A be the positions of that color. If $u < v$ are in A , properness forces (v, u) to have another color, because (u, u) , (v, u) , (v, v) form a directed L. Hence one coloring of the square gives a coloring of the strict triangular source on A using at most $k - 1$ colors. This is the diagonal reduction used in the published L-problem arguments [4, 5].

The reduction suggests studying these triangular sources directly. Their corners correspond to repeated differences between marks of A . However, the color constraints cannot be checked one corner at a time: two corners may require colors at the same cell. We therefore ask how many repeated differences can coexist with a single proper coloring of the whole source.

Setting and question. For finite $A \subset \mathbb{Z}$, write

$$\begin{aligned} V(A) &= \{(v, u) \in A^2 : u < v\}, & W(A) &= \{(v, u) \in A^2 : u \leq v\}, \\ F_d(A) &= \{u \in A : u + d \in A\}, & r_d(A) &= |F_d(A)| \quad (d > 0), \\ n &= |A|, & M(A) &= \binom{n}{2}, & L(A) &= \max A - \min A. \end{aligned}$$

We set $L(\emptyset) = 0$ and call $V(A)$, $W(A)$ the strict and weak sources. A single coloring must be defined on the whole source. A *Golomb set* has all positive differences distinct; put $\Delta B = \{v - u : u < v, u, v \in B\}$. With $T(A)$ the number of directed corners in $V(A)$ and $E(A) = \#\{(a, b, c, e) \in A^4 : a - b = c - e\}$, the basic counts are

$$T(A) = \sum_{d>0} \binom{r_d(A)}{2}, \quad E(A) = 2n^2 - n + 4T(A).$$

Proposition 3.1 proves these identities. A uniform bound $T(A) \leq CM(A)$ for properly two-colored strict sources would give a span bound by Corollary 3.4. We prove such estimates for several specified families. The corresponding statement for arbitrary sources remains open.

Relation to previous work. The corners theorem of Ajtai–Szemerédi [1] belongs to the same density-theoretic setting as Szemerédi’s progression theorem [2]. Graham–Solymosi [3] study monochromatic right triangles on grids. These results establish a general framework, but small numbers of colors require more explicit counts. Canakci et al. [4] use diagonal exclusion, color forcing, and SAT search to prove $21 \leq R_3 \leq 2593$. Mammel–Smith–Yerger [5] improve the upper bound to 493 using Golomb restrictions and interval counts. Lemma 2.1 checks that their orientation convention is equivalent to ours.

Mammel–Smith–Yerger use the Golomb restriction to bound the number of forced points on a subdiagonal. Here we count equal differences across both Golomb classes, and then study how the resulting corners can share source cells. Boolean cubes provide small configurations where all the required colors can be listed. Their Golomb capacity belongs to the established binary Sidon problem: $a(4) = 7$ is recorded in [12], and Wood [13] discusses binary pair sums and the base-three transfer. We use this known capacity in the source-coloring argument; we do not claim the value seven as a new result. Appendix D compares these starting points with the results proved below.

Main results.

1. **Joint energy counting.** Theorem 4.1 proves $T(B \sqcup C) \leq \min\{2M, M + 5|\Delta B \cap \Delta C|\}$ and $E(B \sqcup C) \leq 6n^2 - 5n$. The deficit formula records losses from unequal class sizes, and Corollary 4.3 retains them in each fiber. Propositions 4.4 and 4.5 disprove stronger proposed counts. The benchmark $8n^2$ is the separate-class estimate shown in Section 4.
2. **Exact cube colorings.** Corollary 5.4 excludes every injective five-cube with positive integer generators and supplies a proper coloring for every separated four-cube. Thus the maximum attainable dimension is four. Theorem 5.5 proves a mixed-corner lower bound of nine on every injective three-cube, attained under 2-dissociation. The five-cube exclusion uses the known seven-mark Golomb capacity.
3. **Structured sources and constructions.** Section 6 proves $T \leq CM$ for four families, with the assumptions and constants listed in Table 5. Table 6 gives explicit properly colored examples with repeated differences. The two-distance example extends to arbitrarily large sources with $T/M \rightarrow 1/2$.

The graph in Theorem 5.6 has one vertex for each actual source cell. Thus corners sharing a completion must use the same color there. Its identity relates the corner count to edges and fiber imbalances. The three groups of results above are independent of the open assertions in Section 7.

Organization. Sections 2 and 3 fix the orientation and basic counts. Sections 4–6 prove the results above, and Section 7 gives conditional Ramsey consequences. Appendices A and B contain finite certificates and SAT formulations. Appendix C derives a general bound from the external density theorem of Jaber et al. [6]; Appendix D records the sources and attribution.

We first check the orientation convention and compute a small example. This will make the difference between a fiber and a diagonal color class explicit before the general proofs.

2 Background and the orientation convention

A 4×4 example is enough to illustrate diagonal exclusion and the weak coloring inherited by a fiber. Before examining it, we compare the orientation in this paper with the one used in the published small-color bounds.

The cited L-problem papers [4, 5] use

$$\mathcal{P}(x, y; d) = \{(x, y), (x, y + d), (x + d, y + d)\}, \quad d > 0.$$

Define $R_k(\mathcal{P})$ by this configuration in the same way as $R_k(\mathcal{L})$. Coordinate transposition gives the following equivalence.

Lemma 2.1 (Transposition). *For every $n, k \geq 1$, $\sigma(x, y) = (y, x)$ gives a bijection between proper k -colorings of $[n]^2$ for $\mathcal{L}$ and for $\mathcal{P}$. As a consequence $R_k(\mathcal{L}) = R_k(\mathcal{P})$ for every $k \geq 1$.*

Proof. Direct substitution gives $\sigma(\mathcal{L}(x, y; d)) = \mathcal{P}(y, x; d)$ with the same $d > 0$. Transposition maps the square onto itself and satisfies $\sigma^2 = \text{id}$. Hence, for any coloring χ , the coloring $\chi \circ \sigma$ gives each transposed triple the same colors that the original triple had under χ ; a forbidden triple is monochromatic on one side exactly when its image is monochromatic on the other. Since applying σ twice recovers χ , the correspondence between proper colorings is a bijection. Thus the least excluded side length is the same for both orientations. $\square$

The cited papers give $21 \leq R_3(\mathcal{L}) \leq 493$: Canakci et al. [4, Theorem 3.1 and Figure 1] supply the lower bound and the earlier upper bound 2 593, while Mammel–Smith–Yerger [5, Theorem 6] supply 493. These are the documented bounds used for comparison in Table 7. The numerical improvements considered in Section 7 retain their stated hypotheses.

For the density-theoretic background, Ajtai–Szemerédi [1] prove that for every $\delta > 0$, sufficiently large N forces a standard corner $(a, b), (a + h, b), (a, b + h)$, with a nonzero integer h , in every $S \subseteq [N]^2$ with $|S| \geq \delta N^2$. Szemerédi [2] similarly proves that, for every integer $r \geq 3$ and $\delta > 0$, all sufficiently large subsets of $[N]$ of density at least δ contain an r -term arithmetic progression with nonzero difference. Both are density statements; when we pass to our prescribed orientation, the sign of a standard corner still has to be tracked.

Shkredov’s quantitative work [7, 8] gives quantitative density thresholds. In particular, for a sufficiently large finite abelian group G , density at least $(\log \log |G|)^{-c_0}$, for an absolute $c_0 > 0$, forces a nonzero group corner [8]. Jaber–Liu–Lovett–Ostuni–Sawhney [6] replace this by a function that decays exponentially in a power of $\log |G|$; their precise theorem is the external input in Appendix C. For lattice colorings, Graham–Solymosi [3] give the context, and Shkredov–Solymosi [9] study both axis-parallel and tilted triangles. Allowing tilted triangles enlarges the family of configurations, but it does not by itself force the positive orientation used here.

For explicit small-color bounds, Canakci et al. [4, proof of Theorem 2.3] and Mammel–Smith–Yerger [5, proof of Theorem 2] remove a color common to a diagonal, while the latter’s interval counts and Golomb restrictions [5, Section 2.1 and proof of Theorem 6] constrain the cells that remain. We draw on both devices with attribution. The surveys of O’Bryant [10] and Drakakis [11] provide the Sidon and Golomb background; under our convention, unordered pair sums are likewise unique once doubled endpoints are allowed.

A small example, computed in full. Table 1 crops displayed rows 8–11 and columns 12–15 from Canakci et al.’s Figure 1 [4]. If a_{rc} is their array, our crop is $f_4(x, y) = a_{x+7, y+11}$; the first coordinate indexes rows. R, G, B name the visible colors. With this indexing, a forbidden triple in our convention is a transposed triple in theirs.

Table 1: A proper crop on $A = [4]$, together with its fibers. Boxes identify red diagonal cells; the starred entries form $(1, 1), (3, 1), (3, 3)$, a step-2 corner of colors R, B, R. The right-hand lists are the colors of $(u + d, u)$ in increasing order of u .

$x \backslash y$	1	2	3	4				
1	R *	G	B	R	d	F_d	Colors	r_d
2	G	R	G	B	1	$\{1, 2, 3\}$	G, B, B	3
3	B*	B	R *	G	2	$\{1, 2\}$	B, G	2
4	G	G	B	R	3	$\{1\}$	G	1

We verify this directly from the printed data. For fixed d , list the three colors at (x, y) , $(x + d, y)$, $(x + d, y + d)$ in lexicographic order of $(x, y) \in [4 - d]^2$:

d	Complete list of color triples
1	RGR, GRG, BGB, GBB, RBR, GRG, BGG, BGB, RBR
2	RBR, GBG, GGB, RGR
3	RGR

There are $9 + 4 + 1 = 14$ candidates and none is monochromatic. All six strict-source cells use G or B, as the red diagonal requires. The inherited weak coloring $w_1(v, u) = f_4(v + 1, u)$ has successive rows G; B,B; G,G,B. Its diagonal classes are $\{1\}$ and $\{2, 3\}$, both Golomb, although their union F_1 is not. The starts $1 < 2$ give the corner $(2, 1)$, $(3, 1)$, $(3, 2)$ with colors G,B,B. For this example $M = 6$, $L = 3$, $T = \binom{3}{2} + \binom{2}{2} + \binom{1}{2} = 4$, and $E = 4^2 + 2(3^2 + 2^2 + 1^2) = 44$. Note that the fiber offset d and the corner step $v - u$ are in general different quantities; they happen to agree here only by coincidence. $\square$

The example distinguishes the full fiber F_1 from its two diagonal color classes. The classes are Golomb, but their union need not be. We now prove the identities and coloring restrictions used in this calculation.

3 Notation and the source formalism

We first count corners without imposing a coloring. A corner corresponds to one positive difference with two distinct starts. Additive energy also includes the zero difference and the negative differences, so these contributions must be included separately.

We then apply the diagonal-exclusion and Golomb arguments of Canakci et al. [4] and Mammel–Smith–Yerger [5]. A color shared by diagonal marks is excluded from the strict source on those marks. Figure 2 separates the positive offset d from the corner step $h = v - u$ and shows why a weak source sits inside a strict one.

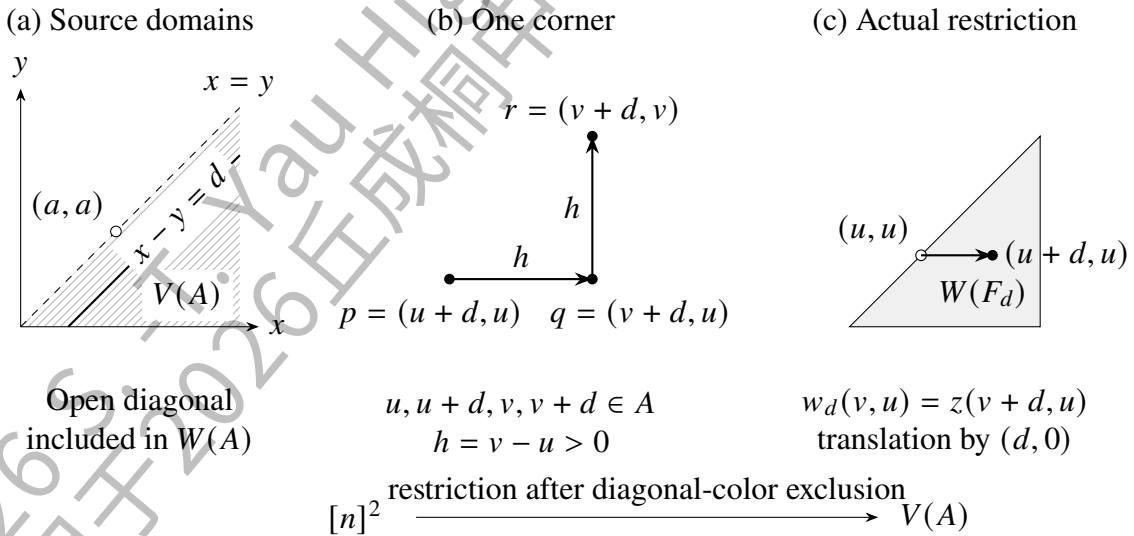


Figure 2: Source geometry and inherited colors. On the hatched side $x > y$, and only cells with both coordinates in A are included. The corner in (b) has equal legs h , and its parallel endpoints have offset d ; the case $u + d = v$ is allowed. In (c), translating the first coordinate sends the whole weak source into the strict source and sends its diagonal to offset d . The bottom arrow records a restriction only; it asserts nothing more.

Throughout this section $A \subset \mathbb{Z}$ is finite. We write $m = |A|$, $M = \binom{m}{2}$, and $L = L(A)$, with the span of the empty set taken to be zero. The remaining source symbols are as in Section 1, and $T(A)$

counts every corner. When discussing colors, we identify the start set F_d with its cells $(u + d, u)$ for $u \in F_d$. We recall that a Golomb set contains at most one pair at each positive difference; in particular, a three-term progression fails to be Golomb, because its two adjacent intervals have the same length. Empty and singleton sets are included everywhere below.

Proposition 3.1 (Exact source counts). *For every finite $A \subset \mathbb{Z}$,*

$$\sum_{d>0} r_d = M, \quad T(A) = \sum_{d>0} \binom{r_d}{2}, \quad \sum_{d>0} r_d^2 = M + 2T(A), \quad E(A) = 2m^2 - m + 4T(A).$$

The corners correspond bijectively to $d > 0$ and $u < v$ in F_d , through $(u + d, u), (v + d, u), (v + d, v)$; their step is $v - u$.

Proof. Every unordered pair of different marks has one smaller mark u and one positive difference d , so it contributes once to r_d . This proves the first identity. Given $u < v$ in F_d , the four memberships $u, u + d, v, v + d \in A$ put all three displayed cells in $V(A)$: their coordinate differences are $d, v + d - u, d$, all positive. The horizontal and vertical legs both have length $v - u > 0$.

Conversely, a corner $(x, y), (x + h, y), (x + h, y + h)$ in $V(A)$ determines $u = y, v = y + h$, and $d = x - y > 0$. The cell memberships imply $u, u + d, v, v + d \in A$, so $u, v \in F_d$. These parameters recover every cell, and they are unique. If $u + d = v$, the middle marks coincide, but the three cells remain distinct; the bijection therefore includes this case. Counting pairs of starts proves the formula for T , and $r_d^2 = r_d + 2\binom{r_d}{2}$ proves the square-sum identity.

For the energy, zero has m ordered representations as a difference and contributes m^2 . Each positive d and each negative $-d$ has r_d representations. Hence $E = m^2 + 2\sum_{d>0} r_d^2 = 2m^2 - m + 4T$, using $2M = m^2 - m$. The empty set has energy zero and a singleton has energy one; their positive-difference sums and corner counts are empty, so the formulas hold there too. $\square$

We next recall the published diagonal mechanism of Canakci et al. [4, Theorem 2.3] and Mammel–Smith–Yerger [5, Theorem 2], and we state it in source notation.

Lemma 3.2 (Diagonal exclusion and weak restriction). *Let $k \geq 1$, let $\chi : [n]^2 \rightarrow [k]$ be proper, and let $A \subseteq [n]$ satisfy $\chi(a, a) = c_0$ for all $a \in A$. Then $\chi(v, u) \neq c_0$ for $u < v$ in A , and its restriction to $V(A)$ is proper in at most $k - 1$ colors. More generally, for a proper coloring $z : V(A) \rightarrow \Omega$ with finite palette Ω and each $d > 0$,*

$$w_d(v, u) = z(v + d, u) \quad (u, v \in F_d, u \leq v)$$

is a proper coloring of $W(F_d)$, whose diagonal color at u is $z(u + d, u)$.

Proof. Fix $u < v$ in A . The cells $(u, u), (v, u), (v, v)$ lie in the square and form $\mathcal{L}(u, u; v - u)$, because both legs have length $v - u > 0$. Their diagonal cells have color c_0 , so properness excludes c_0 at (v, u) . Restricting the same coloring cannot introduce a monochromatic corner, which proves the first assertion (the empty-domain case $k = 1$ included).

For the second assertion, $u \in F_d$ gives $u \in A$, and $v \in F_d$ gives $v + d \in A$. Since $v + d > u$ even when $u = v$, the cell on the right exists in $V(A)$. Translation $(x, y) \mapsto (x + d, y)$ preserves both positive equal legs, so any monochromatic weak corner would map to a monochromatic strict corner under z . This proves properness and gives the diagonal formula. This is a restriction of an existing coloring; the lemma gives no extension from a source to a square. $\square$

We shall also use the Golomb restriction of Mammel–Smith–Yerger [5, proof of Theorem 6], stated in the pair-sum form that follows the standard convention of O’Bryant [10] and Drakakis [11].

Lemma 3.3 (Golomb classes and pair sums). *If $w : W(S) \rightarrow \{0, 1\}$ is proper, each $B_i = \{a \in S : w(a, a) = i\}$ is Golomb. Consequently, if $z : V(A) \rightarrow \{0, 1\}$ is proper, each $B_{d,i} = \{u \in F_d : z(u + d, u) = i\}$ is Golomb. In any finite Golomb set B , an equality $b_1 + b_2 = b_3 + b_4$ implies equality of the unordered pairs with repetition $\{b_1, b_2\} = \{b_3, b_4\}$.*

Proof idea. Two diagonal marks of one color force the cell between them to the other color. A repeated difference in that diagonal class would produce a monochromatic corner among the forced cells.

Proof. If $u < v$ belong to B_i , the diagonal corner used above forces $w(v, u) = 1 - i$. Note that this conclusion uses the two-element palette. Suppose two distinct pairs in B_i have the same positive difference h . Order their starts to write them $(a, a + h)$ and $(a + t, a + t + h)$ with $t > 0$. Then $(a + h, a)$, $(a + t + h, a)$, $(a + t + h, a + t)$ are three distinct strict cells with endpoints in B_i , so all have color $1 - i$. They form a step- t corner. Even if $h = t$, the three cells are distinct and strict. The contradiction proves Golombness. Applying Lemma 3.2 to w_d gives the fiber assertion, using the same coloring throughout.

For the pair-sum claim, order each pair so that $b_1 \leq b_2$ and $b_3 \leq b_4$. If the pairs differ, their smaller entries differ; interchange the representations to arrange $b_1 < b_3$. The equal sum then gives $b_4 < b_2$ and $b_3 - b_1 = b_2 - b_4 > 0$. These are distinct equal-difference pairs: if they coincided, their smaller entries would give $b_1 = b_4$, contradicting $b_1 < b_3 \leq b_4$. This contradicts Golombness, and it also covers a doubled endpoint. $\square$

Since every positive difference is at most the span, there are at most L possible differences. Applying Cauchy–Schwarz to their multiplicities gives the required relation between M and L .

Corollary 3.4 (Energy to span). *If $A \subset \mathbb{Z}$ is finite, $\kappa \geq 0$, and $T(A) \leq \kappa M(A)$, then*

$$M(A) \leq (2\kappa + 1)L(A), \quad |A| \leq 1 + \sqrt{2(2\kappa + 1)L(A)}.$$

A bound assumed only on a class of sets gives these conclusions only on that class.

Proof. If $M = 0$, then A is empty or a singleton and $L = 0$, so both conclusions hold. Otherwise $L \geq 1$ and every positive difference lies in $\{1, \dots, L\}$. There are L positions, not $L + 1$, because zero is excluded. Proposition 3.1 and Cauchy–Schwarz give

$$M^2 = \left(\sum_{d=1}^L r_d \right)^2 \leq L \sum_{d=1}^L r_d^2 = L(M + 2T) \leq (2\kappa + 1)LM.$$

Division by $M > 0$ proves the first bound. Since $(m - 1)^2 \leq m(m - 1) = 2M$ for $m \geq 1$, the second follows. The estimate for T is a hypothesis of this corollary; properness alone has not established it. $\square$

The problem is therefore to bound repeated differences. A proper weak two-coloring gives two Golomb classes, and the next section counts the energy of their union.

4 Two-Golomb energy and its limits

The Golomb restriction of Mammel–Smith–Yerger [5] controls the repeated differences within each color class. However, a fiber contains both classes, so we also need to count repeated differences formed by pairs between the classes. For example, two pairs from B to C can have the same length even though neither B nor C repeats that length internally.

We separate these pairs according to which class contains the smaller mark. This leads to the two counts N and Q in Theorem 4.1. Pairs with the same orientation give shared differences; pairs with opposite orientations give equal sums. We then test two possible stronger bounds by explicit examples.

To compare the result with a standard estimate, let $F_D(t) = \sum_{a \in D} e^{2\pi i a t}$ on $\mathbb{R}/\mathbb{Z}$. Orthogonality gives $E(D) = \|F_D\|_4^4$. Since a Golomb set of size b has energy $2b^2 - b$, the Fourier representation of energy [14] and Minkowski’s inequality give, for disjoint Golomb classes of sizes b, c and $n = b + c$,

$$\begin{aligned} E(B \sqcup C)^{1/4} &\leq (2b^2 - b)^{1/4} + (2c^2 - c)^{1/4} \\ &\leq 2^{1/4}(\sqrt{b} + \sqrt{c}) \leq 2^{3/4}\sqrt{n}. \end{aligned}$$

Thus estimating the two classes separately gives $E(B \sqcup C) \leq 8n^2$. The joint count below gives $6n^2 - 5n$, with a further saving when b and c are unequal. This comparison is with the separate-class estimate above; we do not determine the optimal coefficient.

Theorem 4.1 (Two-Golomb accounting). *Let $B, C \subset \mathbb{Z}$ be finite disjoint Golomb sets, either possibly empty. Put $S = B \sqcup C$, $b = |B|$, $c = |C|$, $n = b + c$, $U = \binom{b}{2} + \binom{c}{2}$, and $M = \binom{n}{2} = U + bc$. For $h > 0$ define*

$$\begin{aligned} \beta_h &= \mathbf{1}_{h \in \Delta B}, & \gamma_h &= \mathbf{1}_{h \in \Delta C}, & X_h &= \{u \in B : u + h \in C\}, & Y_h &= \{u \in C : u + h \in B\}, \\ x_h &= |X_h|, & y_h &= |Y_h|, & \rho &= |\Delta B \cap \Delta C|, \\ N &= \sum_{h>0} (\beta_h + \gamma_h)(x_h + y_h), & Q &= \sum_{h>0} x_h y_h. \end{aligned}$$

Then

$$\sum_{h>0} \left[\binom{x_h}{2} + \binom{y_h}{2} \right] = \rho, \quad T(S) = 2\rho + N + Q, \quad (1)$$

$$2\rho \leq U, \quad N \leq 2bc, \quad Q \leq U, \quad (2)$$

$$T(S) \leq \min\{2M, M + 5\rho\}, \quad E(S) \leq 6n^2 - 5n. \quad (3)$$

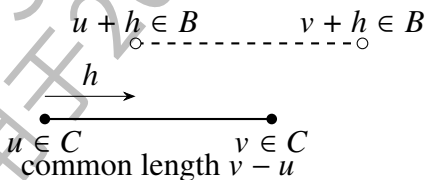
Empty and singleton sets are included.

Proof idea. Count the two orientations separately. Equal lengths with the same orientation correspond to a difference shared by B and C . Opposite orientations give an equal-sum pair in each class. Pair-sum uniqueness then bounds how often an internal pair can occur.

Proof. Because each class is Golomb, β_h, γ_h are exactly its within-class multiplicities. Thus $r_S(h) = \beta_h + \gamma_h + x_h + y_h$, while $\sum_h (x_h + y_h) = bc$, since each cross-class pair has one positive length and one orientation.

Two starts $u < v$ in X_h give the B -pair (u, v) and the C -pair $(u + h, v + h)$ at the common length $v - u$. A collision in Y_h gives the same construction with the classes reversed. Conversely, a length in $\Delta B \cap \Delta C$ has exactly one pair in each class. Their lower endpoints are different because the classes are disjoint. Their displacement has a nonzero sign and absolute value h ; its sign chooses X_h or Y_h . Both endpoints then recover the two starts uniquely. These constructions undo one another, which proves the first identity in (1). Expanding $\binom{\beta_h + \gamma_h + x_h + y_h}{2}$, the term $\beta_h \gamma_h$ sums to ρ , and the two same-orientation collision terms sum to another ρ . The remaining terms are exactly $N + Q$. Also $2\rho \leq |\Delta B| + |\Delta C| = U$, and $\beta_h + \gamma_h \leq 2$ gives $N \leq 2bc$.

For a concrete example, take the sets in Proposition 4.5. The C -pair $(3, 4)$ and the B -pair $(5, 6)$ have shared length one. Translating the first pair by two gives the second, so the starts 3, 4 form one collision in Y_2 , as shown below.



Example: $3, 4 \in C$ and $5, 6 \in B$
lower-start displacement 2; orientation Y_2

An incidence in Q consists of $h > 0$, $u \in X_h$, $v \in Y_h$. It gives equal-sum pairs

$$\{u, v + h\} \subset B, \quad \{u + h, v\} \subset C, \quad u + (v + h) = (u + h) + v.$$

By Lemma 3.3, each pair sum has only one unordered representation in either class, allowing a repeated mark. Indeed, two representations sharing a mark are identical by cancellation. Two different

representations would otherwise repeat a positive difference; this also covers a doubled mark, which would be the midpoint of a three-term progression.

Suppose the matched pairs use four distinct marks $a < b' < c' < e$. Equal sums force the outer pair $\{a, e\}$ to match the inner pair $\{b', c'\}$. Exactly two positive opposite-orientation lengths occur, namely $b' - a = e - c'$ and $c' - a = e - b'$. Charge these two incidences, one each, to the two distinct internal pairs, one in each class. If one pair is doubled, the other has the doubled mark as midpoint and gives just one positive length; charge that incidence to the other, distinct pair. Both pairs cannot be doubled, since this would imply $u = v + h$ and $v = u + h$ with $h > 0$. Pair-sum uniqueness prevents any resource pair from serving a second match. There are only U distinct internal pairs available, so $Q \leq U$. Consequently

$$T \leq U + 2bc + U = 2M, \quad E \leq 2n^2 - n + 8M = 6n^2 - 5n.$$

For the refinement put $D = \Delta B \cap \Delta C$, $J = \sum_{h \in D} (x_h + y_h)$, and $Z = \sum_{h \notin \Delta B \cup \Delta C} (x_h + y_h)$. The multiplier in N is two on D , zero outside the union, and one elsewhere; therefore $N = bc + J - Z$. For integers $k \geq 0$, put $p(k) = \binom{\max\{k-1, 0\}}{2}$. Expansion for $k \geq 1$, and direct evaluation at zero, give

$$k = \mathbf{1}_{k>0} + \binom{k}{2} - p(k).$$

Define the nonnegative quantities

$$\begin{aligned} \nu &= \sum_{h \in D} \left[\binom{x_h}{2} + \binom{y_h}{2} \right], & \eta &= \sum_{h \in D} (2 - \mathbf{1}_{x_h>0} - \mathbf{1}_{y_h>0}), \\ \pi &= \sum_{h \in D} (p(x_h) + p(y_h)), & \mathcal{D}_G &= Z + (U - Q) + \eta + \pi. \end{aligned}$$

The full collision identity gives $\nu \leq \rho$. Summing the integer identity over both orientations on D now gives

$$J = 2\rho + \nu - \eta - \pi, \quad T(S) = M + 4\rho + \nu - \mathcal{D}_G \leq M + 5\rho.$$

All terms in $\mathcal{D}_G$ are nonnegative. The proof uses only Golombness and disjointness, so it applies even when the partition does not come from a coloring. An empty class contributes zero to its counts. For $n \leq 1$, the identities give $T = M = 0$ and the stated energy bound directly. $\square$

Remark. The difference $2M - T$ is exactly $(U - 2\rho) + (2bc - N) + (U - Q)$. Hence equality in $T \leq 2M$ holds if and only if all three nonnegative terms vanish, as they do for $n \leq 1$. Equality in $T \leq M + 5\rho$ holds if and only if $\nu = \rho$ and $\mathcal{D}_G = 0$. When $\rho = 0$, the exact formula becomes $T = M - Z - (U - Q) \leq M$, giving $E \leq 4n^2 - 3n$. Whether the coefficient two is sharp for nontrivial sources remains open here.

Corollary 4.2 (Unequal class sizes). *With the notation of Theorem 4.1,*

$$\begin{aligned} T(S) &\leq \min \left\{ 2M - \frac{(n-1)|b-c|}{2}, M + 5\rho \right\}, \\ E(S) &\leq 6n^2 - 5n - 2(n-1)|b-c|. \end{aligned}$$

If $n \geq 2$, $\varepsilon \geq 0$, and $T(S) \geq (2 - \varepsilon)M$, then $|b - c| \leq \varepsilon n$.

Proof. For $n \geq 1$, the shared-difference count satisfies

$$\rho \leq \min \left\{ \binom{b}{2}, \binom{c}{2} \right\}, \quad U - 2\rho \geq \left| \binom{b}{2} - \binom{c}{2} \right| = \frac{(n-1)|b-c|}{2}.$$

The other two terms in the exact formula for $2M - T$ are nonnegative. Substituting this lower bound gives the first inequality, together with the $M + 5\rho$ bound already proved. The identity $E = 2n^2 - n + 4T$ then gives the energy bound. If $n \geq 2$ and $2M - T \leq \varepsilon M$, division by $(n-1)/2$ gives $|b - c| \leq \varepsilon n$. The empty set satisfies both bounds directly. $\square$

Thus a partition with T close to $2M$ must have nearly equal class sizes. This is only a necessary condition: it gives no construction approaching the bound.

Corollary 4.3 (Fibers and cube moments). *Let $A \subset \mathbb{Z}$ be finite and let $z : V(A) \rightarrow \{0, 1\}$ be proper. Write $m = |A|$, $M = M(A)$, $T = T(A)$, and $L = L(A)$. For the diagonal classes B_d, C_d of the inherited weak coloring w_d , put*

$$\rho_d = |\Delta B_d \cap \Delta C_d|, \quad R_z(A) = \sum_{d>0} \rho_d,$$

$$D_z(A) = \sum_{\substack{d>0 \\ r_d>0}} (r_d - 1) ||B_d| - |C_d||.$$

Both quantities depend on the fixed coloring z , and $D_z(A) \geq 0$. Define, for $j \geq 1$,

$$K_j(A) = \sum_{h_1, \dots, h_j > 0} \#\{x \in \mathbb{Z} : x + \sum_{i \in I} h_i \in A \text{ for every } I \subseteq [j]\}.$$

The generators are ordered and may repeat, and subset sums may coincide. Then

$$K_1 = M, \quad K_2 = T, \quad K_3 = \sum_{d>0} T(F_d) \leq \min \left\{ 2T - \frac{D_z(A)}{2}, T + 5R_z(A) \right\}.$$

If $C_3(A)$ counts ordered tuples $(x, h_1, h_2, h_3) \in \mathbb{Z}^4$ satisfying the same vertex conditions, allowing signed and zero generators, then

$$C_3(A) = m + 6M + 12T + 8K_3$$

$$\leq 7E(A) - 11m^2 + 5m - 4D_z(A),$$

and

$$\sum_{d=1}^L \frac{r_d^4}{2(L-d)+1} \leq M + 12T - 2D_z(A).$$

The last sum is empty if $L = 0$.

Proof. Lemmas 3.2–3.3 give two Golomb classes in each F_d . A one-generator configuration is a pair of marks, so $K_1 = M$. Fixing the first generator d in a larger configuration leaves a positive cube of starts in F_d : every remaining vertex and its translate by d must belong to A . For two generators this counts pairs in F_d , giving $K_2 = T$. For three it gives $K_3 = \sum_{d>0} T(F_d)$.

Apply Corollary 4.2 to each nonempty fiber partition. Since $\sum_{d>0} \binom{r_d}{2} = T$, summing its first bound gives $K_3 \leq 2T - D_z(A)/2$. Summing the other bound from Theorem 4.1 gives $K_3 \leq T + 5R_z(A)$.

For the signed count, choose the j nonzero generator positions and their signs. Replace every negative generator by its absolute value and move the base to $x + \sum_{h_i < 0} h_i$. Reversing these choices recovers the original ordered tuple, even if vertices coincide. There are $\binom{3}{j} 2^j$ choices, namely 1, 6, 12, 8 for $j = 0, 1, 2, 3$. Hence the formula for C_3 is exact. The bound for K_3 gives

$$C_3 \leq m + 6M + 28T - 4D_z(A) = 7E(A) - 11m^2 + 5m - 4D_z(A).$$

For the last inequality, let p_s count ordered pairs with sum s in an r -mark integer set of span at most ℓ . There are at most $2\ell + 1$ possible sums and $\sum_s p_s = r^2$. Energy also counts equal sums, since rearranging $a - b = c - e$ gives $a + e = b + c$. Cauchy–Schwarz therefore gives $E = \sum_s p_s^2 \geq r^4 / (2\ell + 1)$. A nonempty F_d has span at most $L - d$. Corollary 4.2 also gives

$$E(F_d) \leq 6r_d^2 - 5r_d - 2(r_d - 1) ||B_d| - |C_d||.$$

Combine these bounds and sum, using $\sum_d r_d = M$ and $\sum_d r_d^2 = M + 2T$. The result is $6(M + 2T) - 5M - 2D_z(A) = M + 12T - 2D_z(A)$. Empty fibers contribute zero, and a singleton at $d = L$ has denominator one. $\square$

These refinements retain the nonnegative deficits from unequal class sizes in each fiber. Since $D_z(A) \geq 0$, omitting these corrections recovers the coarser bounds. Neither form establishes a uniform inequality $T(A) \leq CM(A)$ for arbitrary properly two-colored strict sources.

For the four-cube with weights $(1, 3, 9, 27)$ and the specific coloring in Table 4, direct fiber counting gives $T = 200$, $D_z(A) = 26$, and $R_z(A) = 25$. The correction lowers the first branch from 400 to 387, while the second branch is $T + 5R_z = 325$. The combined upper bound therefore remains 325 for this coloring; a stronger first branch need not improve the minimum in every example.

The bound $N \leq 2bc$ in (2) suggests a possible strengthening: could $T(S) \leq 2|B||C|$ hold when the two classes come from a proper weak coloring? The following example shows why the internal pairs cannot be omitted.

Proposition 4.4 (A weak-source sharpness limit). *There is a proper two-coloring of $W(S)$ with six marks in each diagonal color class and $T(S) = 74 > 72 = 2|B||C|$.*

Proof. Let $P = \{0, 4, 7, 12, 21, 23\}$, $S = P + \{0, 1\}$, $B = P$, and $C = P + 1$. Every mark has a unique expression $p + \alpha$ with $p \in P$, $\alpha \in \{0, 1\}$, since distinct marks of P differ by at least two. Give $(p + \alpha, p + \alpha)$ color α ; give $(p + 1, p)$ color one for $p \in \{0, 4, 23\}$ and zero for $p \in \{7, 12, 21\}$. Table 2 specifies every remaining cell and prints the resulting full array.

Table 2: All cells of the twelve-mark weak coloring. In the first panel $p < q$ and $\alpha, \beta \in \{0, 1\}$. In the second, rows are v and columns are u ; a dash means that $(v, u) \notin W(S)$.

Type	Pairs (p, q)	$z(q + \beta, p + \alpha)$
A	$(0, 4)$	$1 - \beta$
B	$(0, 7), (0, 23), (4, 12), (4, 23)$	$(1 - \alpha)(1 - \beta)$
C	$(0, 12), (0, 21), (4, 7), (4, 21)$	$1 - \alpha\beta$
D	$(7, 12), (7, 21), (7, 23), (12, 21), (12, 23), (21, 23)$	$1 - \alpha$

$v \backslash u$	0	1	4	5	7	8	12	13	21	22	23	24
0	0	–	–	–	–	–	–	–	–	–	–	–
1	1	1	–	–	–	–	–	–	–	–	–	–
4	1	1	0	–	–	–	–	–	–	–	–	–
5	0	0	1	1	–	–	–	–	–	–	–	–
7	1	0	1	1	0	–	–	–	–	–	–	–
8	0	0	1	0	0	1	–	–	–	–	–	–
12	1	1	1	0	1	0	0	–	–	–	–	–
13	1	0	0	0	1	0	0	1	–	–	–	–
21	1	1	1	1	1	0	1	0	0	–	–	–
22	1	0	1	0	1	0	1	0	0	1	–	–
23	1	0	1	0	1	0	1	0	1	0	0	–
24	0	0	0	0	1	0	1	0	1	0	1	1

Every weak corner has parallel endpoints $(u + d, u), (v + d, v)$ with $u < v$ and $d \geq 0$, and completion $(v + d, u)$. Unequal endpoint colors are already safe. At $d = 0$, the equal-color start sets are P and $P + 1$; the rules give all completions color one and zero, respectively. Table 10 in Appendix A lists every positive-offset equal-color group with at least two starts and all its completions. Each completion has the opposite color. Its fiber counts also show that every omitted group has at most one start, so the check covers every corner of the weak domain.

Grouping the positive fibers by multiplicity gives the following complete histogram:

r_d	1	2	3	4	7
Number of offsets	1	11	8	3	1

Consequently

$$T = 11 \binom{2}{2} + 8 \binom{3}{2} + 3 \binom{4}{2} + \binom{7}{2} = 11 + 24 + 18 + 21 = 74 > 72 = 2 \cdot 6 \cdot 6.$$

Thus $T(S) \leq 2|B||C|$ fails even for a proper weak coloring. This explains why the proof of Theorem 4.1 also counts internal pairs. $\square$

A different strengthening would be $N \leq 2bc - \min\{b, c\}$. One possible argument would assign at least one lost incidence to every mark of the smaller class. However, the losses can be concentrated at a single mark, as the next example shows.

Proposition 4.5 (Failure of a separate linear deficit). *For $B = \{0, 2, 5, 6\}$ and $C = \{1, 3, 4, 8\}$, both sets are Golomb and $N = 29 > 28 = 2bc - \min\{b, c\}$. No weak coloring is asserted for this partition.*

Proof. The six positive differences of B , in increasing-pair order, are 2, 5, 6, 3, 4, 1; those of C are 2, 3, 7, 1, 5, 4. Each list is repetition-free, with supports $\{1, 2, 3, 4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 7\}$. Table 3 prints all sixteen mixed distances and the oriented counts, so every term of N is recoverable.

Table 3: The concentrated-deficit example. The left panel lists $|p - q|$ for $p \in B, q \in C$. The right panel records $n_h = (\beta_h + \gamma_h)(x_h + y_h)$ and $q_h = x_h y_h$. The last two columns sum to $N = 29$ and $Q = 11$, and five rows have $\beta_h = \gamma_h = 1$, giving $\rho = 5$.

					h	β_h	γ_h	x_h	y_h	n_h	q_h
$p \backslash q$	1	3	4	8	1	1	1	2	2	8	4
					2	1	1	2	2	8	4
0	1	3	4	8	3	1	1	2	1	6	2
2	1	1	2	6	4	1	1	1	1	4	1
5	4	2	1	3	5	1	1	0	1	2	0
6	5	3	2	2	6	1	0	1	0	1	0
					7	0	1	0	0	0	0
					8	0	0	1	0	0	0

The sum is $N = 2(4 + 4 + 3 + 2 + 1) + 1 = 29$, whereas $2bc - b = 32 - 4 = 28$. A mixed pair at length h loses $2 - \beta_h - \gamma_h$ incidences. Fourteen pairs lose none; only (2, 8) loses one and (0, 8) loses two. Thus $2bc - N = 3 < 4$, and all losses meet the same mark $8 \in C$. In particular, the marks $5, 6 \in B$ meet no deficient pair. Therefore Golombness and order alone do not guarantee the proposed linear deficit. $\square$

The examples rule out two different stronger bounds; only the first is asserted to admit a proper weak coloring. Theorem 4.1 applies to every fiber partition, but it does not by itself bound $T(A)/M(A)$ for all proper strict sources. We next use the Golomb restrictions to study cubes contained in a source.

5 Cubes and the forcing graph

A Boolean cube gives a simple way to produce repeated differences: adding one generator to different subset sums gives pairs of the same length. The corresponding ordinary binary Sidon problem has known maximum size seven in dimension four [12]. We give a short pair count for Golomb subsets of arbitrary injective weighted cubes, then use the bound with an explicit coloring to determine the largest attainable dimension of a properly two-colored strict cube source.

For three-cubes, the Golomb restriction also forces at least nine mixed corners, and we give a coloring attaining this number. To compare such local restrictions across a source, we introduce a graph whose vertices are the actual source cells. Its edges record the color forced at the completion of a corner.

5.1 Golomb capacity and the attainable dimension

Call a Boolean cube with generators $w_1, \dots, w_d$ *injective* when its 2^d subset sums are distinct, and *separated* when $w_{i+1} > 2 \sum_{j \leq i} w_j$ for every $1 \leq i < d$. We call the generators *2-dissociated* if $\sum_i a_i w_i = 0$, with every $a_i \in \{-2, -1, 0, 1, 2\}$, implies that all a_i are zero. Separation implies 2-dissociation, which implies injectivity. The exclusions below need only injectivity; the four-cube construction uses separation.

Lemma 5.1 (Pair capacities). *Let $d \geq 1$, let $w_1, \dots, w_d$ be positive real numbers with distinct subset sums, and let $a \in \mathbb{R}$. Put $Q_d = a + \{\sum_i \epsilon_i w_i : \epsilon \in \{0, 1\}^d\}$. If $B \subseteq Q_d$ is Golomb and $e_J(B)$ counts pairs whose labels differ exactly on $\emptyset \neq J \subseteq [d]$, then*

$$e_J(B) \leq \min\{2^{|J|-1}, 2^{d-|J|}\}, \quad \binom{|B|}{2} \leq \sum_{j=1}^d \binom{d}{j} \min\{2^{j-1}, 2^{d-j}\}.$$

In particular $|B| \leq 7$ for $d = 4$.

Proof idea. Fix the coordinates on which a pair differs. Its signs on these coordinates determine its difference, while its common bits on the remaining coordinates determine its sum. Since both differences and unordered sums are unique in a Golomb set, each count gives an upper bound.

Proof. Write $j = |J|$. First, orient a pair and record its signs in $\{-1, 1\}^J$. Reversing the orientation negates every sign, leaving 2^{j-1} types for unordered pairs. A type determines one absolute numerical difference, independently of the common bits outside J . Injectivity makes this difference nonzero, and Golombness permits at most one pair of each type. If different types give the same numerical difference, this only reduces the number of possible pairs.

Second, fix the common bits outside J ; there are 2^{d-j} possible choices. The pair sum is $2a + \sum_{i \in J} w_i + 2 \sum_{i \notin J} \epsilon_i w_i$, so each choice fixes one sum. The pair-sum argument of Lemma 3.3 applies equally to real marks and makes this unordered representation unique. Hence there is at most one pair for each choice of common bits. Every distinct pair belongs to exactly one nonempty J , so summing gives the second bound. At $d = 4$ its right side is

$$4 \cdot 1 + 6 \cdot 2 + 4 \cdot 2 + 1 \cdot 1 = 25 < 28 = \binom{8}{2},$$

so a Golomb subset cannot contain eight marks. □

Remark (The known binary Sidon value). A set $C \subseteq \{0, 1\}^d$ is binary Sidon if every sum of two labels has a unique unordered representation, allowing a repeated label. Here addition is ordinary coordinatewise addition, not addition modulo two. This setting and its relation to integer representations are discussed by Wood [13].

Write $\phi(\epsilon) = a + \sum_i \epsilon_i w_i$. If $\phi(C)$ is Golomb and the cube is injective, then C is binary Sidon: an equality between vector sums gives the same equality between scalar sums, and Golombness makes the unordered scalar pairs, hence their labels, identical. Conversely, with $w_i = 3^{i-1}$, pair sums have ternary digits in $\{0, 1, 2\}$ with no carries. Equality of scalar pair sums therefore implies equality of vector sums. Thus every binary Sidon set gives a Golomb set for these weights. The largest attainable Golomb size over injective d -cubes is consequently the binary Sidon number $a(d)$.

The value $a(4) = 7$ is recorded in [12]; Lemma 5.1 supplies a proof of the upper bound suited to the cube restrictions used here. For attainment, take

$$B = \{0, 1, 4, 13, 28, 30, 36\} \subseteq \left\{ \sum_{i=1}^4 \epsilon_i 3^{i-1} : \epsilon_i \in \{0, 1\} \right\}. \quad (4)$$

The binary masks are 0, 1, 3, 7, 9, 10, 12, and the $\binom{7}{2} = 21$ positive differences are exactly

$$\Delta B = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 15, 17, 23, 24, 26, 27, 28, 29, 30, 32, 35, 36\}. \quad (5)$$

They are all distinct, so B is Golomb. This example is equivalent to the seven-mask example in [12]: in low-to-high bit order, $(x_1, x_2, x_3, x_4) \mapsto (x_2, 1 - x_1, x_4, x_3)$ sends its masks to $\{0, 1, 2, 4, 7, 9, 14\}$. We use the known value and this convenient representative below; neither is claimed as a new extremum or construction. Attaining seven does not require equality in every bound for $e_j(B)$.

Corollary 5.2 (Cube occupancy). *Let $A \subset \mathbb{Z}$ be finite. If $W(A)$ admits a proper two-coloring, then $|A \cap Q| \leq 14$ for every injective four-cube Q with integer translation and positive real generators. If $V(A)$ admits a proper two-coloring, then $|A \cap Q| \leq 30$ for every such five-cube Q . In particular, A contains no full injective five-cube.*

Proof. Restrict a weak coloring to the occupied cube marks. Its two diagonal classes are Golomb by Lemma 3.3, so Lemma 5.1 gives at most seven marks in each class, or fourteen in total.

For a strict source, pair each of the sixteen labels in one four-dimensional face with its translate by the fifth generator d . If d is not an integer, no pair can have both marks in the integer source, so at most sixteen vertices are occupied. Otherwise, the start of each complete pair belongs to F_d . Lemma 3.2 gives a proper weak coloring on this fiber, so the first part shows that at most fourteen pairs are complete. If $k \leq 14$ pairs are complete, the occupied vertices number at most $2k + (16 - k) = 16 + k \leq 30$. Injectivity makes all sixteen pairs disjoint, even if the two faces interleave in numerical order. $\square$

Theorem 5.3 (A separated four-cube coloring). *If positive integers $w_1, \dots, w_4$ satisfy $w_{i+1} > 2 \sum_{j \leq i} w_j$ for $1 \leq i < 4$, then $A = \{\sum_{i=1}^4 \epsilon_i w_i : \epsilon_i \in \{0, 1\}\}$ admits the proper strict two-coloring in Table 4. The resulting coloring attains the dimension bound in Corollary 5.4.*

Proof idea. Split the cube into two three-dimensional faces and give tables for the cells within each face and between the faces. Under separation, two differences are equal exactly when their signed coordinate expressions agree. Thus a finite check of the tables applies to every weight sequence in the theorem.

Proof. Let s_u be the sum encoded by the three-bit mask $u \in \{0, \dots, 7\}$, with bit $i - 1$ representing w_i , and put $S = \{s_u\}$. Mask order agrees with numerical order, because the largest differing weight exceeds the sum of all lower weights. To verify 2-dissociation, suppose $\sum_i a_i w_i = 0$ with $a_i \in \{-2, -1, 0, 1, 2\}$. A largest nonzero index would contribute at least its weight in absolute value, strictly more than all possible lower contributions. This is impossible, so all a_i vanish.

Table 4 defines the coloring by

$$z(s_v, s_u) = L(v, u), \quad z(w_4 + s_v, w_4 + s_u) = U(v, u) \quad (u < v), \quad z(w_4 + s_r, s_c) = X(r, c).$$

All cross cells are strict since $w_4 > 2L(S)$. The three disjoint types cover the entire domain.

To distinguish labels from numbers, define $\widehat{L}(s_v, s_u) = L(v, u)$, $\widehat{U}(s_v, s_u) = U(v, u)$, and $\widehat{X}(s_r, s_c) = X(r, c)$. Put $\ell = L(S)$. An internal offset is at most ℓ , whereas a cross offset is at least $w_4 - \ell > \ell$. Hence two cells with equal offsets are either both internal or both cross cells. Internal cells lie in the same lower face, the same upper face, or different faces; call the last case a *bridge*.

For the cross case, write the two cells as $(w_4 + r, c)$ and $(w_4 + r', c')$, now with $r, c, r', c' \in S$ denoting numerical marks. Equality of offsets gives $r - c = r' - c'$. After ordering $c < c'$, put

Table 4: The complete four-cube assignment. Row v in the first panel lists columns $0, \dots, v-1$. The second panel gives the whole cross matrix. Indices are binary-mask labels, and all numerical additions use their subset sums s_u .

v	$L(v, 0) \cdots L(v, v-1)$	$U(v, 0) \cdots U(v, v-1)$	$r \backslash c$	0	1	2	3	4	5	6	7
1	1	0	0	1	0	0	1	0	1	1	0
2	10	01	1	1	0	1	1	1	1	0	0
3	100	111	2	1	1	0	1	1	0	1	0
4	1001	0110	3	0	0	0	1	0	0	0	1
5	10110	11001	4	1	1	1	0	0	1	1	0
6	110101	101010	5	0	0	1	0	0	1	0	0
7	0001111	1000000	6	0	1	1	0	1	0	0	0
			7	0	1	1	0	1	0	1	0

$h = c' - c > 0$; then $r' = r + h$. We therefore have four families: lower, upper, cross, and bridge. The cross and bridge color words are, respectively,

$$\widehat{X}(r, c), \widehat{X}(r + h, c), \widehat{X}(r + h, c + h); \quad \widehat{L}(u + h, u), \widehat{X}(v + h, u), \widehat{U}(v + h, v).$$

All displayed arguments belong to S , including $u, u + h, v, v + h$ in a bridge.

In S , a positive signed support with t active coordinates has its highest active sign positive. There are $n_t = \binom{3}{t} 2^{t-1}$ such supports, namely 3, 6, 4 for $t = 1, 2, 3$. Each has $m_t = 2^{3-t}$ representations, obtained by choosing the common bits. By 2-dissociation, two numerical differences agree exactly when their signed supports do. Each internal family therefore has $\sum_t n_t \binom{m_t}{2} = 24$ corners. A cross corner chooses one row representation and one column representation of the same positive difference. A bridge similarly chooses a lower and an upper representation. Thus each of these families has $\sum_t n_t m_t^2 = 3 \cdot 16 + 6 \cdot 4 + 4 = 76$ corners.

For completeness, Table 11 in Appendix A records every color-word multiplicity. It can be reproduced without selecting numerical weights: for each signed support put a lower bit zero and upper bit one at a positive coordinate, reverse these at a negative coordinate, and run through all common bits at the other coordinates. Pair the resulting representations as just described and read off Table 4. For example, support $(1, 0, 0)$ gives lower endpoint colors 1, 0, 0, 1 at common masks 0, 2, 4, 6; their six pair words are 101, 101, 110, 001, 010, 011. Summing the three primitive supports gives its first row. The same finite rule determines every subsequent row; absent words have multiplicity zero.

Every possible corner belongs to one of these supports and families, and every resulting word uses both colors. Hence the coloring is proper for every weight sequence in the theorem. Together with Corollary 5.2, this gives the dimension threshold below. $\square$

Corollary 5.4 (Exact attainable cube dimension). *Among injective Boolean cubes with positive integer generators, the largest dimension whose strict source admits a proper two-coloring is exactly four.*

Proof. Every cube of dimension at least five contains an injective five-coordinate face. Restricting a proper source coloring to that face contradicts Corollary 5.2. Conversely, Theorem 5.3 supplies a four-dimensional example, for instance with generators $(1, 3, 9, 27)$; restrictions supply examples in every smaller positive dimension. Translation preserves properness. Thus the largest attainable dimension is exactly four. $\square$

The assertion is existential in dimension four. We do not claim every injective four-cube admits a proper coloring; the construction above assumes separation.

5.2 The minimum mixed-corner count on a three-cube

Theorem 5.5 (The sharp three-cube cost). *Let $Q = x + \{\sum_{i=1}^3 \epsilon_i w_i : \epsilon_i \in \{0, 1\}\}$ have integer x , positive integer generators, and eight distinct subset sums. Every proper two-coloring of $V(Q)$ has at least nine mixed corners, where the two parallel endpoints have different colors. If the generators are 2-dissociated, the construction below attains*

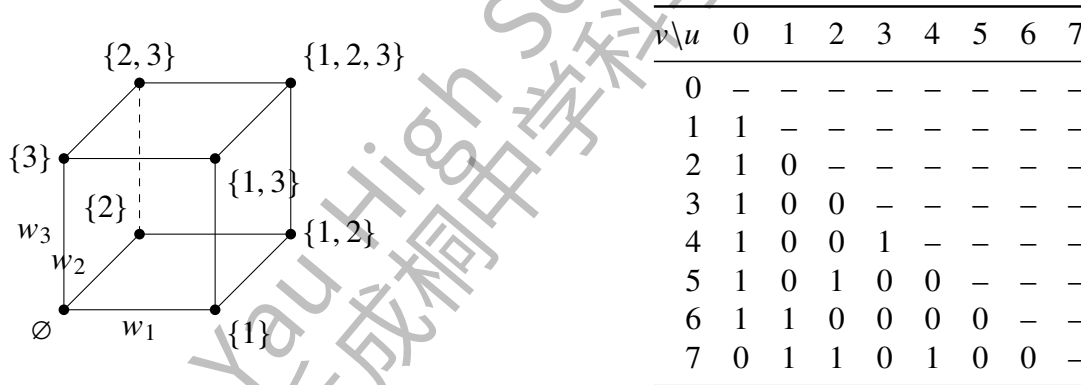
$$M = 28, \quad T = 24, \quad B_{\text{mix}} = 9, \quad H_{\text{force}} = T - B_{\text{mix}} = 15.$$

Thus nine is the exact minimum for every 2-dissociated three-cube, including every separated three-cube.

Proof idea. Each generator gives a fiber containing four starts with a repeated difference. These four cells must use both colors, giving at least three mixed corners per generator. To attain nine, we make each of these three fibers split 1 + 3 and check that every other repeated fiber has just one color.

Proof. We first prove the lower bound. At offset w_i , the four starts $x + \{0, w_j, w_k, w_j + w_k\}$ belong to the fiber, where $\{i, j, k\} = [3]$. They have a repeated w_j -difference and are not Golomb. By Lemma 3.3, their four cells must therefore use both colors. If k_0 of them have one color, then $1 \leq k_0 \leq 3$ and there are $k_0(4 - k_0) \geq 3$ mixed pairs. Each gives a mixed corner. Injectivity makes w_1, w_2, w_3 distinct, so the three families have distinct offsets and contribute at least nine different corners. Any additional representations of these differences only add corners.

To attain the bound, label marks by subsets of $[3]$ and orient each pair by its numerical values. Give color one exactly to an empty/singleton or empty/doubleton pair, a singleton/full-set pair, or a singleton and its complementary doubleton. Give every other strict cell color zero. Figure 3 gives the full array for separated weights. The rule itself uses unordered subset labels, so it also defines a coloring when positive 2-dissociated weights do not follow mask order.



0 : $\emptyset$, 1 : $\{1\}$, 2 : $\{2\}$, 3 : $\{1, 2\}$, 4 : $\{3\}$, 5 : $\{1, 3\}$, 6 : $\{2, 3\}$, 7 : $[3]$.

At each w_i : $\boxed{1} 0 0 0$; the boxed cell pairs with each zero. The nine mixed pairs are

- w_1 : (1, 0) with (3, 2), (5, 4), (7, 6),
- w_2 : (2, 0) with (3, 1), (6, 4), (7, 5),
- w_3 : (4, 0) with (5, 1), (6, 2), (7, 3).

Figure 3: The sharp three-cube construction. Vertex I represents $x + \sum_{i \in I} w_i$ in binary coordinates. The strict-cell table and the mixed-pair lists use separated weights and the mask legend above. Each listed endpoint pair has its unique outer completion. The 1 + 3 primitive splits give the nine mixed corners; Table 12 in Appendix A also records the other fifteen.

At offset w_i , the common-bit sets $\emptyset, \{j\}, \{k\}, \{j, k\}$ give colors 1, 0, 0, 0. The three pairs involving the first entry are mixed. For each pair of zero entries, the completion joins a singleton to its complementary doubleton or to the full set, so it has color one. Thus all six corners at this offset are proper.

We distinguish the remaining supports by their size. A two-coordinate support has two representations. For equal signs they are empty/doubleton and singleton/full cells, both one, with empty/full completion zero. For opposite signs they join two singletons and two doubletons, both zero, with complementary singleton/doubleton completion one. A three-coordinate support has one representation and makes no corner. By 2-dissociation these cases exhaust the differences. There are three fibers of size four, six of size two, and four of size one. Therefore $M = 3 \cdot 4 + 6 \cdot 2 + 4 = 28$, $T = 3 \binom{4}{2} + 6 \binom{2}{2} = 24$, and only the primitive fibers contribute mixed corners, giving $B_{\text{mix}} = 9$ and $H_{\text{force}} = 15$.

Table 12 gives the complete representative certificate for weights $(1, 3, 9)$, with the translation suppressed. A pair (i, j) denotes the cell (s_i, s_j) ; the indices are not added. The symbolic proof above justifies its use throughout the claimed family. $\square$

Remark. The nine-corner lower bound needs only injectivity, whereas the stated exact construction needs 2-dissociation. It excludes a full injective three-cube in an all-forcing proper source. A bound on how often mixed corners are reused across different cubes is still missing.

5.3 Counting corners with a shared completion

We next write the two colors as -1 and 1 . For a fixed completion $q = (v, u)$, the two parallel inputs have a common offset $0 < d < v - u$ and form a corner of step $v - u - d$. Their half-sum is zero when their signs differ and equals their common sign when they agree. This gives a convenient way to count all corners forcing the color of q . In the graph below, q remains one vertex even if several corners have this completion.

Theorem 5.6 (Signed completion and physical forcing). *Let $A \subset \mathbb{Z}$ be finite and $z : V(A) \rightarrow \{-1, 1\}$ be proper. Write $M = M(A)$ and $T = T(A)$. For each $q = (v, u) \in V(A)$, put*

$$D_q = \{d \in \mathbb{Z} : 0 < d < v - u, u + d, v - d \in A\}, \quad p_{q,d} = (u + d, u), \quad r_{q,d} = (v, v - d),$$

$$(Kz)_q = \frac{1}{2} \sum_{d \in D_q} (z(p_{q,d}) + z(r_{q,d})), \quad H_q = \#\{d \in D_q : z(p_{q,d}) = z(r_{q,d})\}, \quad H_{\text{force}} = \sum_q H_q.$$

Then $H_q = -z(q)(Kz)_q = |(Kz)_q|$. On the physical cells $V(A)$, insert edges pq, rq for each forcing corner with inputs p, r and output q , and no edge pr . The resulting graph Γ is simple, bipartite under z , and satisfies

$$|E(\Gamma)| = 2H_{\text{force}}, \quad H_{\text{force}} = \frac{2T - M + \sum_d s_d^2}{4}, \quad 2T = M + 2|E(\Gamma)| - \sum_d s_d^2,$$

where $s_d = \sum_{u \in F_d} z(u + d, u)$. If $b(p)$ counts the cells of p 's own color in its fiber, including p , then $\deg_\Gamma(p) = b(p) - 1 + 2H_p$.

Proof idea. At a fixed output q , all equal-sign input pairs must have sign $-z(q)$. This gives the pointwise identity. Each row or column edge determines its corner uniquely, so a forcing corner contributes exactly two distinct graph edges. Counting equal-sign pairs in each fiber then gives the global identity.

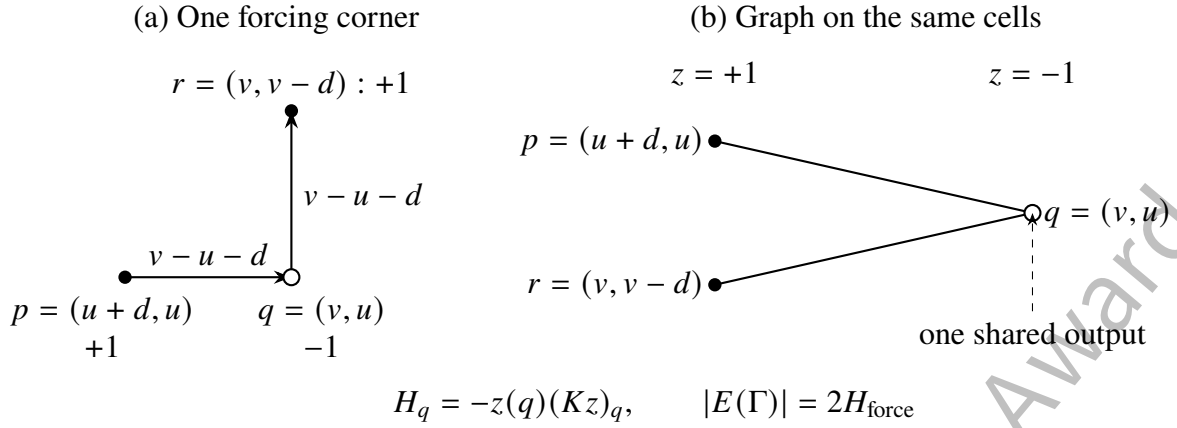


Figure 4: A forcing corner contributes two edges. Here $0 < d < v - u$ and both legs have length $v - u - d > 0$. Filled and open circles distinguish the signs. There is no input–input edge. The output is one vertex even when several corners demand its color; interchanging both signs gives the other case.

Proof. For $d \in D_q$, both input cells are strict and their displacement is $(v - u - d, v - u - d)$, with positive coordinates. Together with q they form a corner, and conversely any corner completed at q determines this unique d . Note that if $u + d = v - d$, the inputs still have different first and second coordinates and remain distinct.

Opposite input signs contribute zero to Kz and nothing to H_q . Equal signs must both be $-z(q)$, or their corner would be monochromatic. Summing gives $(Kz)_q = -z(q)H_q$, and hence both pointwise equalities, including the case $H_q = 0$.

The edge from p to q is horizontal and the edge from r to q is vertical. A horizontal pair $(a, u), (v, u)$ with $a < v$ determines the third cell $(v, u + v - a)$; a vertical pair determines its third cell in the same way. Thus two corners cannot insert the same edge. Each new edge joins opposite signs, and the inputs are distinct from each other and from the output. Hence the graph is simple and bipartite, and every forcing corner contributes exactly two edges. As an input, a cell p pairs with its $b(p) - 1$ other own-color fiber cells, giving that many incident edges. As an output it receives $2H_p$. The two roles have disjoint edges, because every output has larger coordinate difference than either input. This proves the degree formula.

Finally, a fiber has $(r_d + s_d)/2$ positive and $(r_d - s_d)/2$ negative cells, both nonnegative integers. Its equal-sign pair count is

$$\binom{(r_d + s_d)/2}{2} + \binom{(r_d - s_d)/2}{2} = \frac{r_d^2 + s_d^2 - 2r_d s_d}{4}.$$

Every pair has one valid completion by Proposition 3.1. Sum over fibers, substitute $\sum r_d = M$ and $\sum r_d^2 = M + 2T$, and then use $|E(\Gamma)| = 2H_{\text{force}}$. This yields both global identities, with the empty source included. $\square$

The identity shows what a global estimate must control: the net quantity $2|E(\Gamma)| - \sum_d s_d^2$. A linear bound on $|E(\Gamma)|$ would suffice, but it is not necessary if the imbalance term can offset a larger edge count. The following assertion is one possible route.

Open assertion 5.7 (G9). For every finite integer A and every proper two-coloring of $V(A)$, the forcing graph satisfies $|E(\Gamma)| \leq 9M(A)$. This assertion is open.

Proposition 5.8 (Conditional consequence of G9). *Assume G9. Every properly two-colored strict source then satisfies $T \leq 19M/2$ and $M \leq 20L$. It is enough for S21 that G9 hold only when $L \leq 99994$.*

Proof. Assuming G9, Theorem 5.6 gives $2T \leq M + 18M = 19M$, since $\sum_d s_d^2 \geq 0$. Applying

Corollary 3.4 with $\kappa = 19/2$ gives $M \leq 20L$, including the case $M = 0$. If G9 is assumed only for $L \leq 99\,994$, the same argument gives $M \leq 20L \leq 21L$ on this range, as required by S21. $\square$

The cube bounds and the forcing identities are unconditional. G9 remains open: bipartiteness alone does not give a linear edge bound, and the cube arguments do not control how often the same cells occur in different cubes. In the next section, we obtain bounds for T/M by placing additional assumptions on the structure of the source.

6 Structured sources and endpoint deletion

We next ask whether the number of corners can be bounded by a constant times the number of cells. We can prove this for four families: complete progression products, complete direct sums, translation matchings, and sources with two-distance representations. In a separated product, an equality of differences can be checked in each factor. For a translation matching, the two copies may overlap in their difference ranges, so we count the resulting correlations explicitly. The fourth argument uses Euclidean distances to count additive relations.

Table 5 states the hypotheses and conclusions. We also study the number of corners lost when one endpoint is deleted. The endpoint estimate is sharp under its anchor assumptions, but these assumptions need not hold after the deletion. Throughout this section we write $n = |A|$ for the number of marks.

Table 5: Bounds for the structured families. All proper colorings in this table use two colors, and a complete product contains every pair of factor marks. The matching span bound uses positive-difference counting in its two Golomb classes. The endpoint row concerns a single deletion; EP10 and H10 remain open.

Class	Hypotheses	Corner/energy bound	Span bound
Progression product	$A = B + Q\{0, \dots, t-1\}$ complete; $B \neq \emptyset$, integers $t \geq 2$, $Q > 2L(B)$; proper $V(A)$	$T \leq 7M/2$; $E \leq 9n^2 - 8n$	$M \leq 8L$
Direct sum	$A = B + QC$ complete; $ B , C \geq 2$; integer $Q > 2L(B)$; proper $V(A)$	$T \leq 7M$; $E \leq 16n^2 - 15n$	$M \leq 15L$
Translation matching	$A = S \sqcup (S + d)$, integer $d > 0$; proper $V(A)$; no separation; $S \neq \emptyset$ for strict span bound	$T \leq 10M$; $E \leq 22n^2 - 21n$	$M < 18L$; radical in Theorem 6.5
Two distances	Injective map preserving all pair-sum equalities, including repetitions; nonzero squared distances $\tau, 2\tau$, $\tau > 0$	$T \leq 7M$; explicit proper coloring	$M \leq 15L$
Restricted endpoint	$\min A = 0$, $n \geq 2$; proper all-forcing coloring; uniform anchor or exactly one exceptional anchor	$e_0 = 0$ if uniform; otherwise $e_0 \leq n - 2$	No iterative span bound

Theorem 6.1 (Complete progression products). *Let $B \subset \mathbb{Z}$ be finite and nonempty, let $t \geq 2$ and $Q > 2L(B)$ be integers, and put $A = B + Q\{0, \dots, t-1\}$, $n = t|B|$. If $V(A)$ admits a proper two-coloring, then*

$$E(A) \leq 9n^2 - 8n, \quad T(A) \leq \frac{7}{2}M, \quad M \leq 8L.$$

The same conclusions follow from the weaker hypothesis that $B + Q\{0, \dots, t-2\}$ is a union of two disjoint Golomb sets.

The fiber at offset Q contains exactly $t - 1$ complete layers. We compare its energy with the energy of all t layers; the common factor $E(B)$ then cancels.

Proof. Write $b = |B|$. Since $Q > 2L(B)$, all layers are disjoint. Comparing two representations of marks differing by Q gives

$$F_Q = B + Q\{0, \dots, t-2\}.$$

Indeed, the difference of two base marks has magnitude less than Q . Thus two marks differing by Q must have the same base mark and consecutive layer indices. Lemma 3.2 gives a proper weak coloring of this fiber. Apply the Golomb restriction of Mammel–Smith–Yerger [5], stated in Lemma 3.3, and then Theorem 4.1 to obtain

$$E(F_Q) \leq 6[(t-1)b]^2 - 5(t-1)b.$$

The estimate also holds when the fiber has one mark. Under the weaker hypothesis, its two-Golomb partition gives the same estimate directly.

We first compute the energy of an interval. For $P_s = \{0, \dots, s-1\}$, direct difference counting gives

$$e_s := E(P_s) = s^2 + 2 \sum_{j=1}^{s-1} (s-j)^2 = \frac{s(2s^2+1)}{3}.$$

Consider an equality of two differences from $B + QP_s$. The discrepancy of the base differences is at most $2L(B) < Q$, so it cannot cancel a nonzero integer multiple of Q . Therefore both coordinate differences agree, and their multiplicities multiply, including zero coordinates. Summing their squares gives $E(B + QP_s) = e_s E(B)$ for every $s \geq 1$.

Since B is nonempty, $E(B) > 0$, and the ratio of the two energies is e_t/e_{t-1} . Set $D_t = 2(t-1)^2 + 1$, $R_t = (2t^2 + 1)/D_t$, and $\lambda_t = \frac{6(t-1)}{t} R_t$. Substituting the fiber bound gives $E(A) \leq \lambda_t n^2 - 5R_t n$. We must check the linear term as well as the coefficient of n^2 . First,

$$9 - \lambda_t = \frac{3(t-2)(2t^2 - 4t - 1)}{tD_t} \geq 0 \quad (t \geq 2).$$

The expression is zero at $t = 2$ and positive for $t \geq 3$. Subtracting our bound from $9n^2 - 8n$ gives $n[(9 - \lambda_t)n + 5R_t - 8]$. Since $n \geq t$, the bracket is at least

$$(9 - \lambda_t)t + 5R_t - 8 = \frac{6t^3 - 30t^2 + 53t - 13}{D_t} > 0.$$

The numerator is 21, 38, 103 at $t = 2, 3, 4$. For $t \geq 5$, it equals $6t^2(t-5) + 53t - 13 > 0$. Hence $E(A) \leq 9n^2 - 8n$. Proposition 3.1 now gives $4T \leq 7n(n-1)$, and Corollary 3.4 gives $M \leq 8L$. $\square$

Theorem 6.2 (Complete direct sums). *Let $B, C \subset \mathbb{Z}$ be finite with $b = |B| \geq 2$, $c = |C| \geq 2$, and let $Q > 2L(B)$ be an integer. If $A = B + QC$, containing every pair of factor marks, has properly two-colored $V(A)$, then, with $n = bc$,*

$$E(A) \leq 16n^2 - 15n, \quad T(A) \leq 7M, \quad M \leq 15L.$$

Here $E(A) = E(B)E(C)$, but bounding each factor by the two-Golomb estimate is insufficient. We also use the full arrays inside the fibers. A monochromatic rectangle would give two equal differences in one Golomb class. This observation bounds difference multiplicities when a factor is large. The factors with at most six marks require a separate estimate.

Lemma 6.3 (Complete overlap restrictions). *Under Theorem 6.2's hypotheses, $E(A) = E(B)E(C)$ and each factor admits a two-Golomb partition. If one factor has at least seven marks, every positive difference multiplicity of the other is at most two; if one factor has at least five marks, those multiplicities are at most four.*

Proof. The separation argument used above applies to arbitrary integer layer indices. It gives the energy product and the following exact fibers:

$$F_h = (B \cap (B - h)) + QC \quad (h \in \Delta B), \quad F_{Q\delta} = B + Q(C \cap (C - \delta)) \quad (\delta \in \Delta C).$$

By Lemma 3.3, each fiber is partitioned into two Golomb classes. Both difference sets are nonempty. Fix one available base coordinate in the first fiber; the corresponding marks form a translate of QC . Fix one layer in the second fiber; those marks form a translate of B . Translation and nonzero scaling preserve the Golomb property, so both factors have two-Golomb partitions.

A monochromatic rectangle in either fiber array would give two equal nonzero differences in one Golomb class. We first show that every two-colored 7×3 array contains such a rectangle. Each row has a same-color pair among its three entries. There are only $2\binom{3}{2} = 6$ choices of column pair and color, so two of the seven rows share the same choice.

A 5×5 array also has a monochromatic rectangle. Otherwise every row-pair/color label occurs in at most one column, giving at most $2\binom{5}{2} = 20$ same-color row pairs in total. Each column contributes at least four, with equality only at splits $2 + 3$. Thus all five columns attain equality and each color uses all ten row pairs. If k columns contain three red entries, their red-pair count is $5 + 2k$, which cannot equal ten. This contradiction proves the claim. Therefore a factor of size seven forbids a fiber of the other factor with three starts, and a factor of size five forbids five starts. Transposing the arrays gives the conclusions in both directions. $\square$

Lemma 6.4 (Small factors and the six-mark loss). *Every r -mark integer set has $T \leq \binom{r}{3}$ and $E \leq e_r = r(2r^2 + 1)/3$. For $r \geq 2$, equality in the corner bound forces an arithmetic progression. A six-mark factor in Theorem 6.2 therefore has $T \leq 19$ and $E \leq 142$.*

Proof. Order the marks as $a_1 < \dots < a_r$. A difference collision with starts $a_i < a_j$ and endpoint $a_k = a_j + d$ determines a triple $i < j < k$. The triple fixes $d = a_k - a_j$ and the other endpoint $a_i + d$, so different collisions give different triples. Thus $T \leq \binom{r}{3}$. Substituting this bound in the energy identity gives $E \leq e_r$. Both statements also hold for $r = 0, 1$.

For $r = 2$, the two marks already form an arithmetic progression. For $r \geq 3$, equality means that every increasing triple comes from a collision. Put $h = a_2 - a_1 > 0$. The triples $(1, 2, k)$ give $a_k - h \in A$ for $k \geq 3$, and $a_2 - h = a_1$. Thus every nonminimal mark has its predecessor by h in A . Repeated subtraction shows that every mark lies in $a_1 + h\mathbb{Z}$; starting at a_r supplies all intermediate terms. Hence A is an arithmetic progression.

We reproduce the six-term progression obstruction from the prior notes discussed in Appendix D. Translation and positive scaling allow us to use $\{0, 1, \dots, 5\}$. In its strict triangle, the rows, columns, and offset diagonals have lengths 1, 2, 3, 4, 5 in each direction. Every pair on one of these lines completes uniquely to a corner, since the required third cell is still in the triangle. A line of length j has at least $\phi(j) = \binom{j}{2} - \lfloor j^2/4 \rfloor$ equal-color pairs: a split $k + (j - k)$ has $k(j - k) \leq \lfloor j^2/4 \rfloor$ unequal pairs. Every proper two-color corner contributes exactly one equal-color pair among its three sides, and no pair belongs to two corners. The number of equal-color pairs is therefore T . However, summing the line bounds gives

$$3[\phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(5)] = 3(0 + 0 + 1 + 2 + 4) = 21 > 20 = T.$$

This contradiction excludes a proper two-coloring of the six-term progression. Fixing a layer or a base mark in A gives a proper coloring of a translate of B or of a translate of QC , respectively. Thus each factor has a proper strict source. A six-mark factor cannot attain $\binom{6}{3} = 20$, because equality would make it a progression. Since T is an integer, $T \leq 19$, and hence $E \leq 2 \cdot 6^2 - 6 + 4 \cdot 19 = 142$. $\square$

Proof of Theorem 6.2. Apply Theorem 4.1 to the partitions from Lemma 6.3. This gives $E(B) \leq 6b^2 - 5b$ and $E(C) \leq 6c^2 - 5c$. All the numerical estimates are symmetric in the two factors, so for

these estimates we label the smaller size $b \leq c$. This relabeling is used only after proving factorization; the original separation hypothesis remains the same.

For $2 \leq r \leq 5$, we have $e_r/r^2 = (2r + 1/r)/3 \leq 17/5$. Indeed, this expression increases on the range, since its derivative is $(2 - r^{-2})/3 > 0$. We first handle the cases in which both sizes are at most six.

Case 1: $b, c \leq 5$. Then

$$\frac{E(A)}{n^2} \leq \frac{289}{25} < \frac{49}{4} \leq 16 - \frac{15}{n} \quad (n \geq 4).$$

Case 2: $c = 6$ and $b \leq 5$. The six-mark loss gives

$$\frac{E(A)}{n^2} \leq \frac{17}{5} \frac{71}{18} = \frac{1207}{90} < 14 < 16 - \frac{15}{n} \quad (n \geq 12).$$

Case 3: $b = c = 6$. The full comparison is

$$E(A) \leq 142^2 = 20\,164 \leq 20\,196 = 16 \cdot 36^2 - 15 \cdot 36.$$

Here the six-mark estimate leaves a positive difference of 32.

These three cases exhaust $c \leq 6$.

Now let $c \geq 7$. The cap two on B gives $E(B) \leq 3b^2 - 2b$. In general a cap $r_S(h) \leq k$ implies

$$E(S) = |S|^2 + 2 \sum_{h>0} r_S(h)^2 \leq |S|^2 + 2k \binom{|S|}{2} = (k+1)|S|^2 - k|S|.$$

We again distinguish cases, this time on the smaller size b .

Case 4: $b \geq 7$. Both factors have cap two, so $E(A) \leq 9n^2 < 16n^2 - 15n$, since $n \geq 49$.

Case 5: $b = 2$. Here $E(B) = 6$, whence $E(A) \leq 6(6c^2 - 5c) = 9n^2 - 15n$.

Case 6: $b = 3, 4$. Combine the cap on B with the two-Golomb bound on C :

$$E(A) \leq (18 - 12/b)n^2 - (15b - 10)n \leq 15n^2 - 35n < 16n^2 - 15n.$$

Case 7: $b = 5, 6$. The cap four on C instead gives

$$E(A) \leq (15 - 10/b)n^2 - (12b - 8)n \leq \frac{40}{3}n^2 - 52n < 16n^2 - 15n.$$

These coefficient comparisons use the largest quadratic coefficient and the smallest subtracted linear coefficient on the stated ranges. We have now covered every possible pair of sizes. Subtracting $2n^2 - n$ from the energy bound gives $4T \leq 14n(n-1)$, or $T \leq 7M$. Corollary 3.4 then yields $M \leq 15L$. $\square$

Remark. We have not determined the best constants for these product families. For example, the six-mark bound 142 follows from excluding equality in a general estimate. It does not identify the largest energy of a proper six-mark source. Completeness is also used in the proofs: adding missing factor pairs to an incomplete product may destroy properness.

When shifted difference ranges overlap, the energy need not factor. For translation matchings, we can still expand it exactly by keeping the shifted correlations. We then combine the two Golomb classes with the fourth-root energy triangle inequality, whose finite counting proof is included below.

Theorem 6.5 (Translation matchings). *Let $S \subset \mathbb{Z}$ be finite, let $d > 0$ be an integer, and suppose $A = S \sqcup (S + d)$ has properly two-colored $V(A)$. With $n = 2|S|$,*

$$E(A) \leq 22n^2 - 21n, \quad T(A) \leq 10M.$$

If $S \neq \emptyset$, then $L \geq 1$ and

$$M \leq 16L - 13 + 3\sqrt{8L - 7} < 18L.$$

No separation of the translates or their difference ranges is assumed. If $S = \emptyset$, $E = T = M = 0$ and no radical assertion is made.

Proof. Partition $S = B \sqcup C$ according to the colors of $(x + d, x)$. Since $S \subseteq F_d$, Lemma 3.3 makes both classes Golomb. The fiber may contain other starts, but restriction to S is enough. The assumption $S \cap (S + d) = \emptyset$ makes $B, C, B + d, C + d$ pairwise disjoint.

For a nonempty Golomb set B of size b , let $r_B(t)$ count signed differences $(x, y) \in B^2$ with $x - y = t$. Define $C_h(B) = \sum_{t \in \mathbb{Z}} r_B(t)r_B(t + h)$ for $h > 0$. Each of the $b(b - 1)$ nonzero signed differences has multiplicity one, whereas $r_B(0) = b$. Thus $E(B) = 2b^2 - b$.

If $r_B(h) = 0$, the terms at $t = 0, -h$ vanish, and $C_h(B) \leq b(b - 1)$. If $r_B(h) = 1$, those two terms contribute $2b$. Among the nonzero differences, $-h$ has already been counted, and the largest positive difference contributes nothing because shifting it by h leaves the support. These are distinct differences, so

$$C_h(B) \leq 2b + b(b - 1) - 2 = b^2 + b - 2.$$

This is also an upper bound in the first case, since $(b^2 + b - 2) - b(b - 1) = 2b - 2 \geq 0$. For $b = 1$, all these correlations vanish and the bound is zero.

For $P_B = B \sqcup (B + d)$, classify its ordered pairs by their two translates to get

$$r_{P_B}(t) = 2r_B(t) + r_B(t - d) + r_B(t + d), \quad E(P_B) = 6E(B) + 8C_d(B) + 2C_{2d}(B).$$

The second identity follows by expanding the square and shifting the summation index; the two adjacent correlations contribute eight, and the outer one contributes two. As disjointness gives $r_B(d) = 0$, it follows that

$$E(P_B) \leq 6(2b^2 - b) + 8(b^2 - b) + 2(b^2 + b - 2) = 22b^2 - 12b - 4.$$

To combine P_B and P_C , first consider any finite disjoint sets X, Y . Let $q_{X,Y}(t)$ count pairs $(x, y) \in X \times Y$ with $y - x = t$. Rearranging the equality $y - x = y' - x'$ gives $x' - x = y' - y$. Counting by this common difference and applying Cauchy–Schwarz gives

$$\sum_t q_{X,Y}(t)^2 = \sum_t r_X(t)r_Y(t) \leq \sqrt{E(X)E(Y)}.$$

The four ordered part types give $r_{X \sqcup Y} = r_X + r_Y + q_{X,Y} + q_{Y,X}$. The Euclidean triangle inequality for these finite vectors yields

$$E(X \sqcup Y)^{1/2} \leq E(X)^{1/2} + E(Y)^{1/2} + 2(E(X)E(Y))^{1/4},$$

and taking square roots proves $E(X \sqcup Y)^{1/4} \leq E(X)^{1/4} + E(Y)^{1/4}$.

Now suppose $b, c \geq 1$, and put $s = b + c$, $n = 2s$. The positive function $g(t) = 22t^2 - 12t - 4$ on $t \geq 1$ has concave fourth root: its second derivative has the sign of

$$48g'' - 3(g')^2 = -1936(t - 3/11)^2 - 992 < 0.$$

Apply the norm inequality to P_B, P_C and then concavity at b, c to obtain

$$E(A) \leq [g(b)^{1/4} + g(c)^{1/4}]^4 \leq 16g(s/2) = 88s^2 - 96s - 64 = 22n^2 - 48n - 64 \leq 22n^2 - 21n.$$

If one class is empty and $s \geq 1$, the single-class bound gives $E(A) \leq 22s^2 - 12s - 4 = (11/2)n^2 - 6n - 4$. Its difference from the target is $(33/2)n^2 - 15n + 4 > 0$ for $n \geq 2$. The empty matching gives zero directly. Thus $4T \leq 20n(n - 1)$, proving $T \leq 10M$.

For the span bound, assume $S \neq \emptyset$. Since $\min A = \min S$ and $\max A = \max S + d$, the span of S is $L - d \leq L - 1$. Each Golomb class has distinct positive integer differences in this interval. The positive-difference count used in the Golomb method of Mammel–Smith–Yerger [5] therefore gives

$$\binom{b}{2}, \binom{c}{2} \leq L - 1, \quad b, c \leq \frac{1 + \sqrt{8L - 7}}{2}, \quad s \leq 1 + \sqrt{8L - 7}.$$

The size bounds remain valid for an empty class. Since $s(2s - 1)$ increases for $s \geq 1$, substituting gives $M = s(2s - 1) \leq 16L - 13 + 3\sqrt{8L - 7}$. Both quantities compared in the following squaring are nonnegative, and

$$(2L + 13)^2 - 9(8L - 7) = 4(L - 5/2)^2 + 207 > 0.$$

Hence $3\sqrt{8L - 7} < 2L + 13$, proving $M < 18L$. $\square$

Remark. The calculation allows the two translates to interlace. Applying Corollary 3.4 to the energy bound alone would give $M \leq 21L$. The smaller coefficient in $M < 18L$ comes from counting differences in the two Golomb classes. The best constants for this family remain undetermined.

We now consider a geometric representation. A map preserving pair-sum equalities is a Freiman homomorphism of order two; see Tao–Vu [14]. If its distinct image points have only the squared distances τ and 2τ , the parallelogram identity forces every nondegenerate parallelogram to be a square. Counting squares will therefore give a bound on integer corners.

Theorem 6.6 (Two-distance representations). *Let $A \subset \mathbb{Z}$ be finite. Suppose an injective map $\phi : A \rightarrow \mathbb{R}^r$ into a finite-dimensional Euclidean space and a number $\tau > 0$ satisfy: every equality $a + b = c + e$ in A , including repeated marks, gives $\phi(a) + \phi(b) = \phi(c) + \phi(e)$; and every nonzero squared image distance belongs to $\{\tau, 2\tau\}$. Then $T(A) \leq 7M$ and $M \leq 15L$. Coloring (v, u) by $\|\phi(v) - \phi(u)\|^2$ is proper. No common sphere or converse preservation of image equalities is assumed.*

Proof. Put $X = \phi(A)$. There are no three distinct image points in arithmetic progression: their outer squared distance would be four times a side squared distance, exceeding 2τ . Thus a nontrivial equality between two unordered pair sums uses four distinct points. Let $\mathbf{u}, \mathbf{v}$ be adjacent sides of the resulting parallelogram. Then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).$$

Each squared side length is at least τ , and each squared diagonal length is at most 2τ . Hence both sides above must have squared length τ and both diagonals squared length 2τ . Subtracting the two diagonal identities gives $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. The parallelogram is therefore a square. In particular, two distinct pairs of squared length 2τ with the same midpoint are its diagonals.

Call a pair *long* if its squared distance is 2τ . Let $k_{\mathbf{c}}$ count the long unordered pairs with midpoint $\mathbf{c}$. Different pairs at a midpoint have disjoint endpoints, since one endpoint determines the other. Choosing two of them gives one square. Thus twice the number of squares is

$$T_{\text{geom}}(X) = 2 \sum_{\mathbf{c}} \binom{k_{\mathbf{c}}}{2} = \sum_{\mathbf{c}} k_{\mathbf{c}}(k_{\mathbf{c}} - 1).$$

Suppose a midpoint has $k = k_{\mathbf{c}} \geq 9$ long pairs, and let $Y = \{\mathbf{c} \pm \mathbf{p}_i : 1 \leq i \leq k\}$ contain all their endpoints. We have $\|\mathbf{p}_i\|^2 = \tau/2$. For $i \neq j$, the quantities $\|\mathbf{p}_i - \mathbf{p}_j\|^2$ and $\|\mathbf{p}_i + \mathbf{p}_j\|^2$ sum to 2τ and are both at least τ . Hence both equal τ , and the radii $\mathbf{p}_i$ are mutually orthogonal.

Take any outside point $\mathbf{c} + \mathbf{q} \in X \setminus Y$, and put $t = \|\mathbf{q}\|^2 + \tau/2$. Its squared distances to the two endpoints of pair i are $t \mp 2\langle \mathbf{q}, \mathbf{p}_i \rangle \in \{\tau, 2\tau\}$. Taking their average shows that $t \in \{\tau, 3\tau/2, 2\tau\}$. If $t = 3\tau/2$, then $\|\mathbf{q}\|^2 = \tau$ and $|\langle \mathbf{q}, \mathbf{p}_i \rangle| = \tau/4$ for every i . Since orthogonal projection cannot increase squared length, we would have

$$\|\mathbf{q}\|^2 \geq \sum_i \frac{\langle \mathbf{q}, \mathbf{p}_i \rangle^2}{\|\mathbf{p}_i\|^2} = \frac{k\tau}{8} > \tau.$$

This is a contradiction. In either remaining case, both distances equal their average, so $\mathbf{q}$ is orthogonal to every radius.

We next show that no square has some vertices in Y and some outside Y . Translate $\mathbf{c}$ to the origin and project sums of endpoints onto the span of the radii. A long pair within Y must be antipodal, so

its projection is zero. A pair outside Y also has projection zero. A pair with one endpoint in each set has projection $\pm \mathbf{p}_i$. Thus any other long pair with the same sum would have to use the same signed endpoint of Y , after which its outside endpoint is fixed. It would be the same pair and would not form a square. The only remaining possibility would pair an antipodal pair in Y with a long pair outside Y of midpoint $\mathbf{c}$. This is impossible because Y contains all endpoints of long pairs at $\mathbf{c}$. Therefore

$$T_{\text{geom}}(X) = k(k-1) + T_{\text{geom}}(X \setminus Y).$$

Remove such a set Y whenever a midpoint has at least nine long pairs. The process stops because X is finite, and the distance assumptions hold in every remaining subset. In the final set X_0 , every midpoint has at most eight pairs. Hence $k_{\mathbf{c}}(k_{\mathbf{c}} - 1) \leq 7k_{\mathbf{c}}$, and counting each long pair once gives $T_{\text{geom}}(X_0) \leq 7 \binom{|X_0|}{2}$. Each removed $2k$ -point set contributes $k(k-1) \leq 7 \binom{2k}{2}$. These sets and X_0 are disjoint, so their internal pair counts sum to at most $\binom{|X|}{2}$. It follows that $T_{\text{geom}}(X) \leq 7 \binom{|X|}{2}$.

To return to the integer source, let $D(A)$ count patterns $a < b < c < e$ with $a + e = b + c$, and let $P_3(A)$ count three-term arithmetic progressions with distinct marks. A repeated positive difference either uses four marks or comes from two intervals meeting at one endpoint. A four-mark pattern gives two collisions, namely $b - a = e - c$ and $c - a = e - b$; a three-term progression gives one. Therefore $T(A) = 2D(A) + P_3(A)$. Since ϕ preserves equalities with repeated marks, a progression in A would map to a progression in X , which we have excluded. Injectivity also sends different four-mark patterns to different squares. Hence $P_3(A) = 0$ and $T(A) \leq T_{\text{geom}}(X) \leq 7M$. Equalities present only in the image cause no difficulty, since we need an upper bound.

For either integer collision in a four-mark pattern, its two parallel endpoint cells correspond to square sides, while its outer completion corresponds to a diagonal. Thus the corner has squared-distance colors $\tau, 2\tau, \tau$, in the usual input–output–input order. This proves that the stated coloring is proper. Corollary 3.4 gives $M \leq 15L$, including the cases with no pairs. $\square$

Remark. The coefficient seven comes from bounding the remaining midpoint multiplicities by eight; it may be possible to improve it. The representation assumption is restrictive. For example, a three-term progression has a proper two-coloring of its three-cell strict source, but cannot have the stated distance representation.

Remark (Examples with repeated differences). We can check all four sets of hypotheses on small sources. Put $D = \{0, 1, 3, 4\}$ and $P = \{0, 1\}$. Table 6 gives the exact counts and the corresponding bounds on T/M .

Table 6: Examples for the four structured families. The first two rows describe the same source in two ways. Every row has repeated positive differences and a proper strict two-coloring.

Family	Parameters	n	M	T	T/M	Bound
Progression	$B = D + 9P, Q = 27, t = 2$	16	120	200	5/3	7/2
Direct sum	$B = C = D, Q = 9$	16	120	200	5/3	7
Matching	$S = \{0, 2\}, d = 1$	4	6	4	2/3	10
Two distances	$A = \{\pm 1, \pm 5, \pm 25, \pm 125\}$	8	28	12	3/7	7

For the first two rows, $A = D + 9D$ is the cube with weights 1, 3, 9, 27, so Table 4 supplies its proper coloring. The required separations are $27 > 2 \cdot 13$ and $9 > 2 \cdot 4$. Its energy is $6^4 = 1296$, giving $T = (1296 - 2 \cdot 16^2 + 16)/4 = 200$.

For the matching, $A = \{0, 1, 2, 3\}$. Reading each row from its smallest lower mark, the color words at upper marks 1, 2, 3 are 0, 01, 100. They give corner words 001, 100, 010, 010. Here the two translates interlace, and shifted difference supports overlap: $E(A) = 44$, whereas $E(S)E(\{0, 1\}) = 36$. Thus the separated product identity would give the wrong count.

The last row belongs to an infinite family. For any integer $r \geq 1$, put $A_r = \{\pm 5^i : 0 \leq i < r\}$ and map $\pm 5^i$ to $\pm e_i$, where the e_i are orthonormal. This map preserves every pair-sum equality, including

repeated marks. Indeed, expanding $a + b - c - e = 0$ in powers of five gives coefficients in $\{-4, \dots, 4\}$. The largest nonzero coefficient cannot cancel, since $5^k > 4 \sum_{j < k} 5^j = 5^k - 1$. Hence all coefficients vanish. The nonzero squared image distances belong to $\{2, 4\}$, as required with $\tau = 2$.

Among the image vectors, zero has the r antipodal pair representations, while every nonzero pair sum has a unique unordered representation. No doubled mark shares its sum with another pair, so A_r contains no three-term progression. Each choice of two distinct antipodal pairs therefore gives exactly two corners, and these account for every corner. For indices $i < j$, the four marks are $-5^j < -5^i < 5^i < 5^j$, and both corners have outer completion $(5^j, -5^j)$. Define the coloring by $z(v, u) = 1$ when $v = -u$, and 0 otherwise. In each corner the parallel endpoints have color zero and the outer completion has color one, so this is a proper coloring of the whole strict source. For $r = 1$ there are no corners. We obtain

$$|A_r| = 2r, \quad M(A_r) = r(2r - 1), \quad T(A_r) = 2 \binom{r}{2} = r(r - 1),$$

$$\frac{T(A_r)}{M(A_r)} = \frac{r - 1}{2r - 1} \rightarrow \frac{1}{2} \quad (r \rightarrow \infty).$$

The case $r = 4$ gives the last row of the table. Thus the example extends to arbitrarily large properly colored sources, although its limiting ratio remains well below the bound seven in Theorem 6.6.

These counts illustrate the hypotheses and leave substantial gaps below the bounds. They do not determine the best constants. The accompanying script `verify_structured_examples.py` checks the full cell colorings and numerical entries in Table 6 independently.

Lemma 6.7 (Exact endpoint deletion). *Let $A \subset \mathbb{Z}$ be finite and nonempty with $\min A = 0$, and put $B = A \setminus \{0\}$. On B , let $J_0(A)$ have an arc $x \rightarrow w$ exactly when $x < w$ and $w - x \in B$, and write $e_0(A) = |E(J_0(A))|$. Then*

$$T(A) - T(B) = e_0(A) = 2D_0(A) + P_0(A),$$

where D_0 counts $x + y = w$ with $0 < x < y$ in B , and P_0 counts $x > 0$ with $x, 2x \in B$. No coloring is required.

Proof. Zero cannot be the first coordinate of a strict cell. Every lost corner therefore uses second coordinate zero and has the unique form $(x, 0), (w, 0), (w, w - x)$, where $0 < x < w$ and $x, w, w - x \in B$. Its step $w - x$ is positive, so this is a bijection with the arcs of $J_0(A)$. Distinct summands give two arcs, while $x = w - x$ gives one, proving both identities, including $B = \emptyset$. For deletion of the maximum a , the map $(v, u) \mapsto (a - u, a - v)$ sends the source onto $V(a - A)$ and sends a positive corner to a positive corner with its listed order reversed. It preserves properness and gives the corresponding maximum-endpoint identity. $\square$

Lemma 6.8 (A sharp restricted endpoint bound). *Let $|A| = n \geq 2$, $\min A = 0$, and let one proper coloring $z : V(A) \rightarrow \{-1, 1\}$ be all-forcing: the parallel endpoints of every corner have equal signs. Set $B = A \setminus \{0\}$ and $f(a) = z(a, 0)$ for $a \in B$. If all anchor signs equal $s \in \{-1, 1\}$, then $e_0(A) = 0$. If exactly one $r \in B$ has sign $-s$, then $e_0(A) \leq n - 2$, sharply for arbitrarily large sources. These are single-deletion conclusions with the stated anchor hypotheses.*

Proof. Every pair of cells in a repeated fiber completes a corner. The all-forcing assumption therefore makes every such fiber monochromatic; a singleton fiber already has a unique sign. Extend f from the anchor differences B to all represented positive differences by this sign. This is consistent with $f(a) = z(a, 0)$ and gives $z(v, u) = f(v - u)$.

If $x, y, x + y \in B$, the corner $(x, 0), (x + y, 0), (x + y, y)$ gives $f(x + y) = -f(x)$. Interchanging unequal summands then gives $f(x) = f(y) = -f(x + y)$. For equal summands, the relation is $f(2x) = -f(x)$.

If all anchor signs equal s , these relations permit no addition $x + y \in B$, so $e_0 = 0$. Now suppose r is the unique anchor with sign $-s$, and call the other anchors regular. Unequal summands must have

equal signs; since only one anchor is exceptional, both summands must be regular and their sum must be r . A doubling can only have the form $2x = r$ with $x \neq r$, or $r + r = 2r$.

For additions ending at r , the map $x \mapsto r - x$ gives one arc into r for each participating regular mark, including a possible fixed point. If $2r \notin A$, there are at most $n - 2$ regular marks. If $2r \in A$, it is regular but cannot be a summand of the smaller number r . There are then at most $n - 3$ arcs into r , together with the one possible arc $r \rightarrow 2r$. In both cases $e_0 \leq n - 2$.

For sharpness, choose integers $k \geq 1$, $R \geq 5$, and set

$$x_0 = 0, \quad x_i = R^i \quad (1 \leq i \leq k), \quad r = R^{k+2}, \quad A = \{x_i, r - x_i : 0 \leq i \leq k\}.$$

Since each $x_i < r/2$, the lower and upper marks are disjoint and $|A| = 2k + 2$. To identify the repeated differences, compare two expressions $a - b = c - e$ with marks in A . After expanding the four marks, every base- R coefficient has magnitude at most four. If R^i is the largest power with a nonzero coefficient, its contribution has magnitude at least R^i , while all smaller powers contribute at most

$$4 \sum_{j=0}^{i-1} R^j = \frac{4(R^i - 1)}{R - 1} \leq R^i - 1 < R^i \quad (R \geq 5).$$

Thus every coefficient in a zero relation must vanish, including the coefficient of $r = R^{k+2}$. Comparing these coefficients shows that the repeated positive differences are exactly the following, for $0 \leq i < j \leq k$:

$$\begin{aligned} \delta_{ij} = x_j - x_i & : (x_j, x_i), (r - x_i, r - x_j), \\ h_{ij} = r - x_i - x_j & : (r - x_j, x_i), (r - x_i, x_j). \end{aligned}$$

Each has exactly the two displayed representations. The remaining differences $h_{ii} = r - 2x_i$ have one representation each. Indeed, every pair lies within the lower marks, within the upper marks, or between the two sets; these are precisely the cases above.

Give the fibers δ_{ij} and h_{ij} sign s for $i < j$, and give every h_{ii} fiber sign $-s$. Each cell is now colored by its difference. For either repeated fiber indexed by $i < j$, the outer cell is $(r - x_i, x_i)$, whose offset is h_{ii} . It has the opposite sign to the two inputs. These are all the corners, so the coloring is proper and all-forcing.

For $i > 0$, the anchors at x_i and $r - x_i$ have sign s , while the anchor at r has sign $-s$. Coefficient uniqueness also shows that the positive additions in A are exactly $x_i + (r - x_i) = r$, $1 \leq i \leq k$. Each has two distinct summands and contributes two arcs. Hence $e_0 = 2k = |A| - 2$. $\square$

Remark. The uniform bound zero is attained by a two-mark source. The reflected family attains the one-exception bound for every $k \geq 1$. All-forcing survives restriction, but the anchor condition need not survive a change of minimum, so neither conclusion supplies an iterative endpoint theorem.

Open assertion 6.9 (EP10). Every finite integer A with $|A| \geq 2$ and properly two-colored $V(A)$ has an endpoint $a \in \{\min A, \max A\}$ such that $T(A) - T(A \setminus \{a\}) \leq 10(|A| - 1)$. This assertion is open.

Open assertion 6.10 (H10). Every finite integer A admitting a proper two-coloring of $V(A)$ satisfies $T(A) \leq 10M(A)$. This assertion is open.

Proposition 6.11 (Conditional endpoint reduction). *Assume EP10. Then H10 holds, and every properly two-colored source has $M \leq 21L$, hence satisfies S21 when in its finite span range. It suffices for S21 to assume EP10 only on sources of span at most 99 994.*

Proof. At each deletion, the surviving cells keep the same proper coloring. Assuming EP10, choose the promised endpoint successively until one mark remains. Apply Lemma 6.7 after translating the current minimum to zero, and reflecting if the maximum is deleted. Telescoping these costs gives $T(A) \leq 10 \sum_{j=1}^{n-1} j = 10M$; empty and singleton sources already have zero corners. Corollary 3.4 gives $M \leq 21L$. Deletions never increase span, so the finite-range premise suffices as well. $\square$

The four structured bounds and the restricted endpoint estimate do not prove H10 for arbitrary proper sources. In particular, applying the anchor estimate repeatedly would require an anchor condition after every deletion, which we have not established. EP10 and H10 remain open. In Section 7, we use a finite-span assertion to state precisely which additional bound would improve the small-color estimates.

7 Conditional consequences and finite questions

To connect the source estimates with small-color Ramsey bounds, we examine two additional assumptions: a pair bound for every properly two-colored source in a fixed span range, and a rainbow lower bound for a small complete triangle. The first uses the diagonal exclusion and interval-counting arguments of Canakci et al. [4] and Mammel–Smith–Yerger [5]; the second counts dilated copies of the triangle. Neither assumption is used in Sections 4–6. Both remain open, and the results below establish their conditional consequences.

Table 7 separates these questions from the documented bounds $21 \leq R_3(\mathcal{P}) \leq 493$. A thirteen-mark example at the end of the section shows why a proposed local lower bound must also be tested against explicit colorings.

Table 7: Status of the estimates used below. The documented bounds transfer to $\mathcal{L}$ by Lemma 2.1.

Statement	Status	Attribution or qualification
$R_3(\mathcal{P}) \geq 21$	Documented bound	Canakci et al. [4], Theorem 3.1 and Figure 1.
$R_3(\mathcal{P}) \leq 493$	Published bound	Mammel–Smith–Yerger [5], Theorem 6.
$S21 \Rightarrow 379, 99\,996$	Conditional, proved	Theorems 7.2 and 7.4; S21 is open.
$F_{11}(5) \Rightarrow 459; F_{12}(7) \Rightarrow 455$	Conditional, proved	Theorem 7.9; both premises are open.
S21; H10; EP10	Open	Finite-span bound; $T \leq 10M$; an endpoint of cost at most $10(m-1)$ for $m \geq 2$.
G9: $ E(\Gamma) \leq 9M$	Open	All proper strict two-color sources. It would give $T \leq 19M/2$, $M \leq 20L$, and S21.
EP10 $\Rightarrow$ H10 $\Rightarrow$ S21	Proved implications	Endpoint restrictions remain proper; no converse is established.
$F_{13}(17)$	Refuted	Proposition 7.10: exactly fourteen rainbows; no minimality claim.

Open assertion 7.1 (S21). Every finite set $A \subset \mathbb{Z}$ for which $V(A)$ admits a proper two-coloring and $L(A) \leq 99\,994$ satisfies $M(A) \leq 21L(A)$. Empty and singleton sources have $M = 0$.

Theorem 7.2 (Three colors under S21). *Assume S21. Then $R_3(\mathcal{L}) \leq 379$.*

Proof idea. A largest diagonal color class has at least 127 marks. Its strict source uses only two colors, so S21 would bound its number of pairs by its span.

Proof. Suppose $[379]^2$ has a proper three-coloring. At least $\lceil 379/3 \rceil = 127$ main-diagonal cells share a color; choose exactly that many positions to form A , with $L(A) \leq 378$. For $u < v$ in A , the positive corner $(u, u), (v, u), (v, v)$ excludes this color at (v, u) . This is the diagonal exclusion of Canakci et al. [4] and Mammel–Smith–Yerger [5], proved in Lemma 3.2. The restriction to $V(A)$ is therefore a proper two-coloring. S21 applies within its span range, but

$$\binom{127}{2} = 8\,001 > 7\,938 = 21 \cdot 378.$$

Thus $M(A) > 21L(A)$, contradicting S21. □

The next lemma is the interval-counting method of Mammel–Smith–Yerger [5], with an arbitrary cap W on the difference multiplicities.

Lemma 7.3 (Finite shift count). *Let $D \subset \mathbb{Z}$ have $a \geq 0$ marks and span at most an integer $\ell \geq 0$. If a real $W \geq 0$ satisfies $r_D(h) \leq W$ for every positive integer h , then for every integer $t \geq 1$,*

$$a^2 t \leq (\ell + t)[a + W(t - 1)].$$

Proof idea. We add the indicators of t translates of D . Cauchy–Schwarz gives a lower bound for the sum of squares, while the overlap counts give an upper bound.

Proof. If $D = \emptyset$, the left side is zero and the right side is nonnegative, so the inequality holds. Otherwise set $g(x) = \sum_{i=0}^{t-1} \mathbf{1}_D(x - i)$ on $\mathbb{Z}$. Its mass is at , and its support lies between $\min D$ and $\max D + t - 1$, in at most $\ell + t$ integer positions. Thus

$$\frac{a^2 t^2}{\ell + t} \leq \sum_x g(x)^2 = at + 2 \sum_{h=1}^{t-1} (t - h) r_D(h) \leq at + Wt(t - 1).$$

The diagonal terms give at ; for each positive displacement h , the $t - h$ shift pairs occur in both orders. The sum is empty for $t = 1$. Multiplying by $\ell + t > 0$ and dividing by $t > 0$ proves the result. $\square$

Theorem 7.4 (Four colors under S21). *Assume S21. Then $R_4(\mathcal{L}) \leq 99\,996$.*

Proof idea. After restricting to a diagonal color class, we apply diagonal exclusion once more inside each fiber. S21 then bounds every fiber, allowing us to use Lemma 7.3.

Proof. Suppose a proper four-coloring exists on $[N]^2$, where $N = 99\,996$. Choose $a = N/4 = 24\,999$ main-diagonal positions D of one color and put $\ell = N - 1 = 99\,995$. The diagonal exclusion of Canakci et al. and Mammel–Smith–Yerger [4, 5] gives a proper three-coloring of $V(D)$.

For $h > 0$, partition $F_h(D)$ according to the colors of $(x + h, x)$ into at most three parts. If E is one part and $u < v$ lie in it, the equal-colored cells $(u + h, u)$, $(v + h, v)$ force $(v + h, u)$ to avoid their color. All three cells belong to $V(D)$, already avoiding the original diagonal color. Hence the restriction $(v, u) \mapsto (v + h, u)$ gives a proper two-coloring on $V(E)$, as in Lemma 3.2. Translating the first coordinate preserves the two equal positive steps of each corner.

If $1 \leq h \leq \ell$, every nonempty E has span at most $\ell - h \leq 99\,994$. Under S21 its size s satisfies $(s - 1)^2 \leq s(s - 1) \leq 42L(E)$, hence $s \leq 1 + \sqrt{42L(E)}$. Empty parts obey the relaxed bound as well. Consequently

$$r_D(h) \leq 3(1 + \sqrt{42(\ell - h)}) \leq 3(1 + \sqrt{42\ell}) < 6\,153,$$

because

$$\begin{aligned} 2\,050^2 &= 4\,202\,500 > 4\,199\,790 = 42 \cdot 99\,995, \\ 4\,202\,500 - 4\,199\,790 &= 2\,710 > 0, \quad 3(1 + 2\,050) = 6\,153. \end{aligned}$$

Fibers beyond ℓ are empty. Keep the convenient cap $W = 6\,153$. Lemma 7.3, with $t = 550$, now gives $a^2 t \leq (\ell + t)[a + W(t - 1)]$. However,

$$\begin{aligned} 24\,999^2 \cdot 550 &= 343\,722\,500\,550, \\ (99\,995 + 550)[24\,999 + 6\,153 \cdot 549] &= 100\,545 \cdot 3\,402\,996 = 342\,154\,232\,820, \\ 343\,722\,500\,550 &> 342\,154\,232\,820, \\ 343\,722\,500\,550 - 342\,154\,232\,820 &= 1\,568\,267\,730 > 0. \end{aligned}$$

This contradicts Lemma 7.3. Notice that S21 was applied only to the color parts E , whose spans are at most 99 994. The set D itself may have span 99 995. $\square$

For the second approach, consider the complete strict triangle $\mathcal{T}_N = \{(v, u) : 0 \leq u < v < N\}$. We first bound its number of rainbow corners. The required identity comes from counting equal-color pairs in rows, columns, and slope-one diagonals.

Lemma 7.5 (Three-color line identity). *In any three-coloring of $\mathcal{T}_N$, let $\mathcal{R}_N, \mathcal{B}_N$ count rainbow and monochromatic corners. For its nonempty rows, columns, and slope-one diagonals λ , let $n_{\lambda,i}$ count color $i \in \{0, 1, 2\}$ and put*

$$\mathcal{D}_N = \sum_{\lambda} \sum_{i=0}^2 (n_{\lambda,i} - |\lambda|/3)^2.$$

For every integer $N \geq 1$,

$$\mathcal{D}_N + 2\mathcal{R}_N - 4\mathcal{B}_N = N(N-1).$$

In particular, put $M_N = \binom{N}{2}$. A proper coloring satisfies $\mathcal{D}_N + 2\mathcal{R}_N = N(N-1)$ and

$$\mathcal{R}_N \leq U_N := M_N - (N-1) + \left\lfloor \frac{N-1}{3} \right\rfloor.$$

Proof idea. Each pair on one of the three line directions completes to exactly one corner. We count the equal-color pairs first by corner type, and then by expanding the squares in $\mathcal{D}_N$.

Proof. For a row pair $(v, u), (v, w)$ with $u < w < v$, its completion $(v - w + u, u)$ has first coordinate strictly between u and N . For a column pair $(v, u), (w, u)$ with $u < v < w$, the completion $(w, u + w - v)$ has second coordinate between zero and $w - 1$. For an offset pair $(u + d, u), (v + d, v)$ with $u < v$, the completion $(v + d, u)$ also remains in the triangle. Each pair direction fixes this unique completion, and no two directions describe the same distinct pair.

The triangle has $\binom{N}{3}$ corners by Proposition 3.1. A monochromatic corner contributes three equal-color pairs, a two-color corner one, and a rainbow none. Their total is therefore $P = \binom{N}{3} - \mathcal{R}_N + 2\mathcal{B}_N$. Each line direction has one line of every length $1, \dots, N-1$, so

$$\sum_{\lambda} |\lambda| = 3M_N, \quad \sum_{\lambda} |\lambda|^2 = \frac{N(N-1)(2N-1)}{2}, \quad P = \frac{\mathcal{D}_N}{2} + \frac{N(N-1)(2N-1)}{12} - \frac{3M_N}{2}.$$

The last equality expands $\sum_i \binom{n_{\lambda,i}}{2}$. Equating the two expressions for P and substituting $\binom{N}{3} = N(N-1)(N-2)/6$ cancels the cubic terms and proves the identity. Empty line families at $N = 1$ give zero throughout. For a proper coloring, $\mathcal{B}_N = 0$. The integer color counts give a little more than $\mathcal{D}_N \geq 0$. If two counts on a line differ by at least two, moving one unit from the larger count to the smaller decreases the sum of squares. Thus, for a line of length ℓ , the minimum occurs when its three counts differ by at most one, and

$$\sum_{i=0}^2 (n_{\lambda,i} - \ell/3)^2 \geq \begin{cases} 0, & 3 \mid \ell, \\ 2/3, & 3 \nmid \ell. \end{cases}$$

There are $(N-1) - \lfloor (N-1)/3 \rfloor$ lengths not divisible by three, in each of the three directions. Hence

$$\mathcal{D}_N \geq 2 \left((N-1) - \left\lfloor \frac{N-1}{3} \right\rfloor \right).$$

Substitution into the identity gives $\mathcal{R}_N \leq U_N$. We use only a lower bound obtained separately on each line; the argument does not assert that all these minima can occur together. The completion argument uses the full triangle and need not hold for a sparse source. $\square$

We now count smaller triangles inside $\mathcal{T}_N$. A rainbow corner may belong to several copies, so adding the local counts alone is not enough. We assign a nonnegative weight to each dilation and bound the total weight of the copies containing any one corner.

Lemma 7.6 (Weighted embedding implication). *Fix integers $N \geq m \geq 3$ and $\rho \geq 0$. Assume $F_m(\rho)$: every proper three-coloring of $\mathcal{T}_m$ has at least ρ rainbow corners. Let $a_d \geq 0$ vanish outside $1 \leq d \leq \lfloor (N-2)/(m-2) \rfloor$, and define*

$$J = \sum_d a_d \binom{N-d(m-2)}{2}, \quad \Lambda_h = \sum_{\substack{j|h \\ 1 \leq j \leq m-2}} \binom{m-j}{2} a_{h/j}.$$

If $K \geq 0$ and $\Lambda_h \leq K$ for $1 \leq h \leq N-2$, then every proper three-coloring of $\mathcal{T}_N$ satisfies $\rho J \leq K\mathcal{R}_N \leq KU_N$, where U_N is defined in Lemma 7.5. Thus, under $F_m(\rho)$, the strict inequality $\rho J > KU_N$ implies $R_3(\mathcal{L}) \leq N$.

Proof idea. Three gap coordinates describe a triangle as the nonnegative integer triples with a fixed sum. This makes the embeddings easy to count. For a fixed host corner and a fixed local corner, the step determines the dilation and the position determines the translation.

Proof. Map $(v, u) \in \mathcal{T}_m$ to $(u, v - u - 1, m - 1 - v)$. These coordinates are nonnegative with sum $m - 2$, and the inverse recovers u, v . A corner of step j consists of a common nonnegative vector with j added separately to each coordinate. Dilate by $d > 0$ and add $(\alpha, \beta, \gamma) \geq 0$ of sum $N - 2 - d(m - 2)$. The physical map is

$$\phi_{d,\alpha,\beta}(v, u) = (\alpha + \beta + 1 - d + dv, \alpha + du).$$

The gap coordinates verify membership in $\mathcal{T}_N$; the formula verifies injectivity and multiplies the positive step by d . A nonnegative triple summing to q has $\binom{q+2}{2}$ choices, by placing two separators among q identical units. Hence this dilation has exactly $\binom{N-d(m-2)}{2}$ labeled embeddings. No quotient by symmetries is taken.

Under the stated local hypothesis, their weighted rainbow count is at least ρJ . Fix a host corner of step h . A local corner of step j can map to it only at $d = h/j$, requiring $j \mid h$. Its first cell then fixes the two translation parameters uniquely, if a valid embedding exists. There are $\sum_{v=1}^{m-1-j} v = \binom{m-j}{2}$ local corners of step j . Thus this host corner is charged at most Λ_h , with the omitted boundary restrictions only reducing its charge. Summing over host rainbows and applying Lemma 7.5 gives the result. A proper $N \times N$ square restricts, after shifting indices by one, to a proper $\mathcal{T}_N$, so impossibility of that triangle proves the Ramsey conclusion. $\square$

Open assertion 7.7 (Two alternative local hypotheses). $F_{11}(5)$ asserts that every proper three-coloring of $\mathcal{T}_{11}$ has at least five rainbow corners. Separately, $F_{12}(7)$ asserts at least seven for every proper three-coloring of $\mathcal{T}_{12}$. Both assertions are open.

Proposition 7.8 (Integer weight certificates). *Take $N_{11} = 459$ and $N_{12} = 455$. For $m \in \{11, 12\}$ put $c_j = \binom{m-j}{2}$ for $1 \leq j \leq m-2$ and $K_m = 1000c_1$, so $K_{11} = 45\,000$, $K_{12} = 55\,000$. Define successively*

$$a_n^{(m)} = \left\lfloor \frac{K_m - \sum_{\substack{j|n \\ 2 \leq j \leq m-2}} c_j a_{n/j}^{(m)}}{c_1} \right\rfloor \quad (1 \leq n \leq 42), \quad a_n^{(m)} = 0 \quad (n > 42).$$

All weights lie in $[0, 1000]$, all positive-integer loads satisfy $\Lambda_h^{(m)} \leq K_m$, and

$$J_{11} = \sum_{d=1}^{42} a_d^{(11)} \binom{459-9d}{2} = 944\,436\,978, \quad J_{12} = \sum_{d=1}^{42} a_d^{(12)} \binom{455-10d}{2} = 809\,226\,900.$$

These are weight and load assertions, independent of either local rainbow hypothesis.

Proof idea. The recurrence controls loads up to 42. For larger steps, removing prime factors outside $\{2, 3, 5, 7\}$ reduces most checks to this first range. We check the remaining smooth integers directly and use a coefficient bound for the tail.

Proof. The local step coefficients are

$$(c_j)_{11} = (45, 36, 28, 21, 15, 10, 6, 3, 1), \quad (c_j)_{12} = (55, 45, 36, 28, 21, 15, 10, 6, 3, 1).$$

Table 13 in Appendix A evaluates every recurrence term and every copy-count summand. It proves nonnegativity and the bound 1000. For $h \leq 42$, the recurrence gives $0 \leq K_m - \Lambda_h^{(m)} < c_1$, so all these loads are within budget. For example the size-eleven recurrence at six has numerator $45\,000 - 36 \cdot 377 - 28 \cdot 200 - 10 \cdot 1\,000 = 15\,828$, giving $a_6 = 351$ and remainder 33.

Let $s(n)$ keep only the prime-power factors of n drawn from $\{2, 3, 5, 7\}$; call n smooth when $s(n) = n$. For $n \leq 42$, induction gives $a_n = a_{s(n)}$ for each weighting. Every allowed divisor $j > 1$ uses only those four primes, so it divides n exactly when it divides $s(n)$. The proper-divisor weights in the two recurrences agree by induction; hence so do their floors. Smooth n give the identity directly, and $s(n) = 1$ gives empty divisor sums where appropriate.

For $m = 11$, a nonsmooth $43 \leq h \leq 168$ has $s(h) \leq 15$, since its remaining factor is at least eleven. For $m = 12$, the analogous range ends at 210 and gives $s(h) \leq 19$. Every contributing j divides $s(h)$; if $h/j \leq 42$, then $a_{h/j} = a_{s(h)/j}$. Adding the missing nonnegative terms, including the possibly absent $j = 1$ term, gives $\Lambda_h \leq \Lambda_{s(h)} \leq K_m$.

Table 14 in Appendix A treats all remaining smooth integers in the ranges $43 \leq h \leq 168$ for $m = 11$ and $43 \leq h \leq 210$ for $m = 12$. Its list is obtained by taking $2^a 3^b 5^c 7^e$ in the indicated intervals; each load is the displayed divisor sum. Thus there are no untested integers between the small range and the tail.

For $m = 11$, if $h \geq 169$, a nonzero term requires $h/j \leq 42$, hence $j \geq 5$. For $m = 12$, if $h \geq 211$, it requires $j \geq 6$. In either case the sum of the remaining coefficients is $15 + 10 + 6 + 3 + 1 = 35$. Since every weight is at most 1000,

$$\Lambda_h^{(m)} \leq 35\,000 < K_m.$$

This proves the load bound for every positive integer, including steps beyond the host. Adding the printed summands proves both values of J_m . $\square$

Theorem 7.9 (Two conditional rainbow routes). *Assume $F_{11}(5)$, that every proper three-coloring of $\mathcal{T}_{11}$ has at least five rainbow corners. Then $R_3(\mathcal{L}) \leq 459$. Separately, assume $F_{12}(7)$, the corresponding seven-rainbow assertion for $\mathcal{T}_{12}$. Then $R_3(\mathcal{L}) \leq 455$. Both antecedents remain open.*

Proof idea. We substitute the two weight certificates into Lemma 7.6. The integer correction in the line bound reduces the upper rainbow count. Each local hypothesis then gives a lower count exceeding this upper count.

Proof. The largest dilation used is 42, and

$$42 \cdot 9 = 378 \leq 459 - 2, \quad 42 \cdot 10 = 420 \leq 455 - 2.$$

Thus both weightings satisfy the support requirement in Lemma 7.6, at their respective host sizes. The upper rainbow counts are

$$\begin{aligned} U_{459} &= \binom{459}{2} - 458 + \left\lfloor \frac{458}{3} \right\rfloor = 105\,111 - 458 + 152 = 104\,805, \\ U_{455} &= \binom{455}{2} - 454 + \left\lfloor \frac{454}{3} \right\rfloor = 103\,285 - 454 + 151 = 102\,982. \end{aligned}$$

Assuming $F_{11}(5)$, Lemma 7.6 would give $5J_{11} \leq 45\,000U_{459}$. Instead,

$$\begin{aligned} 5J_{11} &= 5 \cdot 944\,436\,978 = 4\,722\,184\,890, \\ 45\,000U_{459} &= 45\,000 \cdot 104\,805 = 4\,716\,225\,000, \\ 5J_{11} - 45\,000U_{459} &= 5\,959\,890 > 0. \end{aligned}$$

Under the alternative $F_{12}(7)$, the contradiction is

$$\begin{aligned} 7J_{12} &= 7 \cdot 809\,226\,900 = 5\,664\,588\,300, \\ 55\,000U_{455} &= 55\,000 \cdot 102\,982 = 5\,664\,010\,000, \\ 7J_{12} - 55\,000U_{455} &= 578\,300 > 0. \end{aligned}$$

The first hypothesis therefore excludes a proper coloring of $\mathcal{T}_{459}$ and hence of $[459]^2$; the second excludes $\mathcal{T}_{455}$ and hence $[455]^2$. We do not claim that these host sizes or the chosen weights are optimal. $\square$

To distinguish parameter adjustment from the integer correction, keep all forty-two weights fixed. With the coarser bound $\mathcal{R}_N \leq M_N$, the same argument already gives $N = 460$ under $F_{11}(5)$ and $N = 457$ under $F_{12}(7)$; the corresponding gaps $\rho J - K_m M_N$ are 1 575 945 and 8 045 239. The integer correction supplies the further improvements $460 \rightarrow 459$ and $457 \rightarrow 455$. Thus the change from the earlier common choice $N = 464$ includes both a separate choice of host for each route and the stronger line bound. These comparisons use the printed weights and retain the same open local premises.

A larger local lower bound would strengthen this argument, but it must hold for every proper coloring. The following example shows that the proposed bound $F_{13}(17)$ is false.

Proposition 7.10 (Refutation of $F_{13}(17)$). *There is a proper three-coloring of $\mathcal{T}_{13}$ with exactly fourteen rainbow corners. Thus $F_{13}(17)$ fails. Fourteen is not asserted to be the minimum.*

Proof. Table 8 defines every cell color $c(v, u) \in \mathbb{F}_3$ by its row word, without permuting positions or colors. For step $d > 0$ define

$$g_d(v, u) = c(v, u) + c(v + d, u) + c(v + d, u + d) \in \mathbb{F}_3.$$

A triple sums to zero exactly when it is monochromatic or rainbow. These cases give $3a = 0$ and $0 + 1 + 2 = 0$, whereas two copies of a and a different color b give $2a + b = b - a \neq 0$.

Table 8: The complete coloring of $\mathcal{T}_{13}$ and all its zero-sum corners. Row v lists colors at $u = 0, \dots, v-1$ in that order. Every zero sum has the three distinct colors shown on the right; the sums of all other corners appear in Table 15 in Appendix A.

v	$c(v, 0) \cdots c(v, v-1)$	Base (v, u)	Step d	Colors p, q, r
1	1	(3, 0)	6	(0, 1, 2)
2	00	(4, 2)	2	(2, 1, 0)
3	020	(5, 2)	4	(0, 1, 2)
4	2022	(5, 2)	5	(0, 2, 1)
5	21021	(5, 3)	5	(2, 1, 0)
6	121000	(6, 0)	1	(1, 2, 0)
7	2000202	(6, 3)	5	(0, 2, 1)
8	21110021	(6, 4)	5	(0, 2, 1)
9	121212210	(7, 2)	4	(0, 1, 2)
10	1021221102	(7, 5)	2	(0, 2, 1)
11	01122022111	(8, 4)	1	(0, 1, 2)
12	010120120121	(8, 4)	2	(0, 2, 1)
		(8, 5)	2	(0, 2, 1)
		(8, 6)	2	(2, 1, 0)

Table 15 prints every corner sum. For each d , its words run through $v = 1, \dots, 12 - d$ and then $u = 0, \dots, v - 1$, so all $\binom{13}{3} = 286$ corners occur once. The zero entries are exactly those listed, with their colors, in Table 8.

Every listed zero has three distinct colors, and every nonzero sum excludes monochromaticity. Thus the coloring is proper and its rainbow count is

$$2 + 5 + 0 + 2 + 4 + 1 + 0 + 0 + 0 + 0 + 0 = 14 < 17.$$

□

This coloring does not determine the minimum rainbow count in $\mathcal{T}_{13}$, settle either smaller local assertion, or give an extension to a full square.

Appendix B encodes the negations of $F_{11}(5)$ and $F_{12}(7)$ as finite SAT instances. A satisfying assignment would provide a counterexample; an independently checked unsatisfiability proof would establish the assertion. Either outcome would settle the corresponding local question. The weight certificates prove the implications in Theorem 7.9, while these decisions and the more general S21 remain questions for further work.

Closing discussion

For two Golomb classes, the repeated differences can be counted by the classes containing their endpoints. Theorem 4.1 uses this split to obtain a joint energy bound and an exact deficit formula. Unequal class sizes produce a loss, which Corollary 4.3 retains in the fiber estimates. The two counterexamples show why internal pairs and the distribution of these losses matter.

The cube arguments give exact conclusions. Every injective five-cube with positive integer generators is excluded, while separated four-cubes admit a proper coloring. On an injective three-cube, every proper coloring has at least nine mixed corners, and the construction attains nine under 2-dissociation. The five-cube exclusion uses the known binary Sidon capacity $a(4) = 7$; the full source colorings require the additional constructions and corner checks.

The four structured families give further cases in which T is bounded by a constant times M . The proofs use complete fibers for products, correlations for translation matchings, and Euclidean squares for two-distance representations. Their constants leave room for improvement. The family $\{\pm 5^i : 0 \leq i < r\}$ nevertheless shows that a proper two-distance source can have arbitrarily many marks, with $T/M \rightarrow 1/2$.

For arbitrary sources, the forcing identity leaves the quantity

$$2|E(\Gamma)| - \sum_d s_d^2 = 2T(A) - M(A)$$

to be controlled. A linear edge bound would suffice; a bound using both the edges and the fiber imbalances could also work. Section 7 records this general question and the two local rainbow questions as further directions. The joint energy bounds, cube thresholds, and structured-source estimates proved here do not depend on resolving them.

A Finite certificates and reproduction

The companion program `verify_directed_L_manuscript.py` checks the finite data using Python’s standard library. It reads the source tables, reconstructs the colorings, and independently enumerates corners from their coordinates. It also checks the seven-mark witness in (4) and recomputes the divisor weights and loads in exact integer arithmetic. The output file `verification_results.json` records the input source’s SHA-256 digest, the execution environment, and each result. These computations verify the printed certificates; statements for arbitrary sets or generators rely on the corresponding proofs.

The verifier enumerates pairs of source cells sharing a coordinate directly, and checks their positive-step completion. Equivalently, for a finite mark set A , a strict corner is given by an offset $\delta > 0$ and a pair $u < v$ with $u, u + \delta, v, v + \delta \in A$; the cells are $(u + \delta, u)$, $(v + \delta, u)$, $(v + \delta, v)$, and their common leg length is $h = v - u > 0$. Weak sources also permit the offset $\delta = 0$ (i.e. the two parallel endpoints are placed on the diagonal); the leg length remains positive and the three cells remain distinct. For a full strict triangle one may instead list $1 \leq v < n$, $0 \leq u < v$, and $1 \leq h < n - v$, obtaining (v, u) , $(v + h, u)$, $(v + h, u + h)$. Both descriptions use the same directed positive-step convention.

Table 9: Finite certificates checked by the companion program. A mixed corner has differently colored parallel endpoints; a rainbow uses all three colors.

Certificate	Size or enumeration	Verified conclusion
Twelve-mark weak source	78 cells; 140 weak corners	No monochromatic corner; 74 strict corners and $T > 2 \cdot 6 \cdot 6$.
Separate-deficit example	Two four-mark Golomb sets	$\rho = 5$, $N = 29 > 28$, $Q = 11$; no source coloring inferred.
Seven-mark Golomb subset	7 marks; 21 positive differences	Contained in the four-cube with generators $(1, 3, 9, 27)$; all differences in (5) are distinct.
Separated four-cube	120 strict cells; 200 corners	The complete table is proper; the printed color-word counts agree.
Separated three-cube	28 strict cells; 24 corners	Proper coloring with 9 mixed and 15 forcing corners.
Two divisor weightings	42 weights in each system	Integer recurrences, copy totals J_m , and every possibly nonzero load agree with the printed certificates.
Thirteen-mark triangle	78 cells; 286 corners	No monochromatic corner and exactly 14 rainbows; $F_{13}(17)$ is false.

For the weight systems, $a_d = 0$ for $d > 42$ and the local step satisfies $j \leq m - 2$. Hence a nonzero load has $h = jd \leq 42(m - 2)$, so checking every h up to this bound takes care of all nonzero loads. The checks retain the two exact gaps

$$5J_{11} - 45\,000U_{459} = 5\,959\,890, \quad 7J_{12} - 55\,000U_{455} = 578\,300.$$

Here U_N is the integer-refined rainbow cap from Lemma 7.5. The gaps verify the arithmetic in the conditional reductions. They do not establish either local rainbow premise. Similarly, the thirteen-mark coloring refutes a proposed lower bound without determining the minimum rainbow count. The general statements in Sections 4–6 are established by their proofs.

The supplementary program `verify_structured_examples.py` checks the four examples in Table 6; its source-bound report is `structured_examples_results.json`. Both verifiers use Python’s standard library. The SAT encoder, validator, inputs, and reproduction instructions are also included. Deciding an open instance requires a solver and, for an unsatisfiability claim, an independently checked proof.

A.1 Weak-source and cube certificates

The tables below complete the finite constructions used in Proposition 4.4 and Theorems 5.3 and 5.5. The defining colorings remain in the main text; the full completion and multiplicity lists are collected here.

Table 10: Complete positive-offset verification for Table 2. In the upper panel the endpoint color is c , and each listed completion has the last-column color. Below, b_d, c_d count the cells of color zero and one. Both panels follow by reading the full array.

d	c	Starts	All pair completions	Color
1	0	{7, 12, 21, 22}	(13, 7), (22, 7), (23, 7), (22, 12), (23, 12), (23, 21)	1
1	1	{0, 4, 23}	(5, 0), (24, 0), (24, 4)	0
2	1	{5, 21}	(23, 5)	0
3	1	{1, 4, 21}	(7, 1), (24, 1), (24, 4)	0
4	0	{1, 8}	(12, 1)	1
4	1	{0, 4}	(8, 0)	0
5	0	{0, 8}	(13, 0)	1
7	0	{1, 5}	(12, 1)	1
8	0	{0, 5, 13}	(13, 0), (21, 0), (21, 5)	1
9	0	{4, 13}	(22, 4)	1
11	1	{1, 12}	(23, 1)	0
12	1	{0, 12}	(24, 0)	0
16	1	{5, 7}	(23, 5)	0
17	1	{4, 7}	(24, 4)	0

d	r_d	b_d	c_d	$\binom{r_d}{2}$	d	r_d	b_d	c_d	$\binom{r_d}{2}$
1	7	4	3	21	13	2	1	1	1
2	3	1	2	3	14	2	1	1	1
3	4	1	3	6	15	2	1	1	1
4	4	2	2	6	16	3	1	2	3
5	3	2	1	3	17	3	1	2	3
6	2	1	1	1	18	2	1	1	1
7	3	2	1	3	19	2	1	1	1
8	4	3	1	6	20	2	1	1	1
9	3	2	1	3	21	2	1	1	1
10	2	1	1	1	22	2	1	1	1
11	3	1	2	3	23	2	1	1	1
12	3	1	2	3	24	1	1	0	0

Table 11: Exhaustive signed-support certificate for the four-cube. Internal L, U words list the two parallel endpoints followed by the completion; cross and bridge words list endpoint, completion, endpoint. The four family totals are 24, 24, 76, 76, and neither 000 nor 111 occurs.

Family	t	Words and multiplicities
L	1	001(3), 010(4), 011(2), 101(6), 110(3)
L	2	011(3), 100(2), 110(1)
U	1	001(3), 011(6), 100(6), 110(3)
U	2	100(3), 101(3)
X	1	001(7), 010(5), 011(8), 100(15), 101(4), 110(9)
X	2	001(1), 010(9), 011(2), 100(5), 101(7)
X	3	010(1), 011(1), 100(2)
Bridge	1	001(6), 010(12), 011(6), 100(8), 101(12), 110(4)
Bridge	2	010(5), 011(5), 100(6), 101(7), 110(1)
Bridge	3	001(1), 100(1), 110(2)

Table 12: All twenty-four three-cube corners. Each word lists p, q, r , with q the outer completion. Bold M marks the nine mixed corners; F marks forcing corners. The offsets are the true integers for the weights (1, 3, 9), and the pairs use Figure 3's mask legend.

d	p	q	r	Word	Type	d	p	q	r	Word	Type
1	(1, 0)	(3, 0)	(3, 2)	110	M	3	(6, 4)	(7, 4)	(7, 5)	010	F
1	(1, 0)	(5, 0)	(5, 4)	110	M	4	(3, 0)	(7, 0)	(7, 4)	101	F
1	(1, 0)	(7, 0)	(7, 6)	100	M	6	(4, 2)	(5, 2)	(5, 3)	010	F
1	(3, 2)	(5, 2)	(5, 4)	010	F	8	(4, 1)	(6, 1)	(6, 3)	010	F
1	(3, 2)	(7, 2)	(7, 6)	010	F	9	(4, 0)	(5, 0)	(5, 1)	110	M
1	(5, 4)	(7, 4)	(7, 6)	010	F	9	(4, 0)	(6, 0)	(6, 2)	110	M
2	(2, 1)	(6, 1)	(6, 5)	010	F	9	(4, 0)	(7, 0)	(7, 3)	100	M
3	(2, 0)	(3, 0)	(3, 1)	110	M	9	(5, 1)	(6, 1)	(6, 2)	010	F
3	(2, 0)	(6, 0)	(6, 4)	110	M	9	(5, 1)	(7, 1)	(7, 3)	010	F
3	(2, 0)	(7, 0)	(7, 5)	100	M	9	(6, 2)	(7, 2)	(7, 3)	010	F
3	(3, 1)	(6, 1)	(6, 4)	010	F	10	(5, 0)	(7, 0)	(7, 2)	101	F
3	(3, 1)	(7, 1)	(7, 5)	010	F	12	(6, 0)	(7, 0)	(7, 1)	101	F

A.2 Divisor weights and load bounds

We record here the complete numerical certificates for Proposition 7.8. The recurrence, and the proof that the small-step, smooth-step, and tail ranges are exhaustive, remain in Section 7.

Table 13: All dilation weights and weighted-copy summands for the separate hosts $N_{11} = 459$ and $N_{12} = 455$. Here $C_d^{(m)} = \binom{N_m - d(m-2)}{2}$. No local rainbow bound is assumed.

d	$a_d^{(11)}$	$C_d^{(11)}$	$a_d^{(11)}C_d^{(11)}$	$a_d^{(12)}$	$C_d^{(12)}$	$a_d^{(12)}C_d^{(12)}$
1	1 000	101 025	101 025 000	1 000	98 790	98 790 000
2	200	97 020	19 404 000	181	94 395	17 085 495
3	377	93 096	35 097 192	345	90 100	31 084 500
4	373	89 253	33 291 369	342	85 905	29 379 510
5	666	85 491	56 937 006	618	81 810	50 558 580
6	351	81 810	28 715 310	326	77 815	25 367 690
7	866	78 210	67 729 860	818	73 920	60 466 560
8	541	74 691	40 407 831	518	70 125	36 324 750
9	743	71 253	52 940 979	719	66 430	47 763 170
10	400	67 896	27 158 400	407	62 835	25 573 845
11	1 000	64 620	64 620 000	1 000	59 340	59 340 000
12	266	61 425	16 339 050	284	55 945	15 888 380
13	1 000	58 311	58 311 000	1 000	52 650	52 650 000
14	280	55 278	15 477 840	297	49 455	14 688 135
15	459	52 326	24 017 634	463	46 360	21 464 680
16	379	49 455	18 743 445	382	43 365	16 565 430
17	1 000	46 665	46 665 000	1 000	40 470	40 470 000
18	98	43 956	4 307 688	94	37 675	3 541 450
19	1 000	41 328	41 328 000	1 000	34 980	34 980 000
20	244	38 781	9 462 564	218	32 385	7 059 930
21	410	36 315	14 889 150	401	29 890	11 985 890
22	200	33 930	6 786 000	181	27 495	4 976 595
23	1 000	31 626	31 626 000	1 000	25 200	25 200 000
24	178	29 403	5 233 734	131	23 005	3 013 655
25	778	27 261	21 209 058	764	20 910	15 975 240
26	200	25 200	5 040 000	181	18 915	3 423 615
27	529	23 220	12 283 380	510	17 020	8 680 200
28	322	21 321	6 865 362	278	15 225	4 232 550
29	1 000	19 503	19 503 000	1 000	13 530	13 530 000
30	118	17 766	2 096 388	55	11 935	656 425
31	1 000	16 110	16 110 000	1 000	10 440	10 440 000
32	419	14 535	6 090 165	386	9 045	3 491 370
33	377	13 041	4 916 457	345	7 750	2 673 750
34	200	11 628	2 325 600	181	6 555	1 186 455
35	622	10 296	6 404 112	575	5 460	3 139 500
36	323	9 045	2 921 535	263	4 465	1 174 295
37	1 000	7 875	7 875 000	1 000	3 570	3 570 000
38	200	6 786	1 357 200	181	2 775	502 275
39	377	5 778	2 178 306	345	2 080	717 600
40	393	4 851	1 906 443	343	1 485	509 355
41	1 000	4 005	4 005 000	1 000	990	990 000
42	258	3 240	835 920	195	595	116 025
Σ			944 436 978			809 226 900

Table 14: Every remaining smooth-step load. The first weighting requires $43 \leq h \leq 168$; the second requires $43 \leq h \leq 210$. A dash lies beyond the first range, where its tail bound applies. All values are the divisor sums $\Lambda_h^{(m)}$.

h	$\Lambda_h^{(11)}$	$\Lambda_h^{(12)}$	h	$\Lambda_h^{(11)}$	$\Lambda_h^{(12)}$	h	$\Lambda_h^{(11)}$	$\Lambda_h^{(12)}$
45	24 663	33 621	84	31 310	37 306	144	9 236	11 039
48	29 069	37 325	90	9 764	12 839	147	2 460	4 010
49	5 196	8 180	96	20 058	24 998	150	9 550	13 078
50	34 008	43 545	98	1 680	2 970	160	15 270	19 400
54	29 569	38 097	100	19 998	26 377	162	5 388	7 932
56	23 316	30 914	105	26 320	33 751	168	10 936	13 346
60	28 709	35 682	108	21 399	26 010	175	—	19 715
63	16 804	24 080	112	9 876	13 386	180	—	7 096
64	24 666	31 174	120	19 969	22 971	189	—	6 303
70	28 992	37 000	125	11 670	16 044	192	—	6 576
72	24 100	29 311	126	12 192	14 866	196	—	2 780
75	28 669	37 227	128	9 936	13 100	200	—	12 005
80	26 157	32 521	135	8 394	12 099	210	—	13 671
81	15 555	20 517	140	19 356	24 415			

A.3 The thirteen-mark corner sums

The coloring in Table 8 is checked by listing all 286 corner sums below. Note that the fourteen zero sums are verified separately to be rainbow, so no monochromatic corner is hidden among the zeros.

Table 15: All 286 corner sums modulo three. In step row d , slash-separated words run through base rows $v = 1, \dots, 12 - d$; each word uses increasing u . The zero counts total fourteen, and Table 8 verifies every zero is rainbow.

d	Words by increasing base row	Zeros
1	1 /22 /211 /2112 /21121 /021222 /2221222 /21110122 /211222122 / 2222112121 /12221121211	2
2	1 /12 /221 /1202 /11222 /112121 /1121201 /22210001 /221121212 / 1212212121	5
3	2 /12 /111 /1221 /22122 /121122 /1212222 /11221121 /221122121	0
4	1 /12 /122 /1121 /12022 /111221 /1101212 /12212121	2
5	2 /22 /222 /2111 /22002 /121001 /2221221	4
6	2 /12 /021 /1112 /11221 /221211	1
7	1 /22 /221 /1212 /12121	0
8	2 /12 /112 /2211	0
9	1 /12 /121	0
10	2 /22	0
11	2	0

B Exact SAT problems for the local rainbow assertions

Fix integers $n \geq 3$ and $q \geq 0$, and let $\mathcal{H}_n$ be the set of directed corners in $\mathcal{T}_n$. There are $\binom{n}{2}$ cells and $\binom{n}{3}$ corners. To test a proposed lower bound $F_n(q + 1)$, we search for a proper three-coloring having at most q rainbow corners. Introduce Boolean variables $x_{p,c}$ for $p \in \mathcal{T}_n$ and $c \in \{0, 1, 2\}$, with the intended meaning that cell p has color c . Impose

$$\bigvee_{c=0}^2 x_{p,c}, \quad \neg x_{p,c} \vee \neg x_{p,c'} \quad (c < c'),$$

so each cell has exactly one color. For each corner $e = (p_1, p_2, p_3)$, impose

$$\neg x_{p_1,c} \vee \neg x_{p_2,c} \vee \neg x_{p_3,c} \quad (c = 0, 1, 2),$$

which forbids monochromatic corners. Introduce a Boolean variable y_e and, for every permutation π of $\{0, 1, 2\}$, impose

$$\neg x_{p_1,\pi(0)} \vee \neg x_{p_2,\pi(1)} \vee \neg x_{p_3,\pi(2)} \vee y_e.$$

Every rainbow corner forces $y_e = 1$. Finally impose the cardinality constraint

$$\sum_{e \in \mathcal{H}_n} y_e \leq q.$$

The script `directed_L_rainbow_sat.py` turns this last constraint into clauses using a sequential counter and exports a DIMACS formula together with a variable map. The displayed implications for y_e are sufficient: any satisfying assignment bounds the actual rainbow count by q , while any desired coloring extends to a satisfying assignment by setting y_e to its actual rainbow indicator and assigning the counter accordingly. The implementation also includes the three reverse clauses

$$\neg y_e \vee x_{p_1,c} \vee x_{p_2,c} \vee x_{p_3,c} \quad (c = 0, 1, 2),$$

so y_e is exactly the rainbow indicator. These clauses preserve every desired coloring. The default encoding uses no symmetry restriction.

Table 16: The two finite counterexample searches. The fourth column counts only color variables, excluding rainbow indicators and counter variables.

Assertion	Cells	Corners	Color variables	Rainbow cap q
$F_{11}(5)$	55	165	165	4
$F_{12}(7)$	66	220	198	6

A satisfying assignment must be checked against the original geometric definition, not only against the auxiliary-variable clauses. The companion validator reconstructs every cell color, enumerates every corner independently, and checks properness and the rainbow cap. A valid assignment is a finite counterexample to the corresponding assertion. To turn an unsatisfiable solver result into a computer-assisted theorem, one must keep the exact formula, the solver version and command, an unsatisfiability proof, and an independent check of that proof. A timeout, an interrupted run, or a search with additional unproved restrictions settles neither direction. No checked unsatisfiability certificate for either instance is supplied, so both assertions remain open.

An eventual proof of either local assertion would make the corresponding implication in Theorem 7.9 unconditional. A counterexample would close that route at its stated parameters, without refuting S21 or any of the proved structural estimates. These finite questions are also distinct from the unrestricted inequalities G9, EP10, and H10.

C A general bound from the corners theorem

The finite-abelian-group corners theorem of Jaber, Liu, Lovett, Ostuni, and Sawhney [6] gives a general bound for $R_k(\mathcal{L})$. Its quantitative strength is inherited from that theorem. This appendix checks the transfer to our positive-step convention: first reflect a directed L to a standard corner, then pair colors to exclude both signs, and finally embed the square in a cyclic group without creating a corner. The cited paper also discusses the integer-grid transfer. All logarithms below are natural.

Write $C(a, b; h) = \{(a, b), (a + h, b), (a, b + h)\}$ for a standard corner with integer step h .

Lemma C.1 (First-coordinate reflection). *For integers $N, k \geq 1$, put $H_N(x, y) = (N + 1 - x, y)$. A coloring $\chi : [N]^2 \rightarrow [k]$ avoids monochromatic positive directed L's if and only if $f = \chi \circ H_N$ avoids monochromatic standard corners $C(a, b; h)$ with $h > 0$.*

Proof. Reflection sends $\mathcal{L}(x, y; d)$ to $C(N + 1 - x - d, y; d)$. Conversely, it sends $C(a, b; d)$ to $\mathcal{L}(N + 1 - a - d, b; d)$. It is an involution of the square and preserves colors under pullback, so both implications follow. $\square$

Lemma C.2 (Half-turn and paired colors). *Let $N, k \geq 1$ be integers and let $f : [N]^2 \rightarrow [k]$ avoid monochromatic positive standard corners. Put $J_N(a, b) = (N + 1 - a, N + 1 - b)$ and $F(p) = (f(p), f(J_N p))$. Then F has at most k^2 colors, each class avoids standard corners of every nonzero integer step, and one class S has $|S| \geq N^2/k^2$.*

Proof. A positive corner monochromatic for F would be monochromatic for its first component f . If the step h were negative, its image under J_N would be a positive corner of step $-h$, monochromatic for f by equality of the second components. Both contradict the hypothesis. Since there are at most k^2 ordered color pairs, one class has at least N^2/k^2 cells. $\square$

Lemma C.3 (Cyclic embedding without wraparound). *Let $N \geq 1$ be an integer and let $S \subseteq [N]^2$ contain no standard corner of any nonzero integer step. Coordinate reduction $\iota : [N]^2 \rightarrow G^2$, where $G = \mathbb{Z}/(2N\mathbb{Z})$, is injective, and $\iota(S)$ has no group corner $(a, b), (a + \delta, b), (a, b + \delta)$ with $\delta \neq 0$ in G .*

Proof. Reduction is injective because distinct elements of $[N]$ differ by less than $2N$. A group corner would therefore have representatives $(a, b), (a', b), (a, b') \in S$. The increments $h = a' - a$ and $t = b' - b$ satisfy $h \equiv t \pmod{2N}$, while

$$|h - t| \leq 2N - 2 < 2N.$$

Thus $h = t$ as integers. Since the group step is nonzero, this common increment is nonzero. The three representatives give a forbidden integer corner in S . $\square$

The external input is Theorem 1.1 of Jaber–Liu–Lovett–Ostuni–Sawhney [6], using version 2. We restate its finite-abelian-group conclusion in the normalization needed here:

External theorem. There is an absolute $c > 0$ such that every subset A of the square of a finite abelian group G , avoiding standard group corners with nonzero step, satisfies

$$|A| \leq |G|^2 \cdot \exp(-c(\log |G|)^{1/600}).$$

Here $G = \mathbb{Z}/(2N\mathbb{Z})$ is finite abelian, and Lemmas C.2 and C.3 give a set $A = \iota(S)$ avoiding every nonzero group step. Thus the theorem applies with its exponent $1/600$. No restriction on the exponent of G is needed. The stronger exponent $1/178$ from the separate finite-field result does not enter this argument.

Theorem C.4 (General bound from the external corners theorem). *Fix an absolute constant $c > 0$ valid in the cited external Theorem 1.1. Define*

$$B(k) = \left\lceil \exp\left(\left(\frac{\log(8k^2)}{c}\right)^{600}\right) \right\rceil, \quad C = \left(\frac{3}{c}\right)^{600}, \quad D = \exp(C + \log 2).$$

For every integer $k \geq 1$,

$$R_k(\mathcal{L}) \leq B(k) \leq \left\lceil \exp(C(\log(2k))^{600}) \right\rceil \leq D^{k^{600}}.$$

The constants come from the external theorem and are not explicit here.

Proof idea. A proper square gives a paired-color class of density at least $1/(4k^2)$ in the group square. We choose N so that the external theorem would force its density to be at most $1/(8k^2)$, giving a contradiction.

Proof. Put $N = B(k)$ and suppose a proper k -coloring of $[N]^2$ exists. Lemmas C.1–C.3 supply a class S of size at least N^2/k^2 and its corner-free image in a group square of size $4N^2$. The external theorem gives

$$\frac{N^2}{k^2} \leq |S| \leq 4N^2 \exp\left(-c(\log(2N))^{1/600}\right).$$

The definition of N , and monotonicity on positive arguments, give

$$\log(2N) \geq \left(\frac{\log(8k^2)}{c}\right)^{600}, \quad c(\log(2N))^{1/600} \geq \log(8k^2).$$

Thus $|S| \leq 4N^2/(8k^2) = N^2/(2k^2) < N^2/k^2$, contradicting the choice of S . Hence $R_k(\mathcal{L}) \leq B(k)$.

Since $8k^2 \leq (2k)^3$ for $k \geq 1$,

$$\left(\frac{\log(8k^2)}{c}\right)^{600} \leq \left(\frac{3 \log(2k)}{c}\right)^{600} = C(\log(2k))^{600}.$$

Taking exponentials and ceilings proves the middle bound. Also, $k - \log(2k)$ is positive at $k = 1$ and has derivative $1 - 1/k \geq 0$ for $k \geq 1$, so $\log(2k) \leq k$. Using $\lceil x \rceil \leq 2x$ for $x \geq 1$, we obtain

$$\left\lceil e^{C(\log(2k))^{600}} \right\rceil \leq \left\lceil e^{Ck^{600}} \right\rceil \leq 2e^{Ck^{600}} = e^{Ck^{600} + \log 2} \leq e^{(C + \log 2)k^{600}} = D^{k^{600}}.$$

□

Hence $R_k(\mathcal{L}) \leq \exp(O((\log(2k))^{600}))$. This quasipolynomial bound describes the dependence on k , but its unspecified constant gives no new numerical bound for three or four colors.

D Sources and attribution

The comparison $21 \leq R_3(\mathcal{L}) \leq 493$ uses the exhibited coloring of Canakci et al. [4, Theorem 3.1 and Figure 1] and Theorem 6 of Mammel–Smith–Yerger [5]. The latter appeared in *Integers* **25** (2025), A21. Lemma 2.1 checks the orientation convention. These are the specified sources for the numerical comparison; the conditional bounds in Section 7 are not replacements for them.

Table 17: Relation of the main arguments to the cited starting points. The table specifies what is proved and used here; it does not assert that a literature search establishes priority.

Topic	Cited starting point	Result or use in this paper
Two Golomb classes	The forced same-color marks on a subdiagonal are Golomb [5, Theorem 6].	Joint collision and deficit counts in Theorem 4.1; comparison with the displayed separate-class estimate.
Four-cube Golomb capacity	The ordinary binary Sidon number is $a(4) = 7$ [12]; binary pair sums and base-three encoding are discussed in [13].	A pair-capacity proof adapted to weighted cubes. The value seven is treated as known.
Cube source colorings	The binary Sidon restriction bounds each diagonal color class; it does not itself color all strict source cells.	Five-cube exclusion and an explicit separated four-cube coloring; mixed-corner lower bound nine, attained under 2-dissociation.
Shared forcing	Diagonal exclusion and forced colors are used in [4, 5].	A graph on the actual source cells and an exact identity involving edge count and fiber imbalance. A uniform bound remains open.
General growth in k	The density estimate of Jaber et al. [6].	Verification of reflection, positive orientation, and cyclic embedding for this convention; the asymptotic strength is inherited.

The binary Sidon connection uses ordinary coordinatewise addition, not addition modulo two. An injective weighted image that is Golomb has Sidon binary labels. Conversely, weights 1, 3, 9, ... preserve binary pair sums without carries. Hence the maximum attainable Golomb capacity agrees with the binary Sidon number. The seven-point witness in (4) is also equivalent, under a permutation and a reflection of cube coordinates, to the example in [12]; it is included to make the argument self-contained, not as a new construction.

Earlier supplied notes report the upper bounds $R_3(\mathcal{L}) \leq 465$ and $R_4(\mathcal{L}) \leq 21\,991\,300$. In the material accompanying this revision we have not identified a named author, a dated version, or a retrievable proof for these numbers. We therefore record them as prior reports only, not as established baselines. None of the conditional reductions or unconditional structural proofs uses them. The elementary six-term exclusion from those notes is different: Lemma 6.4 includes its complete proof, so its use is self-contained while retaining the supplied-note attribution.

The external finite-abelian-group theorem is cited as version 2 of Jaber et al. [6], dated 14 July 2025, and its exponent $1/600$ is used unchanged. The reflection, the paired-color argument, and the representative-interval check in Appendix C verify the hypotheses for its application; the density theorem itself is not a contribution of this paper.

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A separate contribution and assistance statement accompanies the manuscript for completion and confirmation by the student and supervisor. It identifies the personal research contributions, supervision arrangements, and the record of AI use, which is limited to language polishing, LaTeX code proofreading, literature and source checks, and program preparation.