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Title: Sharper Bounds for Quantitative Exact Approximation and Rational Exponential Scale

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Abstract

Exact approximation problem in convergent fraction approximations asks how narrowly convergence approximation error can follow a prescribed rate function. It is clear that in most cases, the output of the target function and the real convergence approximation has some small differences infinitely often even when denominators get sufficiently large. Therefore, a relative tolerance scale is included. If an irrational number can be approximated by the prescribed rate function, with the difference of the function and approximation less than the tolerance scale, it is considered as an exact approximation. Quantitatively, we write set $E(\Psi, \omega)$ for real numbers that satisfies exact approximation, namely the target function is Ψ and tolerance function is ω . Baker-Ward's paper already gave a sufficient condition for $E(\Psi, \omega)$ being uncountable and separately, being empty. The central question is how small this tolerance can be while the resulting set remains nonempty. A smaller tolerance places the successful approximation errors in a narrower interval and thus imposes a stronger requirement.

This paper mainly focuses on two parts. First is that in case where the prescribed rate function Ψ as exponential decay function with a rational base $\lambda = r/s$ ($r, s = 1$ which is greater than 1. We established function $h_\lambda(q) = \lambda^{q-qr^q}$, and find the existence and emptiness criteria are both occurring at this scale, specifically, when the fixed multiplicative constant is greater than 6, $E(\Psi, \omega)$ contains a Cantor set and is thus uncountable, and if the constant is less than 1, $E(\Psi, \omega)$ is empty. This makes the criteria up to multiplicative constants' differences. The critical point of constant is yet to be determined. This scale is substantially smaller than the sufficient tolerance obtained from the general result for exponential decay. Our paper is not merely making constants better, but also is working better when Ψ decays fast.

More, in general, we also sharpened Baker-Ward's sufficient condition of $E(\Psi, \omega)$ being uncountable. Assuming that $q \mapsto q^2\Psi(q)$ is eventually non-increasing, $q^2\Psi(q) \rightarrow 0$, and $\omega(q) \rightarrow 0$, we strengthened the bad interval estimation in multiplicative constants and proved our construction to be admissible. Therefore, in exponential decay case, the leading negative exponent can be strengthened from $1/2$ to 1 , showing our method works better when Ψ decays very fast.

For the rational-base result, arithmetic separation at shell endpoints is combined with bounded continued-fraction padding construction lemma. The construction enforces the target approximation infinitely often while excluding excessively good approximations at every sufficiently large denominator.

The Cantor set conclusions give continuum many numbers satisfying the required approximation conditions. The rational exponential result further shows that the arithmetic of the decay rate can strengthen the existence criterion beyond what follows from the general monotonicity argument. Any optimal case is not claimed by now. For irrational bases, the shell endpoint need not be rational, so the arithmetic argument cannot immediately apply. Hence exact constant and general case study needs further investigation.

Keywords. Diophantine approximation; continued fractions; exact approximation order; fundamental intervals; Cantor construction; exponential decay.

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1 Introduction

Let $\Psi : (0, \infty) \rightarrow (0, \infty)$ be non-increasing. Define

$$W(\Psi) := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| \leq \Psi(q) \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

For a function $\omega : (0, \infty) \rightarrow (0, 1)$ satisfying $\omega(q) \rightarrow 0$, Baker and Ward define

$$E(\Psi, \omega) := \{ x \in W(\Psi) : \text{for every sufficiently large } q \text{ and every } p \in \mathbb{Z}, \left| x - \frac{p}{q} \right| \geq (1 - \omega(q))\Psi(q) \}.$$

Thus $x \in E(\Psi, \omega)$ has approximations at the scope $\Psi(q)$ infinitely often, but has no approximation below the slightly smaller scale $(1 - \omega(q))\Psi(q)$ once q is sufficiently large. Since $\omega(q) \rightarrow 0$, these two scales are identical asymptotically. This means Ψ is the exact approximation scale wanted for x ; see Baker and Ward [1, Section 1.1].

Now, we recall Theorem 1.1 of Baker–Ward. They assume that $q \mapsto q^2\Psi(q)$ is non-increasing on $(0, \infty)$, $q^2\Psi(q) \rightarrow 0$, $\omega(q) \rightarrow 0$, and that

$$\omega(q) \geq \frac{45}{\Psi(q)} \Psi\left(\frac{1}{2q\Psi(q)}\right) \quad (1)$$

for all sufficiently large q , then members in $E(\Psi, \omega)$ can construct a cantor set, and therefore $E(\Psi, \omega)$ is uncountable [1, Theorem 1.1].

For exponential decaying case $\Psi(q) = \lambda^{-q}$, condition (1) becomes

$$\omega(q) \geq 45\lambda^{q - \lambda^q/(2q)}. \quad (2)$$

The leading negative term in the exponent is $-\frac{\lambda^q}{2q}$. A straightforward corollary of this paper is to refine the local denominator estimate in Baker–Ward’s Lemma 4.4 and replace this leading coefficient 1/2 by 1 and greatly improve the result.

For rational exponential based case, a further improvement beyond the general estimation can be obtained. Theorem 1.5 gives a sufficient tolerance scale that only differs Baker and Ward’s emptiness criterion in terms of multiplicative constants, specifically the function is $\lambda^{q - qr^q}$. Thus, in this case, we determine the transition pf $E(\Psi, \omega)$ by the quantity of constants.

Only sufficiently large denominators are considered in the construction below, and hence the strict condition of non-increasing is not needed and only eventual monotonicity is needed and can be weakened to. Throughout the paper we assume

$$q \mapsto q^2\Psi(q) \text{ is eventually non-increasing,} \quad q^2\Psi(q) \rightarrow 0, \quad \omega(q) \rightarrow 0. \quad (3)$$

The classical continued-fraction estimate lemmas used throughout may be found in Khinchin [4]; broader backgrounds and discussions of exact-order questions include Bugeaud [2] and Bugeaud–Moreira [3].

In the general monotonicity branch of this paper, one of our construction is partly imported from Baker–Ward. However, we first locally keep the previous denominator and the rounding integer-cell phase in the first denominator step, and then verify that our sharper input can be substituted into their construction. Globally, we did change the fundamental geometry separation input from 20 to any fixed constant that is bigger than 2. We also give a new type of construction that does not consider positive ratio length in lemma 5.4.

For sufficiently large q , put

$$\kappa(q) := 1 - q^2\Psi(q), \quad T(q) := \frac{\kappa(q)}{q\Psi(q)} = \frac{1}{q\Psi(q)} - q. \quad (4)$$

Fix any constant $\Gamma > 6$ and define

$$K_\Gamma := \frac{\Gamma}{3} > 2, \quad c_\Gamma := \frac{K_\Gamma - 2}{10K_\Gamma} = \frac{\Gamma - 6}{10\Gamma} > 0.$$

Set

$$R_\Gamma(q) := \Gamma \kappa(q)^2 \frac{\Psi(T(q))}{\Psi(q)}. \quad (5)$$

Theorem 1.1 (Cantor set construction and uncountability). *Assume (3). Suppose, in addition, that for all sufficiently large q , the condition*

$$\omega(q) \geq \frac{R_\Gamma(q)}{1 + R_\Gamma(q)}. \quad (6)$$

Then $E(\Psi, \omega)$ contains a Cantor set. In particular, $|E(\Psi, \omega)| = 2^{\aleph_0}$.

Theorem 1.2 (Exponential sub-case). *Let $\lambda > 1$ and $\Psi(q) = \lambda^{-q}$. Let $\omega : (0, \infty) \rightarrow (0, 1)$ satisfies $\omega(q) \rightarrow 0$. Fix constant $\Gamma > 6$. If*

$$\omega(q) \geq \frac{R_{\lambda, \Gamma}(q)}{1 + R_{\lambda, \Gamma}(q)}, \quad R_{\lambda, \Gamma}(q) := \Gamma(1 - q^2 \lambda^{-q})^2 \lambda^{2q - \lambda^q/q}, \quad (7)$$

for all sufficiently large q , then $E(\Psi, \omega)$ contains a Cantor set and is uncountable and holds the cardinality of continuum.

Corollary 1.3. *Under the hypotheses of Theorem 1.2, fix $0 < \varepsilon < 1$. The condition*

$$\omega(q) \geq \Gamma \lambda^{-(1-\varepsilon)\lambda^q/q} \quad (8)$$

for all sufficiently large q implies the conclusion of Theorem 1.2.

Corollary 1.4 (Power-law approximation). *Let $\tau > 2$ and*

$$\Psi(q) = q^{-\tau}.$$

Suppose that $\omega(q) \rightarrow 0$ and that, for some fixed constant $C > 6$,

$$\omega(q) \geq Cq^{-\tau(\tau-2)}$$

for all sufficiently large q . Then

$$E(\Psi, \omega)$$

contains a Cantor set and therefore is uncountable and holds the cardinality of the continuum.

Theorem 1.5 (Rational Exponential Base Specified). *Let $\lambda = \frac{r}{s} > 1$, where $r > s \geq 1$ are integers that satisfies $(r, s) = 1$. Put $\Psi(q) = \lambda^{-q}$ and let $\omega : (0, \infty) \rightarrow (0, 1)$ satisfy $\omega(t) \rightarrow 0$. For $q \in \mathbb{N}$, define*

$$N(q) := qr^q, \quad h_\lambda(q) := \frac{\Psi(N(q))}{\Psi(q)} = \lambda^{q - qr^q}.$$

Then the following arguments hold:

1. *If there exists a constant $C > 6$ such that $\omega(q) \geq Ch_\lambda(q)$ for every sufficiently large q , then $E(\Psi, \omega)$ contains a Cantor set.*
2. *If there exists a constant $0 < c < 1$ such that $\omega(q) \leq ch_\lambda(q)$ for every sufficiently large integer q , then $E(\Psi, \omega) = \emptyset$.*

The second argument is simply a specialization of Baker-Ward [1, Theorem 1.6(1)].

Notice that $h_\lambda(q)$ is way smaller than the function we gained in theorem 1.2, since

$$\frac{qr^q}{\lambda^q/q} = q^2 s^q \rightarrow \infty.$$

and $q < 2q$. At this scale, constants bigger than 6 guarantee the uncountability of set $E(\Psi, \omega)$ while constant in the range of $(0, 1)$ guarantee its emptiness. The critical endpoint is not claimed but indeed the problem of exact approximation turns into how big the constant is.

We refined the fundamental geometry of Baker–Ward paper by showing the separation input, or the sufficient ratio of 20 is way far from necessary. Our paper puts a sharper replacement for their Lemma 4.4, improves bad-interval constant and the sharpened cutoff cylinder (separation input). We verify that under their framework, the sharper replacement is admissible, and then write out the binary construction of Cantor set. The stronger measure conclusion of their Lemma 4.1 is not used in this paper.

This paper only concerns the general monotonicity branch of Baker–Ward’s paper and does not have improvement and general discussion in their square divisibility result and Hausdorff dimension result. Particularly, when $\Psi(q) = q^{-\tau}$ with an integer $\tau \geq 3$, their divisibility-based argument applies at the strictly smaller scale $q^{-\tau(\tau-1)}$, whereas our general result concerns the scale $q^{-\tau(\tau-2)}$.

2 Set-up for continued fraction

Consider

$$x = [0; a_1, a_2, \dots], \quad \frac{p_n}{q_n} = [0; a_1, \dots, a_n].$$

The denominators satisfy

$$q_{n+1} = a_{n+1}q_n + q_{n-1}. \quad (9)$$

Fix a rational fraction $p/q = [0; a_1, \dots, a_n]$, n is even, $q = q_n$, $q' = q_{n-1}$. We first present some basic lemmas that help our logic to be more explicit and clear.

Lemma 2.1 (Tail coordinate). *For $z > 1$,*

$$[0; a_1, \dots, a_n, z] = \frac{zp_n + p_{n-1}}{zq_n + q_{n-1}}. \quad (10)$$

If n is even, then

$$[0; a_1, \dots, a_n, z] - \frac{p_n}{q_n} = \frac{1}{q_n^2(z + \frac{q_{n-1}}{q_n})}. \quad (11)$$

Proof. Equation (10) can be understood by multiplying the standard continued-fraction matrices. Using

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n, \quad (12)$$

we can hence obtain

$$\frac{zp_n + p_{n-1}}{zq_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{p_{n-1}q_n - p_nq_{n-1}}{q_n(zq_n + q_{n-1})} = \frac{1}{q_n^2(z + \frac{q_{n-1}}{q_n})}$$

when n is even. For odd n , the sign is reversed; this is why we only work on the right-hand shell for even convergents. $\square$

By (11), the points in convergent fractions that are to the right of p/q satisfying

$$(1 - \omega(q))\Psi(q) \leq x - \frac{p}{q} \leq \Psi(q)$$

Here, when dealing with continued prefixes, we call z the tail coordinates and x to be the original coordinates, or the x -coordinates. The $S(p/q)$ and bad intervals defined below is defined in tail coordinates, their image however is in the original coordinates.

By calculation, this means its tail coordinates belong to the interval

$$S(p/q) := \left[\frac{1}{q^2 \Psi(q)} - \frac{q'}{q}, \frac{1}{q^2 \Psi(q)(1 - \omega(q))} - \frac{q'}{q} \right]. \quad (13)$$

Indeed, solving the two error inequalities in (11) gives the endpoints displayed above, with order reversed because the error is decreasing in z . This is the interval appearing in [1, Lemma 4.2]. Its exact length is

$$\mathcal{L}(S(p/q)) = \frac{\omega(q)}{q^2 \Psi(q)(1 - \omega(q))} \geq \frac{\omega(q)}{q^2 \Psi(q)}. \quad (14)$$

If the tail coordinate z with the length of $n + 1$ lies in $S(p/q)$, then the n -th convergent $p/q = p_n/q_n$ approximates x at the desired scale:

$$(1 - \omega(q))\Psi(q) \leq x - \frac{p}{q} \leq \Psi(q).$$

A quick note however, is that this condition controls the current convergent only; a later convergent may still approximate x too accurately that it does not belongs to the range of approximation $E(\Psi, \omega)$ permits.

To control approximations that is unwanted due to later conditions inside $S(p/q)$, we use the bad intervals introduced by Baker and Ward: For a finite word $\mathbf{c} = (c_0, \dots, c_m)$, let

$$I_m(c_0; c_1, \dots, c_m)$$

denote the fundamental interval consisting of tails whose first $m + 1$ continued-fraction digits are $c_0, \dots, c_m$ and the later tail is arbitrary.

Let $b_0 \in \mathbb{N}$ satisfy

$$[b_0, b_0 + 1] \cap S(p/q) \neq \emptyset, \quad (15)$$

and let $\mathbf{b} = (b_1, \dots, b_m)$ be a possibly empty finite word. Define q_{n+m+1} by

$$\frac{p_{n+m+1}}{q_{n+m+1}} = [0; a_1, \dots, a_n, b_0, b_1, \dots, b_m],$$

and set

$$C(\mathbf{b}) := \left\lceil \frac{1}{3q_{n+m+1}^2 \Psi(q_{n+m+1})} \right\rceil.$$

according to Baker and Ward [1] rough estimation. Following Baker and Ward [1, (4.9)], define the bad interval

$$B(p/q, b_0, \mathbf{b}) := \bigcup_{c \geq C(\mathbf{b})} I_{m+1}(b_0; b_1, \dots, b_m, c). \quad (16)$$

Thus $B(p/q, b_0, \mathbf{b})$ consists of tails for which the next partial quotient is bigger than the maximum partial quotient that makes approximation near Ψ enough. This maximum value is called "relevant cutoff" in this paper.

Baker and Ward prove in equations (4.10)–(4.11) that

$$\mathcal{L}(B(p/q, b_0, \mathbf{b})) \leq 9q_{n+1}^2 \Psi(q_{n+1}), \quad (17)$$

where

$$q_{n+1} = b_0 q + q'. \quad (18)$$

The estimate is uniform over all possibly empty words $\mathbf{b}$. When $\mathbf{b}$ is empty, $m = 0$ and $q_{n+m+1} = q_{n+1}$, as required.

Lemma 2.2 (Sharpened bad-interval estimate). *Assume (3). For every sufficiently large and admissible q and a corresponding convergent fraction p/q , every relevant b_0 , and every possibly empty word $\mathbf{b}$,*

$$\mathcal{L}(B(p/q, b_0, \mathbf{b})) \leq 3q_{n+1}^2 \Psi(q_{n+1}). \quad (19)$$

Proof. Define $D = q_{n+m+1}$, and let Q_i be the denominator of the tail convergent $[b_0; b_1, \dots, b_i]$, with $Q_{-1} = 0$ and $Q_0 = 1$. After appending the digit c , the new denominator is $cQ_m + Q_{m-1}$. Consequently,

$$\begin{aligned} \mathcal{L}(B) &= \sum_{c=C(\mathbf{b})}^{\infty} \frac{1}{(cQ_m + Q_{m-1})((c+1)Q_m + Q_{m-1})} \\ &= \frac{1}{Q_m(C(\mathbf{b})Q_m + Q_{m-1})}, \end{aligned}$$

where the last equality is a telescoping sum. Since $C(\mathbf{b}) \geq \frac{1}{3D^2\Psi(D)}$, this exact formula gives

$$\mathcal{L}(B) \leq \frac{1}{C(\mathbf{b})Q_m^2} \leq 3 \left(\frac{D}{Q_m} \right)^2 \Psi(D). \quad (20)$$

Specifically since $Q_m \geq 1$,

$$\mathcal{L}(B) \leq 3D^2\Psi(D).$$

Since $D \geq q_{n+1}$ and $t^2\Psi(t)$ is eventually non-increasing,

$$\mathcal{L}(B) \leq 3D^2\Psi(D) \leq 3q_{n+1}^2\Psi(q_{n+1}),$$

as claimed □

Lemma 2.3 (Sharpened adjacent cutoff cylinder). *Fix $\Gamma > 6$. Let q to be sufficiently large and for the corresponding p/q , let b_0 be relevant, and let $\mathbf{b} = (b_1, \dots, b_m)$ be a possibly empty word. Put*

$$D := q_{n+m+1}, \quad \theta_D := \frac{1}{3D^2\Psi(D)}, \quad C_D := \lceil \theta_D \rceil, \quad F_D := \lfloor \theta_D \rfloor, \quad A_D := C_D - 1$$

Then $A_D \leq F_D$ always holds. The cylinder with final digit A_D is immediately adjacent to the bad interval and satisfies

$$\frac{\mathcal{L}(I_{m+1}(b_0; b_1, \dots, b_m, A_D))}{\mathcal{L}(B(p/q, b_0, \mathbf{b}))} \leq \frac{1}{A_D}.$$

Moreover, globally over all relevant prefixes,

$$\frac{1}{A_D} \leq \frac{K_\Gamma}{2} - 1 = \frac{\Gamma - 6}{6}$$

for all sufficiently large q .

Proof. Let Q_i be the denominator of $[b_0; b_1, \dots, b_i]$, with $Q_{-1} = 0$ and $Q_0 = 1$. The exact telescoping formula used in lemma 2.2 gives

$$\mathcal{L}(B(p/q, b_0, \mathbf{b})) = \frac{1}{Q_m(C_D Q_m + Q_{m-1})}.$$

Since $C_D = A_D + 1$,

$$\mathcal{L}(I_{m+1}(b_0; b_1, \dots, b_m, A_D)) = \frac{1}{(A_D Q_m + Q_{m-1})(C_D Q_m + Q_{m-1})}.$$

Therefore

$$\frac{\mathcal{L}(I_{m+1}(b_0; b_1, \dots, b_m, A_D))}{\mathcal{L}(B(p/q, b_0, \mathbf{b}))} = \frac{Q_m}{A_D Q_m + Q_{m-1}} \leq \frac{1}{A_D}.$$

Since globally,

$$D \geq q_{n+1} \geq T(q).$$

and $t \mapsto t^2 \Psi(t)$ is eventually non-increasing, for sufficiently large q_n ,

$$D^2 \Psi(D) \leq T(q)^2 \Psi(T(q)) \longrightarrow 0.$$

Hence $A_D \rightarrow \infty$. With $\Gamma > 6$ is fixed, we eventually gain

$$A_D \geq \frac{6}{\Gamma - 6},$$

which is equivalent to

$$\frac{1}{A_D} \leq \frac{\Gamma - 6}{6} = \frac{K_\Gamma}{2} - 1.$$

□

3 The local rounding denominator estimate

The standard proof first obtains a rough lower bound for b_0 and then uses $q_{n+1} \geq b_0 q$. Retaining q' in the exact recurrence formula improves this estimate, while retaining the integer rounding reveals the precise location of the remaining loss. Put

$$A(q) := \frac{1}{q^2 \Psi(q)}, \quad r := \frac{q'}{q}, \quad \rho(u) := u - \lceil u \rceil + 1 \in (0, 1]. \quad (21)$$

If $u \notin \mathbb{Z}$, then $\rho(u)$ is the fractional part of u , whereas $\rho(u) = 1$ when $u \in \mathbb{Z}$. Thus $\rho(u)$ measures the distance from u to the left endpoint of the leftmost closed unit cell containing u .

Lemma 3.1 (Exact rounding estimation first step). *Assume $q^2 \Psi(q) < 1$, and let*

$$\rho_{p/q} := \rho\left(A(q) - \frac{q'}{q}\right). \quad (22)$$

If (15) holds, then

$$q_{n+1} \geq \frac{1}{q^2 \Psi(q)} - \rho_{p/q} q. \quad (23)$$

Moreover, if

$$L := A(q) - \frac{q'}{q} > 1,$$

then the smallest admissible integer cell meeting the left endpoint of $S(p/q)$ can meet equality in (23). In particular, this equality case is available for all sufficiently large q .

Proof. We first verify the rounding identity

$$\lceil L \rceil - 1 = L - \rho(L). \quad (24)$$

If $L = k + \theta$, where $k \in \mathbb{Z}$ and $0 < \theta < 1$, then

$$\lceil L \rceil - 1 = k, \quad \rho(L) = \theta,$$

so both sides of (24) equal k . If $L = k \in \mathbb{Z}$, then accordingly $\rho(L) = 1$, and both sides equal $k - 1$.

Now since we already assumed $[b_0, b_0 + 1] \cap S(p/q) \neq \emptyset$ and L is the left endpoint of $S(p/q)$, we must have $b_0 + 1 \geq L$. Since $b_0 \in \mathbb{Z}$,

$$b_0 \geq \lceil L \rceil - 1 = L - \rho(L).$$

Applying convergents recurrence gives us

$$\begin{aligned} q_{n+1} &= b_0 q + q' \\ &\geq q(L - \rho(L)) + q' \\ &= q \left(A(q) - \frac{q'}{q} - \rho(L) \right) + q' \\ &= qA(q) - \rho(L)q \\ &= \frac{1}{q\Psi(q)} - \rho_{p/q}q. \end{aligned}$$

To meet equality, choose

$$b_0^* := \lceil L \rceil - 1.$$

If $L \notin \mathbb{Z}$, then

$$b_0^* < L < b_0^* + 1.$$

If $L \in \mathbb{Z}$, then

$$b_0^* = L - 1$$

Thus, either way, the closed cell $[b_0^*, b_0^* + 1]$ contains the left endpoint L and therefore intersects with $S(p/q)$. If $L > 1$, then $b_0^* \geq 1$, so it is a legitimate continued-fraction digit. Substituting $b_0^* = L - \rho(L)$ into $q_{n+1} = b_0 q + q'$ proves the condition of equality in (23).

Finally, since $q^2\Psi(q) \rightarrow 0$, we have $A(q) \rightarrow \infty$, while $0 \leq \frac{q'}{q} < 1$. Hence $L > 1$ holds for all sufficiently large q . $\square$

Corollary 3.2 (Global/Uniform denominator bound). *Under the premise of Lemma 3.1,*

$$q_{n+1} \geq T(q) = \frac{1}{q\Psi(q)} - q = \frac{1 - q^2\Psi(q)}{q\Psi(q)}. \quad (25)$$

Proof. This follows from Lemma 3.1 and $0 < \rho_{p/q} \leq 1$. $\square$

The improvement can be seen by comparing the rough and exact uses of the denominator recurrence. From Lemma 3.1, we have

$$b_0 \geq A(q) - r - \rho_{p/q}, \quad r = \frac{q'}{q}.$$

Consequently,

Baker-Ward rough recurrence	exact recurrence
$\begin{aligned} q_{n+1} &\geq b_0 q \\ &\geq q(A(q) - r - \rho_{p/q}) \\ &= qA(q) - \underbrace{rq}_{=q'} - \rho_{p/q}q \\ &= \frac{1}{q\Psi(q)} - q' - \rho_{p/q}q \end{aligned}$	$\begin{aligned} q_{n+1} &= b_0 q + q' \\ &\geq q(A(q) - r - \rho_{p/q}) + q' \\ &= qA(q) - \underbrace{rq}_{=q'} - \rho_{p/q}q + q' \\ &= \frac{1}{q\Psi(q)} - \rho_{p/q}q. \end{aligned}$

Thus, in the previous rough estimation, the positive term q' is discarded before it can cancel the term $-rq = -q'$ rising from the lower bound for b_0 . The local rounding estimate therefore improves the rough lower bound by exactly q' .

Proposition 3.3 (Local optimality of the uniform loss). *Coefficient 1 for the negative term q in (25) cannot be decreased using the previous hypotheses that a unit integer cell intersects with $S(p/q)$ and that $q_{n+1} = b_0 q + q'$.*

Specifically, the exact minimum value allowed is the right side of (23), and phases $\rho_{p/q}$ can attain the worst value 1.

Proof. The equality statement in Lemma 3.1 automatically proves the first argument. If $A(q) - \frac{q'}{q}$ is an integer, then $\rho_{p/q} = 1$, and the smallest admissible cell gives

$$q_{n+1} = \frac{1}{q\Psi(q)} - q.$$

Therefore no $q_{n+1} \geq \frac{1}{q\Psi(q)} - (1-c)q$ with any fixed $c > 0$ follows from the local intersection alone. $\square$

Remark 3.4 (Scope of optimality). Proposition 3.3 is a statement of local lemma instead of a global construction. It does not claim that every branch of the Baker–Ward construction realizes phase 1 infinitely often. Proving that the construction can systematically avoid bad phases would be a new selection theorem. The condition claimed this paper is only sufficient but not necessary.

4 Strengthened version of Baker–Ward Lemma 4.4

For a fixed local phase $0 < \rho \leq 1$, define

$$\kappa_\rho(q) := 1 - \rho q^2 \Psi(q), \quad T_\rho(q) := \frac{\kappa_\rho(q)}{q\Psi(q)} = \frac{1}{q\Psi(q)} - \rho q. \quad (26)$$

And given a fixed constant $\Gamma > 6$

$$R_{\rho,\Gamma}(q) := \Gamma \kappa_\rho(q)^2 \frac{\Psi(T_\rho(q))}{\Psi(q)}. \quad (27)$$

Proposition 4.1 (Local rounding offset separation). *Assume that $t \mapsto t^2 \Psi(t)$ is eventually non-increasing and $t^2 \Psi(t) \rightarrow 0$. Fix an admissible convergent p/q such that*

$$q^2 \Psi(q) < 1,$$

and put $\rho = \rho_{p/q}$. Suppose that

$$\omega(q) \geq \frac{R_{\rho,\Gamma}(q)}{1 + R_{\rho,\Gamma}(q)}. \quad (28)$$

Then every b_0 satisfying (15) and every possibly empty but finite word $\mathbf{b}$ satisfy

$$\mathcal{L}(S(p/q)) \geq K_\Gamma \mathcal{L}(B(p/q, b_0, \mathbf{b})). \quad (29)$$

Proof. Since $0 < \rho \leq 1$ and $0 < q^2 \Psi(q) < 1$,

$$\kappa_\rho(q) = 1 - \rho q^2 \Psi(q) \geq 1 - q^2 \Psi(q) > 0.$$

Since $q\Psi(q) > 0$,

$$T_\rho(q) = \frac{\kappa_\rho(q)}{q\Psi(q)} > 0. \quad (30)$$

This ensures $T_\rho(q)$ lies in the domain of Ψ .

For every b_0 satisfying (15), Lemma 3.1 gives

$$q_{n+1} \geq T_\rho(q). \quad (31)$$

Since both sides are positive and the function $t \mapsto t^2\Psi(t)$ is non-increasing, we obtain

$$q_{n+1}^2\Psi(q_{n+1}) \leq T_\rho(q)^2\Psi(T_\rho(q)).$$

Therefore, by Lemma 2.2,

$$\mathcal{L}(B(p/q, b_0, \mathbf{b})) \leq 3T_\rho(q)^2\Psi(T_\rho(q)). \quad (32)$$

Since $0 < \omega(q) < 1$, condition (28) can be rewritten into

$$\frac{\omega(q)}{1 - \omega(q)} \geq R_{\rho, \Gamma}(q).$$

Using the exact equality in (14), rather than its rough lower bound, we can obtain

$$\begin{aligned} \mathcal{L}(S(p/q)) &= \frac{1}{q^2\Psi(q)} \frac{\omega(q)}{1 - \omega(q)} \\ &\geq \frac{\Gamma\kappa_\rho(q)^2}{q^2\Psi(q)^2} \Psi(T_\rho(q)) \\ &= \Gamma \cdot T_\rho(q)^2\Psi(T_\rho(q)) \\ &\geq K_\Gamma \mathcal{L}(B(p/q, b_0, \mathbf{b})). \end{aligned}$$

This proves (29). $\square$

This proposition is entirely local since $\rho_{p/q}$ depends on the current convergents. This means even when case $\rho_{p/q} = 1$ is sharp locally, it may not produce an optimal global result.

Proposition 4.2 (Global/Uniform separation). *Assume (3) and (6). Then, for every sufficiently large admissible denominator q , every b_0 satisfying (15), a fixed constant $\Gamma > 6$, and every possibly empty word $\mathbf{b}$,*

$$\mathcal{L}(S(p/q)) \geq K_\Gamma \mathcal{L}(B(p/q, b_0, \mathbf{b})). \quad (33)$$

Proof. Because $q^2\Psi(q) \rightarrow 0$, we have $0 < \kappa(q) < 1$ and $T(q)/q = 1/(q^2\Psi(q)) - 1 \rightarrow \infty$. Hence both $T(q) > 0$ and $T(q)$ lie beyond the eventual monotonicity threshold for all sufficiently large q . By Corollary 3.2, $q_{n+1} \geq T(q)$. The eventual monotonicity of $t \mapsto t^2\Psi(t)$ therefore gives

$$\mathcal{L}(B(p/q, b_0, \mathbf{b})) \leq 3T(q)^2\Psi(T(q)). \quad (34)$$

The hypothesis (6) gives $\frac{\omega(q)}{1 - \omega(q)} \geq R_\Gamma(q)$. Hence the exact formula (14) yields

$$\mathcal{L}(S(p/q)) \geq \frac{\Gamma\kappa(q)^2}{q^2\Psi(q)^2} \Psi(T(q)) = \Gamma T(q)^2\Psi(T(q)).$$

Comparison with (34) proves (33). $\square$

Remark 4.3. The bad-interval constant is sharpened to 3, while the separation constant defined is

$$K_\Gamma = \frac{\Gamma}{3} > 2.$$

The limiting geometric loss 2 comes from the possibility that only one half of $S(p/q)$ lies in the relevant fundamental interval. Thus the prefactor barrier we gained is $3 \cdot 2 = 6$ as a limitation of our method. The endpoint $\Gamma = 6$ is not obtained.

5 Transferring refined estimate through the old construction

This section is aimed to show that the refined local estimate is admissible in the framework of Baker-Ward's paper. We revisit their analysis only to identify the precise use of their Lemma 4.4 and to verify our estimate (33) can completely replace their original hypothesis.

For a finite prefix $\mathbf{a} = (b_0; b_1, \dots, b_m)$, write

$$I(\mathbf{a}) := I_m(b_0; b_1, \dots, b_m), \quad I(\mathbf{a}, c) := I_{m+1}(b_0; b_1, \dots, b_m, c),$$

and define

$$B(\mathbf{a}) := B(p/q, b_0, (b_1, \dots, b_m)), \quad C(\mathbf{a}) := C(b_1, \dots, b_m), \quad A(\mathbf{a}) := C(\mathbf{a}) - 1.$$

The cylinders $I(\mathbf{a}, c)$ is consecutive contained inside $I(\mathbf{a})$. As $c \rightarrow \infty$, they approach one of the two endpoints; call it the *bad endpoint*. Thus $B(\mathbf{a})$ also strictly approaches the bad endpoint and $I(\mathbf{a}, A(\mathbf{a}))$ is the adjacent safe cylinder. If $P = I(\mathbf{a})$, we also define $B(P) = B(\mathbf{a})$ and $I(P, c) = I(\mathbf{a}, c)$.

Lemma 5.1 (Selection at a bad endpoint). *Assume (3) and (6), fix $\Gamma > 6$, write $S = S(p/q)$, and let $\mathbf{a}$ be any relevant finite prefix. Suppose that $H \subseteq I(\mathbf{a})$ be a closed interval that satisfies*

$$\mathcal{L}(H) \geq \frac{1}{2}\mathcal{L}(S) \quad \text{and} \quad H \cap B(\mathbf{a}) \neq \emptyset.$$

Then, for all sufficiently large and admissible q , there are consecutive integers $M, \dots, A(\mathbf{a})$ such that

$$U := \bigcup_{c=M}^{A(\mathbf{a})} I(\mathbf{a}, c) \subseteq H$$

and

$$\mathcal{L}(U) \geq c_\Gamma \mathcal{L}(S), \quad c_\Gamma = \frac{K_\Gamma - 2}{10K_\Gamma}.$$

All newly selected digits satisfy the required partial-quotient bound, which means they are indeed admissible.

Proof. Put $B = B(\mathbf{a})$ and $A = A(\mathbf{a})$. By Proposition 4.2,

$$\mathcal{L}(B) \leq \frac{1}{K_\Gamma} \mathcal{L}(S).$$

holds Because B is at one side of the parent interval and H meets B ,

$$J := \overline{H \setminus B}$$

is on the safe side of B , adjacent to B . Moreover,

$$\begin{aligned} \mathcal{L}(J) &\geq \mathcal{L}(H) - \mathcal{L}(B) \\ &\geq \left(\frac{1}{2} - \frac{1}{K_\Gamma} \right) \mathcal{L}(S) \\ &\geq \left(\frac{K_\Gamma}{2} - 1 \right) \mathcal{L}(B). \end{aligned}$$

By Lemma 2.3,

$$\mathcal{L}(I(\mathbf{a}, A)) \leq \left(\frac{K_\Gamma}{2} - 1 \right) \mathcal{L}(B) \leq \mathcal{L}(J).$$

Since $I(\mathbf{a}, A)$ and J lie on the same side of, and are both adjacent to, B , the length comparison gives

$$I(\mathbf{a}, A) \subseteq J \subseteq H.$$

Let $M \in \{1, \dots, A\}$ be the smallest integer such that

$$U := \bigcup_{c=M}^A I(\mathbf{a}, c) \subseteq H.$$

If $M = 1$, then $J \subseteq U$, so $\mathcal{L}(U) \geq \mathcal{L}(J)$. If $M \geq 2$, minimality of M gives

$$J \subseteq I(\mathbf{a}, M-1) \cup U.$$

The adjacent-cylinder comparison argument in [1, Lemma 2.9] gives

$$\mathcal{L}(I(\mathbf{a}, M-1)) \leq 4\mathcal{L}(I(\mathbf{a}, M)) \leq 4\mathcal{L}(U).$$

Thus in both cases $5\mathcal{L}(U) \geq \mathcal{L}(J)$. This means:

$$\mathcal{L}(U) \geq \frac{1}{5} \left(\frac{1}{2} - \frac{1}{K_\Gamma} \right) \mathcal{L}(S) = \frac{K_\Gamma - 2}{10K_\Gamma} \mathcal{L}(S) = c_\Gamma \mathcal{L}(S).$$

Finally, every selected digit satisfies

$$c \leq A(\mathbf{a}) < \frac{1}{3D^2\Psi(D)},$$

where D is the denominator for $\mathbf{a}$, which makes every newly selected digits satisfy the required condition. $\square$

Lemma 5.2. *Assume the same hypotheses of Theorem 1.1. More, for every sufficiently large q with the corresponding admissible p/q , there exists a finite collection of words $\{\mathbf{b}_l\}_{l=1}^L$ such that*

$$\bigcup_{l=1}^L I(\mathbf{b}_l) \subseteq S(p/q)$$

and

$$\mathcal{L}\left(\bigcup_{l=1}^L I(\mathbf{b}_l)\right) \geq c_\Gamma \mathcal{L}(S(p/q)), \quad c_\Gamma = \frac{\Gamma - 6}{10\Gamma}.$$

Moreover, the same intermediate partial-quotient bounds requirement in Baker–Ward’s Lemma 4.5 hold.

Proof. Write $S = S(p/q)$. Proposition 4.2 gives

$$\mathcal{L}(S) \geq K_\Gamma \mathcal{L}(B(p/q, b_0, \mathbf{b}))$$

for every relevant prefix, where

$$K_\Gamma = \frac{\Gamma}{3} > 2.$$

We follow the similar case division strategy of Baker–Ward’s Lemma 4.5.

Case 1. Suppose $I_0(b_0) \subset S$ for some b_0 . Baker and Ward take all consecutive level-zero intervals contained in S . Their union has length at least $\frac{\mathcal{L}(S)}{3}$, which is larger than $c_\Gamma \cdot \mathcal{L}(S)$, $\forall \Gamma > 6$.

Case 2a. Suppose that S intersects both adjacent intervals $I_0(b_0)$ and $I_0(b_0 + 1)$ but contains neither, and

$$\mathcal{L}(S \cap I_0(b_0)) \geq \frac{\mathcal{L}(S)}{2}. \quad (35)$$

This is equation (4.16) of Baker–Ward. If $I_1(b_0; 1) \subset S \cap I_0(b_0)$, then

$$\frac{\mathcal{L}(S)}{2} \leq \mathcal{L}(S \cap I_0(b_0)) < \mathcal{L}(I_0(b_0)) = 2\mathcal{L}(I_1(b_0; 1)).$$

Thus the single interval $I_1(b_0; 1)$ has length at least $\mathcal{L}(S)/4 > c_\Gamma \mathcal{L}(S)$.

Otherwise,

$$I_1(b_0; 1) \not\subset S \cap I_0(b_0).$$

Both $S \cap I_0(b_0)$ and $I_1(b_0; 1)$ are intervals reaching the same common endpoint of $I_0(b_0)$ and $I_0(b_0 + 1)$. Their ordering gives

$$S \cap I_0(b_0) \subseteq I_1(b_0; 1).$$

Set

$$J := S \cap I_0(b_0) = S \cap I_1(b_0; 1).$$

Then $\mathcal{L}(J) \geq \mathcal{L}(S)/2$. Its endpoint at $b_0 + 1$ is the bad endpoint of $I_1(b_0; 1)$, so

$$J \cap B(p/q, b_0, (1)) \neq \emptyset.$$

Lemma 5.1, applied to the prefix $(b_0; 1)$ and the interval J , supplies the required consecutive collection inside S . This completes the discussion of Case 2a.

Case 2b. Suppose equation (4.17) of Baker–Ward holds:

$$J := S \cap I_0(b_0 + 1), \quad \mathcal{L}(J) \geq \frac{\mathcal{L}(S)}{2}.$$

The common endpoint $b_0 + 1$ is the bad endpoint of $I_0(b_0 + 1)$, and J reaches this endpoint too. Hence, with

$$B_1 := B(p/q, b_0 + 1, \emptyset),$$

we have $J \cap B_1 \neq \emptyset$. Lemma 5.1, applied to the prefix $(b_0 + 1)$ and the interval J , gives the required consecutive collection.

Indeed, although they are in reverse order, the lemma did not use any condition relevant to the ordering of the children interval; it does not matter whether they are ordered from left to right or from right to left, and only adjacency is used.

Case 3a. Suppose that

$$S \subset I_0(b_0) \quad \text{and} \quad S \cap B(p/q, b_0, \emptyset) \neq \emptyset.$$

Lemma 5.1, applied with $H = S$ and prefix (b_0) , gives the required collection. In this case the proof of that lemma gives the stronger estimate

$$\mathcal{L}(U) \geq \frac{K_\Gamma - 1}{5K_\Gamma} \mathcal{L}(S) \geq c_\Gamma \mathcal{L}(S).$$

Case 3b. Suppose S is disjoint from $B(p/q, b_0, \emptyset)$ and contains at least one complete level-one fundamental interval. Take all consecutive level-one intervals completely contained in S , and take their union as U .

Since S is disjoint from all the bad tail, every selected child is below the cutoff and is therefore considered safe. The set $S \setminus U$ has at most two boundary pieces. Again, by the adjacent-cylinder comparison in [1, Lemma 2.9], each boundary piece has length at most four times that of the adjacent selected cylinder. Therefore

$$\mathcal{L}(S) \leq \mathcal{L}(U) + 4\mathcal{L}(U) + 4\mathcal{L}(U) = 9\mathcal{L}(U).$$

Thus

$$\mathcal{L}(U) \geq \frac{\mathcal{L}(S)}{9} > \frac{1}{10}\mathcal{L}(S) > c_{\Gamma}\mathcal{L}(S).$$

Case 3c. Suppose that S avoids $B(p/q, b_0, \emptyset)$, contains no complete level-one interval, and intersects with two adjacent intervals $I_1(b_0; b_1)$ and $I_1(b_0; b_1 + 1)$. Let P be the one for which

$$H := S \cap P, \quad \mathcal{L}(H) \geq \frac{1}{2}\mathcal{L}(S).$$

The interval H has the common boundary between the two level-one cylinders as an endpoint. Since S avoids the level-zero bad tail, the final digit that defines P is below its cutoff and satisfies the required bound.

First assume that $H \cap B(P) \neq \emptyset$. Then Lemma 5.1 already gives the required union, regardless what P looks like.

Second, if we assume $H \cap B(P) = \emptyset$, the common boundary will be the non-bad endpoint of P . Indeed, if the two endpoint convergents defining P have numerators and denominators P_m, Q_m and P_{m-1}, Q_{m-1} respectively, then this endpoint is

$$\frac{P_m + P_{m-1}}{Q_m + Q_{m-1}}.$$

This is exactly the children of those two endpoints in the Farey tree, and therefore, appending digit 1 produces exactly this convergent. Hence the child cylinder $I(P, 1)$ is adjacent to the non-bad endpoint of P , and the same point is the bad endpoint of $I(P, 1)$.

If H contains at least one complete child of P , let U be the union of all such children. They are consecutive and safe because H avoids $B(P)$. There is at most one boundary piece of H that is not covered, and [1, Lemma 2.9] gives

$$\mathcal{L}(H) \leq 5\mathcal{L}(U).$$

Consequently,

$$\mathcal{L}(U) \geq \frac{1}{5}\mathcal{L}(H) \geq \frac{1}{10}\mathcal{L}(S) > c_{\Gamma}\mathcal{L}(S).$$

If H contains no complete child of P , then

$$H \subseteq I(P, 1).$$

For sufficiently large and admissible q , the digit 1 is the smallest and satisfies the cutoff bound. Moreover, H meets the bad endpoint, and hence determined as the bad tail, of $I(P, 1)$. Lemma 5.1, now applied one level deeper to the sequence of prefix defining $I(P, 1)$, again gives the required consecutive union. Both left and right orientation are considered and treated. Thus, case 3c is all complete now.

Case 3d. By far, level zero and level one interval are all discussed. Now, suppose that S is contained in some level of an admissible nested fundamental interval. Whenever

$$S \cap B(p/q, b_0, b_1, \dots, b_j) = \emptyset$$

and

$$S \subset I_{j+1}(b_0; b_1, \dots, b_{j+1}),$$

is satisfied. Here, the next digit b_{j+1} lies below the cutoff in (4.9), and hence satisfies (4.15).

This iteration must terminate and can terminate. Indeed, Lemma 2.7 and the denominator recurrence (2.1) imply

$$\mathcal{L}(I_m) \leq \frac{1}{q_m^2} \rightarrow 0,$$

whereas interval S has a fixed positive length. Therefore S cannot remain inside one fundamental interval at every depth forever.

At the first stage that terminates, exactly one of Cases 3a, 3b, or 3c can be applied. This can be understood as in the analogues of Cases 3a and 3c, The replacement input c_r is applied to the current finite prefix word; the analogue of Case 3b is geometric and can be applied at any level. Thus, the preceding calculations can produce the required consecutive collection with every previously fixed digit satisfies (4.15).

Thus every case of Baker–Ward’s Lemma 4.5 is available only using the separation conclusion of Lemma 4.4. Replacing that conclusion by (33) preserves both (4.14) and (4.15). $\square$

Remark 5.3. In the terminal event (3′) on p. 24 of [1], the notation $I_2(b_0; \mathbf{b}, b)$ appears although $\mathbf{b}$ has arbitrary length m . We interpret this as $I_{m+1}(b_0; \mathbf{b}, b)$. Since it is in level $m + 1$, which is consistent with the preceding recursive construction.

5.1 From local to global

For a finite word $\mathbf{c}$, let $I(\mathbf{c})$ denote the corresponding continued fraction fundamental interval. If $\mathbf{a}$ is a fixed prefix, then $I(0; \mathbf{a}, \mathbf{c})$ denotes the cylinder constructed by appending $\mathbf{c}$ to $\mathbf{a}$. Here, $\mathbf{a}$ is a finite word and cannot be irrational.

The following construction in the following lemma does not need the proof of lemma 5.2, since it only concerns in a single admissible cylinder inside $S(p/q)$ that touches the bad interval. It does not assume (6), and the local formulation can be used in rational exponential case.

Lemma 5.4 (Extensions). *Assume (3), and fix $K > 2$. Let*

$$\frac{p}{q} = [0; a_1, \dots, a_n], \quad n \text{ even},$$

with the admissible q sufficiently large, and write $S = S(p/q)$. Suppose that, for every relevant b_0 and finite word $\mathbf{u}$,

$$\overline{B(p/q, b_0, \mathbf{u})} \cap S \neq \emptyset \implies \mathcal{L}(S) \geq K \mathcal{L}(B(p/q, b_0, \mathbf{u})). \quad (36)$$

Then there is a finite word $\mathbf{b} = (b_0, \dots, b_{m-1})$, $m \geq 1$, such that

$$I(0; a_1, \dots, a_n, \mathbf{b}) \subseteq \left\{ x : (1 - \omega(q))\Psi(q) \leq x - \frac{p}{q} \leq \Psi(q) \right\}. \quad (37)$$

Moreover,

$$b_i \leq \left\lfloor \frac{1}{3q_{n+i}^2 \Psi(q_{n+i})} \right\rfloor, \quad 1 \leq i \leq m-1. \quad (38)$$

Here q_{n+i} implies the denominator after appending $b_0, \dots, b_{i-1}$ to the previous word. Here, when we consider cylinders and bad tails, we consider their endpoints at both sides too, or in other words, consider their closures. This does not change their lengths; Our argument of the restrictions of digit concern irrational continuations and endpoints are rational.

Proof. For a tail prefix $\mathbf{u}$, let D be its full denominator and define

$$\theta_D := \frac{1}{3D^2\Psi(D)}, \quad C_D := \lceil \theta_D \rceil, \quad A_D := C_D - 1.$$

For any relevant prefixes, $D \geq q_{n+1} \geq T(q) \rightarrow \infty$. Thus $A_D \rightarrow \infty$ uniformly, and hence for sufficiently large n , correspondingly sufficiently large q_{n+1}, D , the condition

$$A_D \geq 1, \quad \frac{1}{A_D} \leq \frac{K}{2} - 1. \quad (39)$$

exists. The child interval $I(\mathbf{u}, c)$ occur consecutively in the interval $P := I(\mathbf{u})$. The bad tail $B(P)$ is terminal at one of the endpoint approached since then $c \rightarrow \infty$. Its adjacent safe child is $I(\mathbf{u}, A_D)$. The telescoping formula, after calculation, gives

$$\frac{\mathcal{L}(I(\mathbf{u}, A_D))}{\mathcal{L}(B(P))} = \frac{Q_m}{A_D Q_m + Q_{m-1}} \leq \frac{1}{A_D}.$$

At one selection step: Suppose that a closed interval $H \subseteq S \cap P$ satisfies $\mathcal{L}(H) \geq \mathcal{L}(S)/2$ and intersects with $B(P)$. By (36), $\mathcal{L}(B(P)) \leq \mathcal{L}(S)/K < \mathcal{L}(H)$. Since the bad tail is terminal at one end, the interval

$$J := \overline{H \setminus B(P)}$$

lies on its safe side and shares its boundary with the adjacent safe child, which is also a boundary of the bad interval. Furthermore,

$$\mathcal{L}(J) \geq \mathcal{L}(H) - \mathcal{L}(B(P)) \geq \left(\frac{K}{2} - 1\right) \mathcal{L}(B(P)) \geq \mathcal{L}(I(\mathbf{u}, A_D)).$$

The adjacency in same direction therefore gives $I(\mathbf{u}, A_D) \subseteq J \subseteq H$. The newly selected digit is safe because $A_D < \theta_D$.

Next consider the endpoint step: suppose that a closed interval $H \subseteq S \cap P$, with $\mathcal{L}(H) \geq \mathcal{L}(S)/2$, has one of the endpoint of P as one of its endpoints. If that is the bad endpoint, the first step that is already demonstrated applies. If it is the other endpoint, let $P_1 = I(\mathbf{u}, 1)$. This child reaches the same endpoint, which is the bad endpoint for itself: if the consecutive tail convergents are P_m/Q_m and P_{m-1}/Q_{m-1} , appending 1 gives $(P_m + P_{m-1})/(Q_m + Q_{m-1})$. If $P_1 \subseteq H$, select P_1 and we are done. Otherwise $H \subseteq P_1$, and apply the preceding selection step in P_1 at next level, with the same interval H . The extra digit 1 is the most imprecise partial quotient, and is safe by (39). This treats both directions.

We say that an interval crosses two adjacent cells when it has positive-length intersection with the interior of each cell, only endpoints do not count. We now needs to select a cylinder in S . If S already contains a complete unit cell, select that one. If S crosses two adjacent unit cells and contains neither, take the intersection of greater length and apply the previous endpoint step. Otherwise S is contained in one unit cell at some determined level; call the current parent P . We now proceed recursively, maintaining $S \subseteq P$.

At each stage, first check whether S intersects $B(P)$. If it does, apply the first selection step with $H = S$. If it does not, every child intersecting S is safe. If a complete child is contained in S , select that one. If S crosses two adjacent children and contains neither, take its intersection of greater length with one child and apply the second endpoint step there. In the remaining case, S is contained in a single safe child; replace P by this child and continue the iteration.

This indeed terminates: by the estimation of interval length,

$$\mathcal{L}(I_n) < \frac{1}{q_n^2} \rightarrow 0$$

Taken the interval S has a fixed positive length, it cannot be contained in a cylinder at infinite level. Every digit appended is safe because the corresponding parent bad tail was avoided. The endpoint steps also append only safe digits. Hence we obtain a finite tail cylinder in S , with the bounds (38). The tail-coordinate identity (11) gives (37). $\square$

5.2 Proof of main theorem

Proof of Theorem 1.1 of this paper. We use the construction in [1, Section 5.1, pp. 36–38]. Proposition 4.2 verifies (36) with $K = K_\Gamma > 2$. We may therefore apply Lemma 5.4 at each selected convergent below.

Fix a rule for choosing the word constructed by Lemma 5.4; For example, whenever it gives more than one possible continuation, choose the one with least length; if several such words remain, choose the lexicographically first one, which makes it unique. Thus every finite prefix can determine one unique continuation.

Let

$$\boldsymbol{\delta} = (\delta_1, \delta_2, \dots) \in \{1, 2\}^{\mathbb{N}}.$$

Choose an even integer N_0 sufficiently large and let

$$\mathbf{a} = (1, \dots, 1) \in \mathbb{N}^{N_0}.$$

Set $\mathbf{c}_0 := \mathbf{a}$. We construct an infinite continued fraction depending on $\boldsymbol{\delta}$.

Suppose that a finite word $\mathbf{c}_{j-1}$ of even length N_{j-1} has been constructed. Denote

$$\frac{p_{N_{j-1}}}{q_{N_{j-1}}} = [0; \mathbf{c}_{j-1}].$$

where $\mathbf{c}_{j-1}$ is a word.

Apply Lemma 5.4 and let

$$\mathbf{b}_j = (b_{j,0}, \dots, b_{j,m_j-1})$$

be the word selected by the rule. We then observe

$$I(0; \mathbf{c}_{j-1}, \mathbf{b}_j) \subset \left\{ x : (1 - \omega(q_{N_{j-1}}))\Psi(q_{N_{j-1}}) \leq x - \frac{p_{N_{j-1}}}{q_{N_{j-1}}} \leq \Psi(q_{N_{j-1}}) \right\}. \quad (40)$$

If $m_j \geq 2$, its intermediate digits satisfy

$$b_{j,i} \leq \left\lfloor \frac{1}{3q_{N_{j-1}+i}^2 \Psi(q_{N_{j-1}+i})} \right\rfloor, \quad 1 \leq i \leq m_j - 1. \quad (41)$$

Choose $\ell_j \in \{1, 2\}$ so that

$$N_j := N_{j-1} + m_j + \ell_j$$

is even, and define

$$\mathbf{c}_j := (\mathbf{c}_{j-1}, \mathbf{b}_j, \delta_j^{\ell_j}),$$

where $\delta_j^{\ell_j}$ means there are ℓ_j consecutive δ_j s. Since the new selected index N_j is again even, the construction can be repeated indefinitely.

Define

$$F(\boldsymbol{\delta}) = [0; \mathbf{a}, \mathbf{b}_1, \delta_1^{\ell_1}, \mathbf{b}_2, \delta_2^{\ell_2}, \dots].$$

We now verify that $F(\boldsymbol{\delta}) \in E(\Psi, \omega)$.

First, at every selected index N_{j-1} , (40) gives

$$0 < F(\boldsymbol{\delta}) - \frac{p_{N_{j-1}}}{q_{N_{j-1}}} \leq \Psi(q_{N_{j-1}}).$$

Since the selected denominators tend to approach infinity, these are infinitely many Ψ -level approximations, so clearly

$$F(\boldsymbol{\delta}) \in W(\Psi).$$

Next, denote

$$x = F(\delta) = [0; a_1, a_2, \dots]$$

and let $\frac{p_n}{q_n}$ be its n -th convergent. If $n = N_{j-1}$ is selected, by (40),

$$\left| x - \frac{p_n}{q_n} \right| \geq (1 - \omega(q_n))\Psi(q_n).$$

Suppose that n is not a selected index. The next partial quotient a_{n+1} is then either a coding digit in $\{1, 2\}$, or an intermediate digit controlled by (41).

If it is a coding digit in 1, 2, Baker–Ward’s Lemma 2.2 gives

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{3a_{n+1}q_n^2} \geq \frac{1}{6q_n^2}.$$

Since $q^2\Psi(q) \rightarrow 0$,

$$\frac{1}{6q_n^2} > \Psi(q_n)$$

for all sufficiently large n .

If it is an intermediate digit controlled by (41),

$$a_{n+1} \leq \left\lfloor \frac{1}{3q_n^2\Psi(q_n)} \right\rfloor,$$

and Lemma 2.2 tells us

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{3a_{n+1}q_n^2} \geq \Psi(q_n).$$

Thus, in both cases,

$$\left| x - \frac{p_n}{q_n} \right| \geq (1 - \omega(q_n))\Psi(q_n)$$

for every sufficiently large n .

Therefore the two continued-fraction conditions in Baker–Ward’s Proposition 2.5 and Remark 2.6 [1, pp. 12–13] holds. That is why:

$$F(\delta) \in E(\Psi, \omega).$$

Now we shall prove injectivity from $\{1, 2\}^{\mathbb{N}}$ to $E(\Psi, \omega)$. Let $\delta \neq \delta'$, and let j be the index of which they first differ. This means before the j -th coding block, the two constructions have the exact same prefix. The selection rule therefore gives the same word $\mathbf{b}_j$ and the same exponent ℓ_j .

At the next digit, one continued fraction contains δ_j and the other contains δ'_j which are different. Hence their infinite continued-fraction expansions are different, and this means

$$F(\delta) \neq F(\delta').$$

Thus $F : \{1, 2\}^{\mathbb{N}} \rightarrow E(\Psi, \omega)$ is injective.

It remains to verify the stronger topological conclusion that the image of F is a Cantor set.

Consider $F(\delta)$ as a sequence with both extension sequence, or the selected word and coding sequence, or the sequence with all members belongs to $\{1, 2\}$. If the first k entries of coding sequence is the same, the first k prefixes are the same. In Each stage, there is at least extension word, including at least one extension digit, and one coding digit too at elast that are added, so the length of the same prefixes is at least $N_0 + 2k$. A fundamental interval with denominator q_n has length at most q_n^{-2} ; since all partial quotients are positive, q_n is at least the n -th Fibonacci number F_n , with equation holding if and only if all partial quotients are 1. Therefore

$$|F(\delta) - F(\delta')| \leq F_{N_0+2k}^{-2} \rightarrow 0$$

uniformly as $k \rightarrow \infty$. This means F is continuous in the product topology.

Cantor space $\{1, 2\}^{\mathbb{N}}$ is compact and $\mathbb{R}$ is Hausdorff, so a continuous injection we proved to be true is a homeomorphism onto its image. Hence $F(\{1, 2\}^{\mathbb{N}})$ is a Cantor subset of $E(\Psi, \omega)$. Since $E(\Psi, \omega) \subseteq \mathbb{R}$, the reverse inequality also holds. Thus

$$|E(\Psi, \omega)| = 2^{\aleph_0}.$$

□

5.3 Comparison with the Baker–Ward argument

This subsection is aimed to show and summarize the division between imported ideas from Baker–Ward’s paper and sharpened or new ideas.

Stage	Imported from Baker–Ward	New results here
Bad interval estimation	The tail coordinate correspondence with the targeted shell in Lemma 4.2, the definition of the bad intervals in (4.9), and the estimate	Lemma 3.1 refines local rounding offset denominator estimate, while Lemma 2.2 sharpens the bound of coefficient bound from 9 to 3.
	$\mathcal{L}(B) \leq 9q_{n+1}^2 \Psi(q_{n+1})$	
Separation estimate	from (4.11) They yield	In Proposition 4.2, we argue any fixed constant $\Gamma > 6$, as long as $\mathcal{L}(S) \geq K_{\Gamma} \mathcal{L}(B)$, the cantor set can be constructed.
	$\mathcal{L}(S) \geq 20\mathcal{L}(B)$	
Selection inside S	in their own paper The ordering and length comparison of fundamental intervals, including Lemmas 2.7 and 2.9, and the case division structure of Lemma 4.5	Lemma 5.2 follows the same case division as Baker–Ward’s Lemma 4.5. Cases 1 and 3b use only the geometric nature of fundamental intervals. In Cases 2a, 2b, 3a, and 3c, sharpened condition
		$\mathcal{L}(S) \geq K_{\Gamma} \mathcal{L}(B).$
Finite admissible extension	Lemma 4.2 identifies $S(p/q)$ with the set of tails for which	is obtained. 3d eventually reduces to one of Cases 3a–3c. The retained length ratio is therefore c_{Γ} instead of $\frac{9}{100}$. Lemma 5.4 directly constructs one admissible cylinder inside S , giving separation for bad intervals that meet S . In this case, target shell condition and intermediate partial quotient bounds are met. No positive-proportion estimation is required in this lemma. For Lemma 5.2, we investigated the cases and find that for any fixed constant $K_{\Gamma} = \Gamma/3 > 2$, with retained proportion $c_{\Gamma} = \frac{\Gamma-6}{10\Gamma}$, is admissible. Ratio goes back to the original 9/100 when Γ reaches 60. The Cantor constructions below use Lemma 5.4.
	$(1 - \omega(q))\Psi(q) \leq x - \frac{p}{q} \leq \Psi(q).$	
Infinite construction	The binary coding strategy of [1, Section 5.1, pp. 36–38], together with Lemma 2.2, Proposition 2.5, and Remark 2.6	The proof of Theorem 1.1 writes out the induction and verifies separately that the resulting point lies in $W(\Psi)$ and depends injectively on the binary coding sequence. The uniform global situation confirms the existence of Cantor-space.

The exact adjacent-cutoff comparison permits the separation constant $K_\Gamma = \Gamma/3$ to be any fixed number greater than 2. The Baker–Ward case analysis then retains a consecutive collection occupying at least

$$c_\Gamma = \frac{\Gamma - 6}{10\Gamma}$$

of the target interval. When $\Gamma = 60$, this recovers the original proportion 9/100.

6 The exponential case

Proof of Theorem 1.2. Let $\Psi(q) = \lambda^{-q}$. Then

$$\kappa(q) = 1 - q^2 \lambda^{-q}, \quad T(q) = \frac{\lambda^q}{q} - q.$$

And,

$$\frac{\Psi(T(q))}{\Psi(q)} = \frac{\lambda^{-T(q)}}{\lambda^{-q}} = \lambda^{q-T(q)} = \lambda^{2q-\lambda^q/q}.$$

Since $R_\Gamma(q) = R_{\lambda,\Gamma}(q)$, the general condition (6) is exactly (7). The function $q \mapsto q^2 \lambda^{-q}$ tends to zero and is decreasing for all q that are large enough. Thus Theorem 1.1 applies. $\square$

Proof of Corollary 1.3. Put $X_q = \frac{\lambda^q}{q}$. Since $\frac{q}{X_q} = \frac{q^2}{\lambda^q} \rightarrow 0$, for every fixed $0 < \varepsilon < 1$ we have $2q \leq \varepsilon X_q$ for q large enough. Therefore

$$2q - X_q \leq -(1 - \varepsilon)X_q.$$

Also $(1 - q^2 \lambda^{-q})^2 \leq 1$ for sufficiently large q . It follows that

$$R_{\lambda,\Gamma}(q) \leq \Gamma \lambda^{-(1-\varepsilon)X_q}.$$

Since $\frac{R_{\lambda,\Gamma}(q)}{1+R_{\lambda,\Gamma}(q)} \leq R_{\lambda,\Gamma}(q)$, (8) can imply (7), and Theorem 1.2 applies. $\square$

6.1 Comparison with the Baker–Ward

The logarithms to base λ in exponential case of the two sufficient thresholds are

$$\log_\lambda \left(45 \lambda^{q-\lambda^q/(2q)} \right) = q - \frac{\lambda^q}{2q} + O(1),$$

$$\log_\lambda \left(\frac{R_{\lambda,\Gamma}(q)}{1+R_{\lambda,\Gamma}(q)} \right) = 2q - \frac{\lambda^q}{q} + O(1).$$

This means the coefficient of the dominant term $-\lambda^q/q$ improves from $1/2$ to 1 . This is an improvement in the double-exponential scale, not merely in the front constant.

6.2 proof and illustration of Corollary 1.4

Proof of Corollary 1.4. Let $\Psi(q) = q^{-\tau}$, where $\tau > 2$. Then

$$q^2 \Psi(q) = q^{2-\tau} \rightarrow 0,$$

and $q \mapsto q^2 \Psi(q)$ is eventually non-increasing. Moreover,

$$\kappa(q) = 1 - q^{2-\tau}$$

and

$$T(q) = \frac{1}{q \Psi(q)} - q = q^{\tau-1} - q = q^{\tau-1} (1 - q^{2-\tau}).$$

Therefore

$$\begin{aligned}\frac{\Psi(T(q))}{\Psi(q)} &= \left(\frac{q}{T(q)}\right)^\tau \\ &= q^{-\tau(\tau-2)}(1 - q^{2-\tau})^{-\tau}.\end{aligned}$$

It follows that

$$R_\Gamma(q) = \Gamma(1 - q^{2-\tau})^{2-\tau} q^{-\tau(\tau-2)}.$$

Choose any Γ such that

$$6 < \Gamma < C.$$

Since

$$(1 - q^{2-\tau})^{2-\tau} \rightarrow 1,$$

for all sufficiently large q ,

$$R_\Gamma(q) \leq C q^{-\tau(\tau-2)} \leq \omega(q).$$

Hence

$$\frac{R_\Gamma(q)}{1 + R_\Gamma(q)} \leq R_\Gamma(q) \leq \omega(q).$$

Theorem 1.1 therefore applies, so $E(\Psi, \omega)$ contains a Cantor set and has cardinality continuum. $\square$

Remark 6.1. For $\Psi(q) = q^{-\tau}$, the Baker–Ward condition (1) becomes

$$\omega(q) \geq 45 \cdot 2^\tau q^{-\tau(\tau-2)}.$$

Thus Corollary 1.4 replaces the coefficient $45 \cdot 2^\tau$ obtained from their condition by any fixed $C > 6$ even if in power-low cases the decay exponent stays the same. This means in this case, the coefficient condition is exponentially improving. No claim of optimality is made.

Remark 6.2 (Quick note on Hausdorff dimension). Let $\Psi(q) = \lambda^{-q}$, where $\lambda > 1$. Then it is determined that $W(\Psi)$ has Hausdorff dimension zero. Indeed, on any fixed bounded interval, the approximation intervals with denominator q number $O(q)$, and each has diameter $2\lambda^{-q}$. For every $s > 0$,

$$\sum_{q=1}^{\infty} q(2\lambda^{-q})^s < \infty.$$

Taking the tails of these covers shows that $\mathcal{H}^s(W(\Psi)) = 0$ for every $s > 0$. Consequently, whenever $E(\Psi, \omega)$ is nonempty,

$$\dim_{\text{H}} E(\Psi, \omega) = 0.$$

In particular, the Cantor sets obtained in the exponential case have Hausdorff dimension zero, despite having the cardinality of the continuum.

6.3 Rational exponential

Lemma 6.3 (Bounded padding of denominators). *Fix an integer $r \geq 2$, and put $R = \prod_{\ell \mid r, \ell \text{ prime}} \ell$. Every finite continued-fraction prefix can be extended, using at most two digits in $\{1, \dots, R\}$, to a prefix of even length whose denominator is coprime to r .*

Proof. Let u, v be the last two denominators with v being the last one. Then $(u, v) = 1$, and appending a new digit d gives new denominator $dv + u$. For each prime $\ell \mid r$, if $\ell \mid v$, every residue $d \pmod{\ell}$ is admissible, since $\ell \nmid u$. Otherwise exactly one residue is not admissible, which is $d \equiv -uv^{-1} \pmod{\ell}$. Choose an allowed residue for every $\ell \mid r$. The Chinese remainder theorem therefore gives a representative $1 \leq d \leq R$ for which $(dv + u, r) = 1$.

If the prefix has odd length, append this digit d directly. If it has even length, first append 1, and then apply the same argument to the resulting pair of consecutive denominators. In either case, the final length is even and the denominator is coprime to r . $\square$

Proof of Theorem 1.5. We first prove the argument of existence. Fix $C > 6$, and choose a K such that

$$2 < K < \frac{C}{3}$$

It suffices to use

$$w(q) = C\lambda^{q-qr^q}$$

for all sufficiently large q , and when q is small, take it elsewhere in $(0, 1)$, for example $\frac{1}{2}$, and this makes $w : (0, \infty) \rightarrow (0, 1)$ well defined. Indeed, $w \leq \omega$ eventually, so $E(\Psi, w) \subseteq E(\Psi, \omega)$.

Take a prefix with even length $p/q = [0; a_1, \dots, a_n]$ with q sufficiently large and $(q, r) = 1$. Put

$$X = \frac{\lambda^q}{q}, \quad N = qr^q, \quad w_q = w(q), \quad y_0 = \frac{p}{q} + \lambda^{-q}.$$

Write $S = S(p/q)$, with w in place of ω . In the original x -coordinate, its target interval is

$$[y_0 - C\lambda^{-N}, y_0]. \quad (42)$$

Moreover, for

$$y_0 = \frac{pr^q + qs^q}{qr^q}$$

the numerator is coprime to both q and r , by $\gcd(p, q) = \gcd(q, r) = \gcd(r, s) = 1$. Thus the denominator of y_0 is exactly N .

We argue that any bad interval with its closure intersecting S has its convergents fraction P/D with the denominator $D \geq N$ once q is big enough. Let B be such a bad interval and y be its image of a point in $\overline{B} \cap S$ in terms of original x -coordinates. Writing $C_D = \lceil \frac{1}{3D^2\Psi(D)} \rceil$, the bad cutoff and the target interval give

$$\left| y - \frac{P}{D} \right| \leq \frac{1}{C_D D^2} \leq 3\lambda^{-D}, \quad |y - y_0| \leq C\lambda^{-N}.$$

Clearly by continuity, the error estimate remains valid at endpoints. Since the unit cell $[b_0, b_0 + 1]$ meets S , we have $b_0 + 1 \geq \inf S$. Hence

$$D \geq q_{n+1} = b_0 q + q' \geq q(\inf S - 1) + q' = \frac{1}{q\Psi(q)} - q = X - q,$$

where q' is the denominator preceding q . Suppose $D < N$, then since P/D and y_0 are distinct reduced rationals, and hence

$$\frac{1}{N^2} < \frac{1}{ND} \leq \left| y_0 - \frac{P}{D} \right| \leq C\lambda^{-N} + 3\lambda^{-(X-q)}.$$

In other words,

$$N^2(C\lambda^{-N} + 3\lambda^{-(X-q)}) > 1$$

must holds.

Since $N = qr^q$ and $X = \lambda^q/q$,

$$\log(N^2\lambda^{-N}) = 2\log q + 2q\log r - qr^q\log \lambda \rightarrow -\infty,$$

$$\log(N^2\lambda^{-(X-q)}) = 2\log q + 2q\log r - \left(\frac{\lambda^q}{q} - q \right) \log \lambda \rightarrow -\infty.$$

Indeed, $r > 1$, $\lambda > 1$, and $\lambda^q/q^2 \rightarrow \infty$, so the negative term in each line dominates the remaining terms once q is big enough. Therefore

$$N^2 \left(C\lambda^{-N} + 3\lambda^{-(X-q)} \right) \rightarrow 0.$$

For sufficiently large q , we consequently obtain

$$N^2(C\lambda^{-N} + 3\lambda^{-(X-q)}) < 1$$

which forms a contradiction. Thus $D \geq N$. We have therefore proved

$$\overline{B} \cap S \neq \emptyset \implies D \geq N. \quad (43)$$

Now, we compare the lengths in the tail coordinate. Let $P_m/Q_m = [b_0; b_1, \dots, b_m]$ define B , and let q' precede q . Since $D = qP_m + q'Q_m$, $P_m/Q_m \leq b_0 + 1$, and $b_0 \leq \sup S$,

$$\frac{D}{Q_m} \leq q(b_0 + 1) + q' \leq q(\sup S + 1) + q' = \frac{X}{1 - w_q} + q.$$

Using (20) and (43), we obtain

$$\mathcal{L}(B) \leq 3 \left(\frac{X}{1 - w_q} + q \right)^2 \lambda^{-N}, \quad \mathcal{L}(S) = \frac{w_q \lambda^q}{q^2(1 - w_q)} = \frac{CX^2 \lambda^{-N}}{1 - w_q}.$$

Consequently, for sufficiently large q ,

$$\frac{\mathcal{L}(S)}{\mathcal{L}(B)} \geq \frac{C}{3(1 - w_q)((1 - w_q)^{-1} + q/X)^2} = \frac{C}{3} + o(1) > K.$$

Denote $\frac{C}{3(1 - w_q)((1 - w_q)^{-1} + q/X)^2}$ as A_q , then A_q approaches $\frac{C}{3}$ infinitely close, so $\Delta_{A_q} = |\frac{C}{3} - A_q| \rightarrow 0$. On the other hand, since $K < \frac{C}{3}$ is a fixed value, $\Delta_K = \frac{C}{3} - K$ is a fixed positive value. This means when q is sufficiently large, $\Delta_K > \Delta_{A_q}$ always holds and the inequality holds too. Thus, the condition of Lemma 5.4 holds with w in place of ω .

Let $R \geq 2$ be as in Lemma 6.3. Since $D^2 \lambda^{-D} \rightarrow 0$, choose D_0 large enough that all preceding estimates apply and

$$3RD^2 \lambda^{-D} < 1 \quad \text{for every } D \geq D_0.$$

Thus, whenever the current denominator is at least D_0 , every digit $d \in \{1, \dots, R\}$ satisfies

$$d \leq R < \frac{1}{3D^2 \Psi(D)}.$$

In particular, since denominators increase when level increases, every coding digit 1 or 2, as well as every padding digit at most R , satisfies the required partial-quotient bound at the stage where it is appended.

To obtain a suitable initial prefix, take a sufficiently long word of 1's so that its denominator is at least D_0 . Apply Lemma 6.3 to make the length even and the final denominator coprime to r . Since denominators strictly increase when digits are appended, every subsequent denominator remains at least D_0 .

Fix a sequence $(\delta_j)_{j \geq 1} \in \{1, 2\}^{\mathbb{N}}$. From the initial prefix word W_1 , we can construct the prefixes recursively with the following method. Given W_j , append a finite word V_j supplied by Lemma 5.4, then the digit δ_j , coding digits, and finally a padding word T_j supplied by Lemma 6.3. This means

$$W_{j+1} = W_j V_j \delta_j T_j.$$

The prefix W_{j+1} again has even length and denominator coprime to r . Moreover, the interval is strictly contained in the interval of $W_j V_j$, so the cylinder preserves the target-shell condition at this level to stay inside $E(\Psi, \omega)$.

For each input prefix, determine one choice in each of the two lemmas. To specify, remember the rule from the proof of theorem 1.1. Whenever the extension digit or padding digit is not unique, choose the minimal length, and among those choose the lexicographically smallest one as the determined one. The infinite construction then gives a unique irrational number x for each sequence (δ_j) .

Let p_n/q_n denote the convergents in the n_{th} level of x , and let n_j be the index of the corresponding numbers of partial quotients, or corresponding level of the prefix W_j . Thus,

$$\frac{p_{n_j}}{q_{n_j}} = [0; W_j].$$

By previous construction, the requirement

$$(1 - w(q_{n_j}))\lambda^{-q_{n_j}} \leq \left| x - \frac{p_{n_j}}{q_{n_j}} \right| \leq \lambda^{-q_{n_j}}$$

is met for every j . Thus the required upper bound holds infinitely often.

For every sufficiently large index n which is not included in $\{n_j\}$, the next digit a_{n+1} is either an intermediate digit supplied by Lemma 5.4, a coding digit, or a padding digit. In each case,

$$a_{n+1} \leq \frac{1}{3q_n^2 \Psi(q_n)}.$$

The continued-fraction error estimate, with calculation, therefore gives

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{3a_{n+1}q_n^2} \geq \Psi(q_n) = \lambda^{-q_n}.$$

If a reduced fraction p/d is not a convergent of x , Legendre's theorem gives

$$\left| x - \frac{p}{d} \right| \geq \frac{1}{2d^2} > \lambda^{-d}$$

for sufficiently large d . Concluding the cases tells us that every reduced fraction p/d with sufficiently large denominator satisfies the following inequality:

$$\left| x - \frac{p}{d} \right| \geq (1 - w(d))\lambda^{-d} \geq \frac{1}{2}\lambda^{-d}.$$

It remains to check nonreduced fractions. If p/d is reduced and is represented with denominator $Q = kd$, where $k \geq 2$, then, for sufficiently large d ,

$$\left| x - \frac{p}{d} \right| \geq \frac{1}{2}\lambda^{-d} > \lambda^{-2d} \geq \lambda^{-Q}.$$

For any bounded d , only finitely many reduced fractions p/d lie in $[x - 1, x + 1]$. Since x is irrational, their distances from x have a positive minimum. They therefore also have distance greater than λ^{-Q} for sufficiently large Q . Hence $x \in E(\Psi, w)$.

It leaves us to prove $E(\Psi, w)$ contains a cantor set.

For each coding sequence $\delta = (\delta_j)_{j \geq 1} \in \{1, 2\}^{\mathbb{N}}$, the extension and padding rules we fixed determine $W_j(\delta)$ whose prefix lengths tend to infinity. These prefixes define a unique infinite continued fraction with irrational value $x(\delta)$. The preceding estimates show that $x(\delta) \in E(\Psi, w)$. We therefore define

$$F : \{1, 2\}^{\mathbb{N}} \longrightarrow E(\Psi, w), \quad F(\delta) = x(\delta).$$

and we only need to prove injectivity and continuity.

First is injectivity, similar to the proof of theorem 1.1, consider $\delta \neq \delta'$, and let j be their first differing index. Since we fixed selection rules, we have the same prefix W_j and extension word V_j in both constructions. Their next digits are δ_j and δ'_j , which are different, respectively. Clearly, since continued fractions for irrational value is unique,

$$F(\delta) \neq F(\delta').$$

Thus F is injective.

Second is continuity, fix δ . If δ' and δ have the same prefix in its first k entries, or in other words, both constructions have the same prefix W_{k+1} . Their values lie in the same cylinder, giving

$$|F(\delta) - F(\delta')| \leq \mathcal{L}(I(0; W_{k+1})) \leq \frac{1}{q_{n_{k+1}}^2},$$

where $q_{n_{k+1}}$ is defined as the denominator of $[0; W_{k+1}]$. Since these denominators tend to infinity, for every $\varepsilon > 0$, as long as the length of same prefixes is sufficiently large,

$$|F(\delta) - F(\delta')| < \varepsilon.$$

can hold. This proves continuity in the product topology.

Since $\{1, 2\}^{\mathbb{N}}$ is compact and $\mathbb{R}$ is Hausdorff, the continuous injection F is a homeomorphism onto its image. Consequently, its image is a Cantor set contained in $E(\Psi, w) \subseteq E(\Psi, \omega)$, completing the existence proof.

For the emptiness argument, suppose $0 < c < 1$ and $\omega(q) \leq c\lambda^{q-qr^q}$ eventually. If $x \in E(\Psi, \omega)$, then for arbitrarily large q there exist a corresponding p with

$$(1 - \omega(q))\lambda^{-q} \leq \left| x - \frac{p}{q} \right| \leq \lambda^{-q}.$$

Choose a $\sigma \in \{-1, 1\}$ to have the sign of $x - p/q$, and put

$$y_0 = \frac{p}{q} + \sigma\lambda^{-q} = \frac{pr^q + \sigma qs^q}{N}, \quad N = qr^q.$$

Then

$$|x - y_0| \leq \omega(q)\lambda^{-q} \leq c\lambda^{-N}.$$

However since x is in $E(\Psi, \omega)$, and y_0 is not required to be reduced by the definition of $E(\Psi, \omega)$ where the bound applies to all q that is sufficiently large with all integer pairs, and hence

$$|x - y_0| \geq (1 - \omega(N))\lambda^{-N}.$$

must also apply. This contradicts $\omega(N) \rightarrow 0$ and $c < 1$. □

Remark 6.4. Theorem 1.5 determines the scale

$$h_\lambda(q) = \lambda^{q-qr^q}$$

up to multiplicative constants: a sufficiently large constant multiple bigger than 6 allows a Cantor set of exact approximations, whereas a sufficiently small constant multiple less than 1 makes the set entirely empty.

The theorem leaves the constant range $1 \leq C \leq 6$ unresolved for $\omega(q) = Ch_\lambda(q)$ eventually. For irrational λ , the endpoint need not be rational, so the same denominator argument does not apply directly.

This argument is proved by additional input of rational endpoints of reduced denominator. This gives a stronger separation estimate of bad interval. A local extension lemma with bounded padding that makes denominator coprime with r allow the construction that makes the estimation admissible.

7 Sources of the lower-order terms

Write

$$X_q := \frac{\lambda^q}{q}. \quad (44)$$

Determine constant $\Gamma > 6$. The rounding offset denominator bound reads

$$q_{n+1} \geq X_q - \rho_{p/q}q. \quad (45)$$

If the construction could guarantee $\rho_{p/q} \leq \eta$ along every selected branch, for some fixed $0 < \eta \leq 1$, then the rounding offset condition in Proposition 4.1 would involve the right-hand side

$$R_q(\rho_{p/q}) := \Gamma(1 - \rho_{p/q}q^2\lambda^{-q})^2\lambda^{(1+\rho_{p/q})q-X_q}.$$

Since

$$(1 - \rho_{p/q}q^2\lambda^{-q})^2 \leq 1$$

and $\rho_{p/q} \leq \eta$, we have

$$R_q(\rho_{p/q}) \leq \Gamma\lambda^{(1+\eta)q-X_q}.$$

Therefore the uniform sufficient condition

$$\omega(q) \geq \Gamma\lambda^{(1+\eta)q-X_q} \quad (46)$$

would be able to imply $\omega(q) \geq \frac{R_q(\rho_{p/q})}{1+R_q(\rho_{p/q})}$ and hence would suffice along all such branches.

Indeed, before replacing the actual phase by a global upper bound η , the phase-dependent exponent is

$$E_q(\rho_{p/q}) := (1 + \rho_{p/q})q - X_q = \underbrace{q}_{\text{scale-comparison term}} + \underbrace{\rho_{p/q}q}_{\text{phase term}} - X_q. \quad (47)$$

Under the assumption of phase control,

$$E_q(\rho_{p/q}) \leq q + \eta q - X_q,$$

which is the exponent used in (46).

The two lower-order terms in (47) have two different origins.

The scale-comparison term $+q$. The separation condition contains the ratio

$$\frac{\Psi(q_{n+1})}{\Psi(q)}.$$

For $\Psi(q) = \lambda^{-q}$, this becomes

$$\frac{\Psi(q_{n+1})}{\Psi(q)} = \lambda^{q-q_{n+1}}.$$

Even if there is a moment that there is no phase loss, so that $q_{n+1} = X_q$. The exponent is then still

$$q - X_q.$$

So this occurrence of $+q$ does not come from integer rounding or $\rho_{p/q}$. It is needed when the target at denominator q is compared with the bad interval at denominator q_{n+1} , and improving the denominator estimate cannot remove this term.

The phase term $+\rho_{p/q}q$. By rounding offset denominator estimation,

$$q_{n+1} \geq X_q - \rho_{p/q}q.$$

Therefore

$$q - q_{n+1} \leq q - (X_q - \rho_{p/q}q) = (1 + \rho_{p/q})q - X_q.$$

Thus the phase contributes the term $+\rho_{p/q}q$ to the exponent. Using only the uniform bound $\rho_{p/q} \leq 1$, we obtain the exponent

$$2q - X_q.$$

If one could arrange $\rho_{p/q} = o(1)$ along every branch of the construction, then the same calculation would instead give

$$q - X_q + o(q).$$

Even this would not give $-X_q$, because the first $+q$ is already present in scale-comparison term $\Psi(q_{n+1})/\Psi(q)$. This means removing that term would require a big change of the estimate for the bad intervals, or comparing their intersection with $S(p/q)$ more directly.

7.1 Importance of phase control

The three denominator bounds give the following exponents in the exponential case:

Estimate used	Lower bound for q_{n+1}	Exponent of λ
Baker-Ward	$q - \frac{X_q}{2}$	$\frac{X_q}{2}$
Our estimate	$X_q - q$	$2q - X_q$
If $\rho_{p/q} = o(1)$	$X_q - o(q)$	$q - X_q + o(q)$

The last row is clearly hypothetical and cannot apply into global theorem. It shows, however, exactly what such phase control could achieve locally. It would remove the second $+q$ from the exponent, but the first $+q$, coming from $\Psi(q_{n+1})/\Psi(q)$, would remain. Thus phase control alone cannot produce the exponent $-X_q$; any further improvement would have to change the treatment of the bad intervals.

8 Distribution of predecessor phases

For a fixed denominator q , the phase depends on the preceding denominator q' through

$$\rho\left(\alpha - \frac{q'}{q}\right).$$

We first count how many reduced residues $q' \pmod{q}$ give a really small phase, and then show that every such residue can occur as a preceding denominator of some reduced rational p/q .

Let $\nu(q)$ denote the number of distinct prime divisors of q , and let φ and μ denote Euler's totient function and the Möbius function.

Proposition 8.1 (Phase count with reduced residue). *For $0 < \eta < 1$, $\alpha \in \mathbb{R}$, and $q \geq 2$, set*

$$N_q(\eta; \alpha) := \#\left\{1 \leq j < q : (j, q) = 1, \rho\left(\alpha - \frac{j}{q}\right) \leq \eta\right\}.$$

Uniformly in α ,

$$N_q(\eta; \alpha) = \eta\varphi(q) + O\left(2^{\nu(q)}\right), \quad (48)$$

with an absolute implied constant.

Proof. Because $0 < \eta < 1$, the phase condition says that the residue $\frac{j}{q} \in \mathbb{R}/\mathbb{Z}$ lies in the half-open circular interval $[\alpha - \eta, \alpha)$. The half-open convention is exactly what excludes the integer-phase case $\rho = 1$; changing either endpoint would affect only an $O(1)$ term.

Using the identity

$$\mathbf{1}_{(j,q)=1} = \sum_{d|(j,q)} \mu(d).$$

For each $d \mid q$, the multiples of d form a uniform grid of $\frac{q}{d}$ points modulo q . A circular interval of length η contains

$$\eta \cdot \frac{q}{d} + O(1)$$

of these point Therefore

$$\begin{aligned} N_q(\eta; \alpha) &= \sum_{d|q} \mu(d) \left(\eta \frac{q}{d} + O(1) \right) \\ &= \eta q \sum_{d|q} \frac{\mu(d)}{d} + O\left(\sum_{d|q} |\mu(d)| \right) \\ &= \eta \varphi(q) + O\left(2^{\nu(q)} \right). \end{aligned}$$

This also covers intervals that wrap around 0, since the counting was performed on the circle. $\square$

Lemma 8.2 (Local realizability of predecessor denominators). *For every $q \geq 2$ and every j with $1 \leq j < q$ and $(j, q) = 1$, there is a reduced rational $\frac{p}{q} \in (0, 1)$ whose even-length continued-fraction expansion has preceding denominator $q' = j$.*

Proof. Choose the unique $p \in \{1, \dots, q-1\}$ satisfying

$$pj \equiv -1 \pmod{q}.$$

since $(j, q) = 1$ Then $(p, q) = 1$ too. Write $\frac{p}{q}$ in its even-length finite continued-fraction and let $\tilde{q}$ be the preceding denominator. The determinant identity for an even convergent can be applied and gives

$$p_{n-1}q - p\tilde{q} = 1, \quad \text{hence} \quad p\tilde{q} \equiv -1 \pmod{q}.$$

Both j and $\tilde{q}$ lie in $\{1, \dots, q-1\}$, so the uniqueness of the modular inverse implies $\tilde{q} = j$. $\square$

Remark 8.3. The proposition counts all reduced residues $j \pmod{q}$, and Lemma 8.2 shows that each of them occurs for some rational $\frac{p}{q}$. Thus small phases are common when $\frac{p}{q}$ may be chosen freely. Indeed,

$$\frac{2^{\nu(q)}}{\varphi(q)} \longrightarrow 0,$$

so for fixed η the main term $\eta \varphi(q)$ dominates the error.

Unfortunately, the Cantor construction does not allow p/q to be chosen freely. Each new convergent must extend the prefix already constructed and must also survive the bad-interval deletion. The local count therefore does not show that every prefix has an admissible continuation with small phase. Such a result would require an additional phase-steering argument.

9 A numerical case study

Take $\lambda = 2$ and $q = 10$. Then

$$\Psi(q) = 2^{-10}, \quad A(q) = \frac{2^{10}}{10^2} = 10.24, \quad X_q = \frac{2^{10}}{10} = 102.4.$$

Make convergents even and thus we get

$$\frac{p}{q} = [0; 3, 3] = \frac{3}{10}, \quad q' = 3, \quad r = 0.3.$$

The left endpoint of the target tail interval is $L = A(q) - r = 9.94$, so the smallest intersecting cell is $[9, 10]$. Its phase is $\rho(L) = 0.94$, and the exact next denominator is

$$q_{n+1} = 9 \cdot 10 + 3 = 93 = 102.4 - 0.94 \cdot 10.$$

So the phase lemma reaches an equality, while the uniform lower bound gives $q_{n+1} \geq 92.4$. This numerical example nearly realizes the worst phase and shows why an unspecified intersecting cell cannot simply be treated as if $q_{n+1} \geq X_q$.

A second example will be taking

$$\frac{p}{q} = [0; 1, 9] = \frac{9}{10}, \quad q' = 1, \quad r = 0.1.$$

Now $L = 10.14$, the minimal cell is $[10, 11]$, the phase is 0.14 , and $q_{n+1} = 101$. The two examples have the same q and $\Psi(q)$ but very different rounding losses.

p/q	q'	L	$\rho(L)$	minimal b_0	q_{n+1}
$3/10 = [0; 3, 3]$	3	9.94	0.94	9	93
$9/10 = [0; 1, 9]$	1	10.14	0.14	10	101

Hence in this exponential case, under base-2 logarithmic scale, Baker–Ward’s denominator estimate can produce the threshold exponent $10 - 51.2 = -41.2$; the present uniform estimate produces $20 - 102.4 = -82.4$; and the exact phase of the first example produces $10 - 93 = -83$. These numbers are only an illustration at one denominator, not an asymptotic proof, but they make the size of the improvement and the rounding loss visible and easier to understand

10 Conclusion and open problems

For the genuine case of improvement for Baker–Ward’s theorem 1.1, Our paper combines five proved additions: the exact rounding offset denominator step, the telescoping bad-interval constant 3, the geometric selection step we have parameterized finding the sufficient pre-factor can be any fixed constant $\Gamma > 6$, the self-normalized threshold $R/(1 + R)$, and the Cantor-space embedding. In the exponential case the leading negative coefficient improved to 1, while the numerical prefactor in the unsimplified quantity R improved to $\Gamma > 6$. The reduced-residue count additionally shows that favorable phases are locally abundant.

For rational based exponential decay case, theorem 1.5 gains better result, where reduced fractions give rational endpoints that can weaken the sufficient bound for ω and therefore gives the same type of estimation function as the empty criteria as Baker–Ward theorem 1.6. This means whether $E(\Psi, \omega)$ is uncountable or empty is based on the multiplicative constant. Specifically if it is bigger than 6, the set is uncountable and contains Cantor set, and if it is less than 1, the set is empty.

This still leave three distinct problems. The constant $\Gamma > 6$ is only a sufficient proof constant and is not claimed to be numerically optimal. More, the endpoint 6 is not claimed to be true, and As $\Gamma \downarrow 6$, the retained proportion

$$c_\Gamma = \frac{\Gamma - 6}{10\Gamma}$$

tends to zero. Nevertheless, it remains positive for every fixed $\Gamma > 6$, which is sufficient for the Cantor construction.

More importantly, local phase abundance is not a rule for steering every branch of the continued-fraction tree. Even perfect steering would leave the scale-comparison correction $+q$; reaching the pure boundary $\lambda^{-\lambda^q/q}$ would require new information or construction about $B \cap S$, overlap among bad intervals, or a different selection geometry. Thus the theorem remains a sufficient criterion rather than a beautiful, optimal matched empty/non-empty threshold.

The third problem is in the rational exponential case, the optimal constant threshold of emptiness or uncountability of $E(\Psi, \omega)$ is yet to be claimed. The paper does not contribute to Hausdorff dimension theory, but cares about general monotonicity branch and gives new rational based exponential results too.

References

- [1] S. Baker and B. Ward, *A quantitative framework for sets of exact approximation order by rational numbers*, arXiv:2510.18451v1 [math.NT], 2025.
- [2] Y. Bugeaud, *Sets of exact approximation order by rational numbers*, Math. Ann. 327 (2003), 171–190.
- [3] Y. Bugeaud and C. G. Moreira, *Sets of exact approximation order by rational numbers III*, Acta Arith. 146 (2011), 177–193.
- [4] A. Ya. Khinchin, *Continued Fractions*, University of Chicago Press, 1964.

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