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论文题目：A Tropical Yamabe-Type Prescription Problem
on Rank-Three Matroid Fans

A Tropical Yamabe-Type Prescription Problem on Rank-Three Matroid Fans

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Abstract

Let M be a simple matroid of rank three with n points and m lines and let Δ_M be its fine Bergman fan. A piecewise linear function on Δ_M is given by its values $x \in \mathbb{R}^n$ on the point rays and $y \in \mathbb{R}^m$ on the line rays, and its tropical divisor is again a weight on those rays, so that the divisor map is a square symmetric integer matrix

$$L_M = \begin{pmatrix} I_n - R & B \\ B^\top & -I_m \end{pmatrix},$$

where B is the point–line incidence matrix and $R = \text{diag}(r_1, \dots, r_n)$ records the number r_i of lines through the point i . Any two points of M lie on a unique line, so $BB^\top = J + R - I$, and this identity yields the unimodular congruence $L_M = P^\top \begin{pmatrix} J & 0 \\ 0 & -I_m \end{pmatrix} P$ with $P = \begin{pmatrix} I & 0 \\ -B^\top & I \end{pmatrix}$. From the congruence we determine the kernel of L_M , which is the space of globally linear functions, the image, the inertia $(1, m, n - 1)$, and the Smith normal form. A separate Schur-complement computation gives the characteristic polynomial, which depends only on m and on the multiset of the degrees r_i . We then solve the prescription problem for the curvature vector $\mathcal{K}(\varphi) := -\text{div}(\varphi)$: a nonzero constant vector is a curvature if and only if the degrees are all equal, and when they are not, the squared distance from the constant vector to the space of curvatures equals

$$\mathcal{Y}(M)^2 = \sum_i r_i - \frac{n^2}{\sum_i 1/r_i} = n(A - H),$$

n times the difference between the arithmetic and harmonic means of the degrees. Evaluating the proved defect formula on all 8,778 labeled simple rank-three matroids on at most seven points shows that near-pencils maximize the defect for $4 \leq n \leq 7$. The matroid $M_{8,2}$ shows that they no longer do so for $n = 8$. Inside the family of matroids with a single long line, the maximizing proportion of points off that line converges to the root of $2\alpha^3 - 2\alpha^2 + 4\alpha - 1$ in $(0, 1)$ and the maximal squared defect grows like n^2 . The equation solved here is linear, and its curvature is defined by the divisor operator rather than by a metric, so the analogy with the Yamabe problem is limited to the prescription question.

Keywords: matroid; Bergman fan; tropical divisor; Yamabe-type equation; Smith normal form; finite linear space.

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1 Introduction and main results

The Yamabe problem asks whether a conformal class of Riemannian metrics on a compact manifold of dimension at least three contains a metric of constant scalar curvature. The equation behind it is nonlinear and the difficulties are analytic. Tropical geometry replaces smooth manifolds by balanced polyhedral complexes and smooth functions by piecewise linear ones, and the operator that takes the place of the Laplacian is the tropical divisor, which measures the failure of the slopes of a function to balance along a codimension-one cone. This paper studies the prescription problem for that operator on the fine Bergman fan Δ_M of a simple matroid M of rank three [3, 6, 11].

The word curvature is used here in a restricted sense, which we now fix. The fan Δ_M carries no Riemannian metric. For a piecewise linear function φ on Δ_M we put $\mathcal{K}(\varphi) := -\operatorname{div}(\varphi)$, a vector of weights on the rays of Δ_M , and call it the curvature of φ . Since φ is allowed to be an arbitrary piecewise linear function, no reference potential enters and there is no analogue of a conformal class. What survives of the Yamabe question is the problem of deciding when the constant ray weight is the divisor of some φ , and of measuring the discrepancy when it is not. Yamabe type equations on graphs were studied by Grigor'yan, Lin and Yang [7]; there the operator is a graph Laplacian and the equation contains a nonlinear term, so neither the operator nor the equation is the one treated below.

Rank three is used in an essential way. In that rank the nonempty proper flats of M are its points and its lines, so the rays of Δ_M and its codimension-one cones are indexed by the same set and the divisor map is given by a square symmetric matrix. Since $\dim \Delta_M = r - 1$ for rank r , rays are codimension-one cones exactly when $1 = r - 2$, that is, when $r = 3$. When $\operatorname{rk} M \geq 4$, the divisor is indexed by flags of flats of length $r - 2$. The domain and codomain then have different indexing sets, the matrix is generally rectangular, and there is no natural symmetric identification of the kind used here.

Throughout, M is a simple matroid of rank three on n points with m rank-two flats, which we call lines, and r_i is the number of lines through the point i . We consider four questions.

- (1) What is the matrix of the divisor map in point–line coordinates?
- (2) What are its kernel, image, inertia, spectrum and Smith normal form?
- (3) For which M is $\mathcal{K}(\varphi)$ a nonzero constant vector for some φ ?
- (4) When no φ has nonzero constant curvature, how far is the constant vector from the set of curvatures?

The answers are as follows. Let $B \in \{0, 1\}^{n \times m}$ be the point–line incidence matrix, let $R = \operatorname{diag}(r_1, \dots, r_n)$ and let $J = \mathbf{1}\mathbf{1}^\top$. In the coordinates $x_i = \varphi(\bar{e}_i)$ and $y_L = \varphi(\bar{e}_L)$ on the point rays and the line rays, the divisor map is

$$L_M = \begin{pmatrix} I_n - R & B \\ B^\top & -I_m \end{pmatrix}$$

(Proposition 3.1). Because two distinct points of M lie on a unique line, $BB^\top = J + R - I_n$ (Lemma 2.3), and with the unimodular matrix $P = \begin{pmatrix} I_n & 0 \\ -B^\top & I_m \end{pmatrix}$ these two facts give

$$L_M = P^\top \begin{pmatrix} J & 0 \\ 0 & -I_m \end{pmatrix} P, \quad \text{equivalently} \quad \begin{pmatrix} x \\ y \end{pmatrix}^\top L_M \begin{pmatrix} x \\ y \end{pmatrix} = \left(\sum_i x_i \right)^2 - \|y - B^\top x\|^2$$

(Theorem 3.2). This congruence and the subsequent calculations give the following results. The kernel of L_M is the space of globally linear functions and has dimension $n - 1$ (Corollary 3.3); the inertia is $(1, m, n - 1)$, so the rank is $m + 1$ (Corollary 3.4); the Smith normal form is $\operatorname{diag}(1^{m+1}, 0^{n-1})$ and the cokernel is free (Corollary 3.5); and a ray weight (p, q) is a divisor if and

only if $p + Bq$ is a constant vector, an integral weight having an integral potential (Theorem 3.7). Question (3) then has the answer that a nonzero constant vector is a curvature if and only if $r_1 = \dots = r_n$ (Theorem 3.9), and the characteristic polynomial of L_M is determined by m together with the multiset $\{r_1, \dots, r_n\}$ (Theorem 3.12). For question (4), the squared distance from the constant vector $\mathbf{1}$ to the space of curvatures is

$$\mathcal{Y}(M)^2 = \sum_{i=1}^n r_i - \frac{n^2}{\sum_{i=1}^n 1/r_i} = n(A - H),$$

where A and H denote the arithmetic and the harmonic mean of the degrees (Theorem 3.16).

The abstract identification of piecewise linear functions modulo global linear functions with codimension-one tropical cycles is a special case of the duality of Gross and Shokrieh [8]. In the fixed fine-fan coordinates used here, this accounts for both the kernel statement modulo linear functions and the surjectivity onto balanced ray weights. Thus the abstract content of Corollary 3.3 and Theorem 3.7 is not claimed as new; we give explicit rank-three coordinate proofs.

The point–line system is also a finite linear space. Its inequality $m \geq n$ is the classical theorem of de Bruijn and Erdős [4], and the incidence identity $BB^\top = J + R - I$ is a direct pair-counting fact. These are background results, not originality claims. The focus of this paper is the explicit factorization of this divisor matrix, its spectral and arithmetic consequences, the equal-ray-weight prescription criterion, and the closed defect formula with its enumerative and extremal analysis.

Section 2 fixes the matroid and tropical conventions and proves the combinatorial facts that are needed later. Section 3 contains the linear algebra: the divisor matrix, the congruence and its consequences, the prescription theorem, the constant curvature criterion, the characteristic polynomial and the defect formula. Section 4 reports an exhaustive enumeration of the labeled simple rank-three matroids on at most seven points, analyzes the family of matroids with a single long line, and lists the invariants of five representative matroids. Section 5 summarizes what has been proved, states what the model does not capture, and lists open problems.

2 Background and definitions

We follow Oxley [10] for matroid terminology. A matroid $M = (E, \text{rk})$ consists of a finite set E together with a function $\text{rk}: 2^E \rightarrow \mathbb{Z}_{\geq 0}$ satisfying $\text{rk}(\emptyset) = 0$, $\text{rk}(A) \leq \text{rk}(A \cup \{x\}) \leq \text{rk}(A) + 1$, and $\text{rk}(A \cup B) + \text{rk}(A \cap B) \leq \text{rk}(A) + \text{rk}(B)$. Every matroid in this paper is simple, that is, it has no loops and no parallel elements, and has rank three.

In a simple matroid the rank-one flats are the singletons, and we call the elements of E points. A rank-two flat is called a line, and a line with at least three points is called long. We write

$$E = \{1, \dots, n\}, \quad \mathcal{L} = \{L_1, \dots, L_m\}, \quad r_i := \#\{L \in \mathcal{L} : i \in L\},$$

and call r_i the degree of the point i .

Lemma 2.1. *Let M be a simple matroid of rank three. Then*

- (i) *any two distinct points lie on a unique line;*
- (ii) *two distinct lines have at most one point in common;*
- (iii) $\sum_{L \ni i} (|L| - 1) = n - 1$ *for every point i ;*
- (iv) $r_i \geq 2$ *for every point i ;*
- (v) $\sum_{L \in \mathcal{L}} \binom{|L|}{2} = \binom{n}{2}$ *and $\sum_{i=1}^n r_i = \sum_{L \in \mathcal{L}} |L|$.*

Proof. If $i \neq j$ then $\text{rk}\{i, j\} = 2$ because M is simple, so $\text{cl}\{i, j\}$ is the unique line containing i and j ; this gives (i), and (ii) follows from it. For a fixed point i the sets $L \setminus \{i\}$, where L runs over the lines through i , partition $E \setminus \{i\}$ by (i), which is (iii). If r_i were 1, the unique line through i would contain all n points by (iii) and would not be a proper flat, so (iv) holds. The two identities in (v) count unordered pairs of points and point–line incidences in two ways. $\square$

By part (i) the pair $(E, \mathcal{L})$ is a finite linear space, and by part (iv) it is not a single line. The converse holds in the following form, which is what makes the enumeration of Section 4 possible.

Lemma 2.2. *Let E be a finite set with $|E| = n \geq 3$ and let $\mathcal{H}$ be a family of subsets of E , each of size at least 3 and at most $n - 1$, such that two distinct members of $\mathcal{H}$ meet in at most one point. Let $\mathcal{L}$ consist of the members of $\mathcal{H}$ together with every pair of points that is contained in no member of $\mathcal{H}$, and define*

$$\text{rk}(A) = \begin{cases} 0, & A = \emptyset, \\ 1, & |A| = 1, \\ 2, & |A| \geq 2 \text{ and } A \subseteq L \text{ for some } L \in \mathcal{L}, \\ 3, & \text{otherwise.} \end{cases}$$

Then rk is the rank function of a simple matroid of rank three on E whose lines are the members of $\mathcal{L}$ and whose long lines are the members of $\mathcal{H}$. Conversely, the long lines of a simple rank-three matroid on E form such a family, and the two constructions are mutually inverse.

Proof. Every pair of points lies in exactly one member of $\mathcal{L}$: it lies in at most one member of $\mathcal{H}$ by hypothesis, and it is adjoined to $\mathcal{L}$ precisely when it lies in none. In particular rk is well defined.

Monotonicity is clear from the definition, and so is the bound $\text{rk}(A \cup \{x\}) \leq \text{rk}(A) + 1$: the only case to check is $\text{rk}(A) = 1$, where $A \cup \{x\}$ has at most two points and therefore lies on a line. For submodularity, note first that $\text{rk}(A \cup B) + \text{rk}(A \cap B) \leq \text{rk}(A) + \text{rk}(B)$ holds whenever $\text{rk}(A \cup B) \leq \max\{\text{rk}(A), \text{rk}(B)\}$, since $\text{rk}(A \cap B) \leq \min\{\text{rk}(A), \text{rk}(B)\}$. Using $\text{rk} \leq 3$ and again $\text{rk}(A \cap B) \leq \min\{\text{rk}(A), \text{rk}(B)\}$, one checks that in all remaining cases the inequality can fail only if $\text{rk}(A) = \text{rk}(B) = \text{rk}(A \cap B) = 2$ and $\text{rk}(A \cup B) = 3$. But $\text{rk}(A \cap B) = 2$ means that $A \cap B$ contains two distinct points i and j , and then the line on which A lies and the line on which B lies both contain i and j , hence coincide; so $A \cup B$ lies on a line and $\text{rk}(A \cup B) = 2$, a contradiction. Thus rk is a rank function.

It has value 1 on singletons and 2 on two-element sets, so the matroid is simple, and it takes the value 3 because no member of $\mathcal{L}$ is all of E . A set has rank at most two if and only if it lies on a member of $\mathcal{L}$, so the rank-two flats are exactly the maximal such sets, which are the members of $\mathcal{L}$ themselves: distinct members of $\mathcal{H}$ meet in at most one point and hence are incomparable, and an adjoined pair lies in no member of $\mathcal{H}$ by construction. The members of $\mathcal{H}$ are the lines with at least three points.

Conversely, let M be a simple rank-three matroid on E and let $\mathcal{H}$ be its family of long lines. Distinct lines meet in at most one point by Lemma 2.1(ii), no line is all of E , and a pair of points lies in no long line if and only if it is itself a line by Lemma 2.1(i). Hence $\mathcal{H}$ satisfies the hypotheses and the construction returns M . $\square$

We write $B \in \{0, 1\}^{n \times m}$ for the point–line incidence matrix, so $B_{iL} = 1$ if $i \in L$ and $B_{iL} = 0$ otherwise, and we put $R := \text{diag}(r_1, \dots, r_n)$ and $J := \mathbf{1}\mathbf{1}^\top$, the $n \times n$ matrix with all entries equal to 1. The vector of degrees is $r = B\mathbf{1}_m$.

Lemma 2.3. *For every simple matroid of rank three,*

$$BB^\top = J + R - I_n. \quad (1)$$

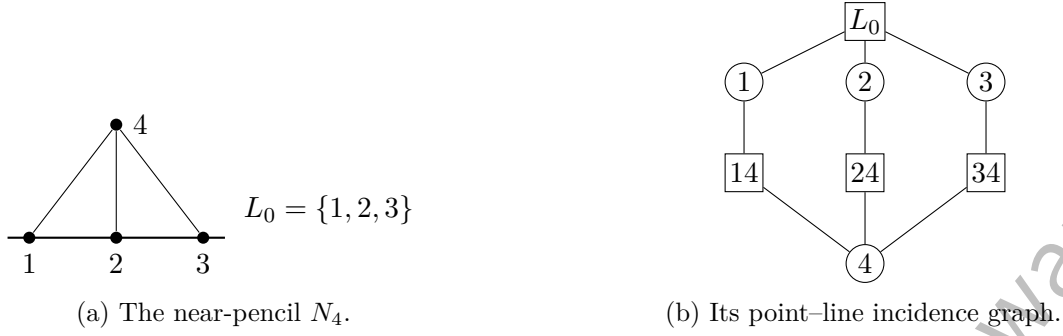


Figure 1: A rank-three matroid and the combinatorics of its fine Bergman fan. Circles are point rays and squares are line rays.

Proof. The entry $(BB^T)_{ii}$ is the number of lines through i , which is r_i . For $i \neq j$ the entry $(BB^T)_{ij}$ is the number of lines containing both points, which is 1 by Lemma 2.1(i). $\square$

We turn to the fan. Put

$$N := \mathbb{Z}^E / \mathbb{Z}\mathbf{1}, \quad N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R},$$

and for $F \subseteq E$ write $e_F := \sum_{i \in F} e_i$ and let $\bar{e}_F$ be its class in N . The fine Bergman fan $\Delta_M \subset N_{\mathbb{R}}$ is the fan whose cones are generated by the flags of nonempty proper flats of M [3, 6, 11]. In rank three the nonempty proper flats are the points and the lines, so the rays of Δ_M are

$$\rho_i = \mathbb{R}_{\geq 0} \bar{e}_i \quad (i \in E), \quad \rho_L = \mathbb{R}_{\geq 0} \bar{e}_L \quad (L \in \mathcal{L}),$$

and the maximal cones are

$$\sigma_{i,L} = \text{cone}(\bar{e}_i, \bar{e}_L) \quad (i \in L).$$

All maximal cones carry weight 1. The link of Δ_M at the origin is the point-line incidence graph of M , also called its Levi graph.

Lemma 2.4. *For every incidence $i \in L$ the pair $(\bar{e}_i, \bar{e}_L)$ extends to a $\mathbb{Z}$ -basis of N . In particular every maximal cone of Δ_M is unimodular.*

Proof. Since L is a proper flat, there is a point $k \notin L$. Since L has rank two, it contains a point $j \in L \setminus \{i\}$. The classes $f_h := \bar{e}_h$ with $h \neq k$ form a basis of $N = \mathbb{Z}^E / \mathbb{Z}\mathbf{1}$, and $\bar{e}_L = \sum_{h \in L} f_h$ in this basis. Replacing f_j by $\bar{e}_L$ is a unimodular change of basis because the coefficient of f_j in $\bar{e}_L$ is 1, and the new basis contains $f_i = \bar{e}_i$ and $\bar{e}_L$. $\square$

Throughout, piecewise linear functions are required to be linear on each cone of the fixed fine Bergman fan. By Lemma 2.4 a function on Δ_M that is linear on every maximal cone is determined by its values on the primitive generators of the rays, these values may be prescribed arbitrarily, and integral values give integral slopes on every maximal cone. Writing

$$x_i := \varphi(\bar{e}_i), \quad y_L := \varphi(\bar{e}_L),$$

we identify the space of piecewise linear functions on Δ_M with $\mathbb{R}^n \oplus \mathbb{R}^m$ and the integral ones with $\mathbb{Z}^n \oplus \mathbb{Z}^m$.

Finally we recall the divisor. Let Σ be a pure balanced fan with unit weights and let φ be piecewise linear on Σ . At a codimension-one cone τ the tropical divisor weight is

$$(\text{div } \varphi)(\tau) = \sum_{\sigma \succ \tau} \varphi_{\sigma}(v_{\sigma/\tau}) - \varphi_{\tau} \left(\sum_{\sigma \succ \tau} v_{\sigma/\tau} \right), \quad (2)$$

where the sum runs over the maximal cones σ containing τ and $v_{\sigma/\tau}$ is a lift to N of the primitive generator of the image of σ in N/N_{τ} ; see [2, 9]. The subtracted term makes the right-hand side independent of the chosen lifts, and it is what forces the divisor of a globally linear function to vanish.

3 The divisor matrix and its linear-algebraic structure

We first evaluate (2) on Δ_M .

Proposition 3.1. *Let M be a simple matroid of rank three and let φ be the piecewise linear function on Δ_M with ray values $(x, y) \in \mathbb{R}^n \oplus \mathbb{R}^m$. Then*

$$(\operatorname{div} \varphi)(i) = \sum_{L \ni i} y_L - (r_i - 1)x_i, \quad (3)$$

$$(\operatorname{div} \varphi)(L) = \sum_{i \in L} x_i - y_L. \quad (4)$$

Proof. The maximal cones containing ρ_L are the cones $\sigma_{i,L}$ with $i \in L$, and the primitive generators in question lift to $\bar{e}_i$. Since $\sum_{i \in L} \bar{e}_i = \bar{e}_L$, the subtracted term in (2) equals $\varphi(\bar{e}_L) = y_L$, which gives (4). The maximal cones containing ρ_i are the cones $\sigma_{i,L}$ with $L \ni i$, with generators lifting to $\bar{e}_L$. The lines through i partition the remaining points, so

$$\sum_{L \ni i} \bar{e}_L = r_i \bar{e}_i + \sum_{j \neq i} \bar{e}_j = (r_i - 1) \bar{e}_i$$

in $N_{\mathbb{R}}$, because $\sum_j \bar{e}_j = 0$ there. The subtracted term is therefore $(r_i - 1)x_i$, which gives (3). $\square$

Ordering the coordinates by points and then by lines, (3) and (4) read

$$\operatorname{div}(\varphi) = L_M \begin{pmatrix} x \\ y \end{pmatrix}, \quad L_M := \begin{pmatrix} I_n - R & B \\ B^\top & -I_m \end{pmatrix}. \quad (5)$$

This matrix is symmetric with integer entries, but it is not the Laplacian of the incidence graph; the diagonal entries and the signs are different, as Example 3.14 shows.

Theorem 3.2. *Let $P := \begin{pmatrix} I_n & 0 \\ -B^\top & I_m \end{pmatrix}$. Then P is unimodular and*

$$L_M = P^\top \begin{pmatrix} J & 0 \\ 0 & -I_m \end{pmatrix} P, \quad (6)$$

or equivalently, for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$,

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top L_M \begin{pmatrix} x \\ y \end{pmatrix} = \left(\sum_{i=1}^n x_i \right)^2 - \|y - B^\top x\|_2^2. \quad (7)$$

Proof. The matrix P is block lower triangular with identity diagonal blocks, so $\det P = 1$. Multiplying out,

$$P^\top \begin{pmatrix} J & 0 \\ 0 & -I_m \end{pmatrix} P = \begin{pmatrix} J - BB^\top & B \\ B^\top & -I_m \end{pmatrix},$$

and $J - BB^\top = I_n - R$ by (1), which proves (6). Evaluating the congruence on (x, y) and using $P(x, y)^\top = (x, y - B^\top x)^\top$ together with $x^\top J x = (\sum_i x_i)^2$ gives (7). $\square$

Corollary 3.3. *Over $\mathbb{R}$,*

$$\ker L_M = \left\{ \begin{pmatrix} a \\ B^\top a \end{pmatrix} : \mathbf{1}^\top a = 0 \right\}, \quad \dim \ker L_M = n - 1, \quad (8)$$

and the same description holds over $\mathbb{Z}$. Under the identification of a piecewise linear function with its ray values, $\ker L_M$ is the space of globally linear functions on $N_{\mathbb{R}}$.

Proof. Since $P^\top$ is invertible, (6) shows that $L_M(x, y) = 0$ if and only if $Jx = 0$ and $y = B^\top x$, and $Jx = 0$ says $\mathbf{1}^\top x = 0$. A linear functional on $N_{\mathbb{R}} = \mathbb{R}^E / \mathbb{R}\mathbf{1}$ is given by coefficients a_i with $\sum_i a_i = 0$, and its value at $\bar{e}_L$ is $\sum_{i \in L} a_i = (B^\top a)_L$, so the two spaces agree. $\square$

Corollary 3.4. *The real inertia of L_M is*

$$(n_+(L_M), n_-(L_M), n_0(L_M)) = (1, m, n - 1), \quad (9)$$

so $\text{rk } L_M = m + 1$ and L_M is indefinite for every simple rank-three matroid.

Proof. The matrix J has one positive eigenvalue and $n - 1$ zero eigenvalues and $-I_m$ has m negative eigenvalues, so this follows from (6) and Sylvester's law of inertia. $\square$

Corollary 3.5. *Over $\mathbb{Z}$,*

$$\text{SNF}(L_M) = \text{diag}(\underbrace{1, \dots, 1}_{m+1}, \underbrace{0, \dots, 0}_{n-1}), \quad (10)$$

and consequently $\text{coker}(L_M: \mathbb{Z}^{n+m} \rightarrow \mathbb{Z}^{n+m}) \cong \mathbb{Z}^{n-1}$ is torsion free.

Proof. Both P and $P^\top$ are unimodular, so (6) is an equivalence over $\mathbb{Z}$. The Smith normal form of J is $\text{diag}(1, 0, \dots, 0)$ and that of $-I_m$ is I_m . $\square$

Remark 3.6. For $\text{rk } M = r \geq 4$ a piecewise linear function is still recorded by its ray values, but its divisor is indexed by the codimension-one cones of the fan, which are flags of flats rather than rays. The matrix of the divisor map is generally rectangular and has no natural symmetry of the kind used above, so the operator studied here is a rank-three phenomenon.

We now describe the image of L_M . Call a vector $w = (p, q) \in \mathbb{R}^n \oplus \mathbb{R}^m$, with p_i on the point ray ρ_i and q_L on the line ray ρ_L , a ray weight. It is a one-dimensional tropical cycle when it balances at the origin, that is, when $\sum_i p_i \bar{e}_i + \sum_L q_L \bar{e}_L = 0$ in $N_{\mathbb{R}}$. Since $\bar{e}_L = \sum_{i \in L} \bar{e}_i$, and since a vector of $\mathbb{R}^E$ vanishes in $\mathbb{R}^E / \mathbb{R}\mathbf{1}$ if and only if it is constant, the balancing condition reads

$$p + Bq = c\mathbf{1} \quad \text{for some } c \in \mathbb{R}. \quad (11)$$

Theorem 3.7. *For a ray weight $w = (p, q)$ the following are equivalent:*

- (i) $w = \text{div}(\varphi)$ for some piecewise linear function φ on Δ_M ;
- (ii) w is balanced;
- (iii) $p + Bq = c\mathbf{1}$ for some $c \in \mathbb{R}$.

If p and q are integral, then φ may be taken integral.

Proof. Writing $L_M(x, y) = (p, q)$ gives $q = B^\top x - y$ and $p = By - (R - I)x$, whence

$$p + Bq = BB^\top x - (R - I)x = Jx = (\mathbf{1}^\top x)\mathbf{1}$$

by (1). This shows that (i) implies (iii), and (ii) is equivalent to (iii) by (11). Conversely, assume $p + Bq = c\mathbf{1}$, choose x with $\mathbf{1}^\top x = c$ and put $y = B^\top x - q$. Then $B^\top x - y = q$ and

$$By - (R - I)x = B(B^\top x - q) - (R - I)x = (J + R - I)x - (R - I)x - Bq = c\mathbf{1} - Bq = p,$$

so $L_M(x, y) = (p, q)$. If p and q are integral then so is c , and the choice $x = (c, 0, \dots, 0)^\top$ makes y integral as well. $\square$

Theorem 3.7 is the rank-three coordinate form of the duality of Gross and Shokrieh [8] mentioned in Section 1: every balanced ray weight on Δ_M is a principal divisor, and by Corollary 3.3 its potential is unique modulo global linear functions.

Remark 3.8. The proof gives an explicit integral solution. If (p, q) is integral and balanced, write $p + Bq = c\mathbf{1}$ with $c \in \mathbb{Z}$, put $x = (c, 0, \dots, 0)^\top$ and $y = B^\top x - q$; then $L_M(x, y) = (p, q)$. It also gives Corollary 3.5 without any computation of Smith normal forms: the homomorphism $\mathbb{Z}^{n+m} \rightarrow \mathbb{Z}^n / \mathbb{Z}\mathbf{1}$ sending (p, q) to the class of $p + Bq$ is surjective, and by Theorem 3.7 its kernel is $\text{im } L_M$, so $\text{coker } L_M \cong \mathbb{Z}^n / \mathbb{Z}\mathbf{1} \cong \mathbb{Z}^{n-1}$.

We can now answer question (3) of Section 1. Recall that the curvature of φ is $\mathcal{K}(\varphi) = -L_M(x, y)$, and call it constant if it equals $\lambda\mathbf{1}_{n+m}$ in the primitive ray basis of Δ_M .

Theorem 3.9. *Let M be a simple matroid of rank three and consider the equation*

$$-L_M \begin{pmatrix} x \\ y \end{pmatrix} = \lambda\mathbf{1}_{n+m}. \quad (12)$$

If $r_1 = \dots = r_n$, then (12) is solvable for every $\lambda \in \mathbb{R}$. If the degrees are not all equal, then (12) is solvable only for $\lambda = 0$. Whenever (12) is solvable, its solutions form a coset of the space of globally linear functions, and for integral λ one of them is integral.

Proof. The prescribed weight is $(p, q) = (-\lambda\mathbf{1}_n, -\lambda\mathbf{1}_m)$, and since $r = B\mathbf{1}_m$ we get $p + Bq = -\lambda(\mathbf{1} + r)$. By Theorem 3.7 the equation is solvable if and only if this vector is constant, that is, if and only if $\lambda = 0$ or r is constant. For $\lambda = 0$ every globally linear function is a solution. The description of the solution set is Corollary 3.3 and the integrality statement is the last clause of Theorem 3.7. $\square$

We call M line-regular of degree ρ if $r_i = \rho$ for every point i . The lines of a line-regular matroid need not all have the same size; the condition constrains the number of lines through a point, not the number of points on a line. The graphic matroid of K_4 , discussed in Section 4, is line-regular with lines of two sizes.

Proposition 3.10. *Let M be line-regular of degree ρ and let $\lambda \in \mathbb{R}$. The unique solution of (12) orthogonal to $\ker L_M$ is*

$$x_i = -\frac{\lambda(\rho+1)}{n}, \quad y_L = \lambda - \frac{\lambda(\rho+1)|L|}{n}, \quad (13)$$

and it is the solution of least Euclidean norm.

Proof. Put $(p, q) = (-\lambda\mathbf{1}_n, -\lambda\mathbf{1}_m)$, so that $p + Bq = -\lambda(\mathbf{1} + r) = -\lambda(\rho+1)\mathbf{1}$. The vector x of (13) satisfies $\mathbf{1}^\top x = -\lambda(\rho+1)$, and $y = B^\top x + \lambda\mathbf{1}_m = B^\top x - q$. This is the pair constructed in the second half of the proof of Theorem 3.7, so $L_M(x, y) = (p, q)$, which is (12). Any other solution differs from it by an element $(a, B^\top a)$ of $\ker L_M$, and

$$\langle (x, y), (a, B^\top a) \rangle = a^\top (x + By).$$

Here x is constant, and so is $By = BB^\top x + \lambda B\mathbf{1}_m$, because $BB^\top = J + (\rho-1)I$ and $B\mathbf{1}_m = r = \rho\mathbf{1}$. As $\mathbf{1}^\top a = 0$, the inner product vanishes, and orthogonality to the kernel characterizes the representative of least norm. $\square$

Remark 3.11. If one starts from a reference potential φ_0 and allows an arbitrary piecewise linear change u , then $\varphi = \varphi_0 + u$ runs over all piecewise linear functions, so the criterion of Theorem 3.9 does not depend on φ_0 . An analogue of a conformal class would require restricting the admissible u ; see Problem 2.

The congruence (6) determines the inertia of L_M but not its individual eigenvalues, which we compute next.

Theorem 3.12. Put $\chi_M(t) := \det(tI_{n+m} - L_M)$. Then $m \geq n$ and

$$\chi_M(t) = t^{n-1}(t+1)^{m-n} \left[t \prod_{i=1}^n (t+r_i) - \sum_{i=1}^n \prod_{j \neq i} (t+r_j) \right]. \quad (14)$$

In particular, two simple rank-three matroids with the same number of lines and the same multiset of degrees have the same characteristic polynomial.

Proof. The inequality $m \geq n$ is the de Bruijn–Erdős theorem for finite linear spaces [4]. Here it also follows directly from the incidence identity: every $r_i \geq 2$ by Lemma 2.1(iv), so $BB^\top = J + R - I$ is positive definite; hence B has row rank n and $m \geq n$. For $t \neq -1$ the Schur complement of the lower right block of $tI - L_M$ gives

$$\begin{aligned} \chi_M(t) &= (t+1)^m \det \left(tI + R - I - \frac{1}{t+1} BB^\top \right) \\ &= (t+1)^{m-n} \det \left((t+1)(tI + R - I) - BB^\top \right) \\ &= (t+1)^{m-n} \det(t(tI + R) - J), \end{aligned}$$

the last equality by (1). Both sides are polynomials in t , so the identity extends to $t = -1$. Applying the matrix determinant lemma to the diagonal matrix $t(tI + R)$,

$$\det(t(tI + R) - J) = t^n \prod_i (t+r_i) \left(1 - \frac{1}{t} \sum_i \frac{1}{t+r_i} \right) = t^{n-1} \left[t \prod_i (t+r_i) - \sum_i \prod_{j \neq i} (t+r_j) \right],$$

The determinant-lemma calculation is initially valid away from $t = 0$ and $t = -r_i$; polynomial identity extends it to those values as well. This proves (14). $\square$

Corollary 3.13. If M is line-regular of degree ρ , then

$$\text{Spec}(L_M) = \left\{ \frac{-\rho + \sqrt{\rho^2 + 4n}}{2}, \frac{-\rho - \sqrt{\rho^2 + 4n}}{2}, 0^{(n-1)}, (-\rho)^{(n-1)}, (-1)^{(m-n)} \right\}. \quad (15)$$

Proof. Substituting $r_i = \rho$ in (14) gives $\chi_M(t) = t^{n-1}(t+1)^{m-n}(t+\rho)^{n-1}(t^2 + \rho t - n)$, and the two roots of the last factor are the first two entries of (15). $\square$

Example 3.14. For the uniform matroid $U_{3,3}$ the lines are the three two-element subsets of E , so the incidence graph is the six-cycle $1 - \{12\} - 2 - \{23\} - 3 - \{13\} - 1$ of Figure 2. Here $n = m = 3$ and $\rho = 2$, so $\text{Spec}(L_M) = \{1, -3, 0, 0, -2, -2\}$ by (15). In this example $L_M = A(C_6) - I$, whereas the Laplacian of the hexagon is $2I - A(C_6)$, whose spectrum is $\{0, 1, 1, 3, 3, 4\}$. The all-ones vector is an eigenvector of L_M for the eigenvalue 1 and of the Laplacian for the eigenvalue 0, and on its orthogonal complement the Laplacian is positive definite while L_M is negative semidefinite with a two-dimensional kernel.

It remains to treat question (4). If M is not line-regular, the constant ray weight is unbalanced and is not a curvature, and we measure the failure in the Euclidean inner product on primitive ray coordinates. This norm is a coordinate normalization attached to the fixed fine fan; we do not claim that the resulting defect is invariant under refinement.

Definition 3.15. The defect of M is

$$\mathcal{Y}(M) := \text{dist}(\mathbf{1}_{n+m}, \text{im } L_M), \quad (16)$$

so that $|\lambda| \mathcal{Y}(M)$ is the least residual in the Euclidean norm when one attempts to prescribe the curvature $\lambda \mathbf{1}_{n+m}$.

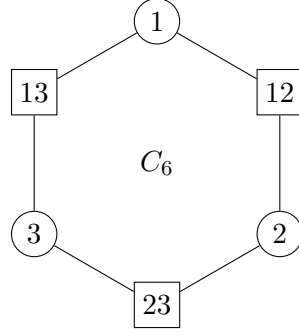


Figure 2: The incidence graph of $U_{3,3}$, for which $L_M = A(C_6) - I$.

Theorem 3.16. *Let A and H be the arithmetic and the harmonic mean of the degrees $r_1, \dots, r_n$. Then*

$$\mathcal{Y}(M)^2 = \sum_{i=1}^n r_i - \frac{n^2}{\sum_{i=1}^n 1/r_i} = n(A - H) = \left(\sum_{i < j} \frac{(r_i - r_j)^2}{r_i r_j} \right) / \sum_{i=1}^n \frac{1}{r_i}. \quad (17)$$

In particular $\mathcal{Y}(M) = 0$ if and only if M is line-regular.

Proof. As L_M is symmetric, $\text{im } L_M = (\ker L_M)^\perp$, so $\mathcal{Y}(M)$ is the norm of the orthogonal projection of $\mathbf{1}_{n+m}$ onto $\ker L_M$. Write $h(a) = (a, B^\top a)$ for a in $H_0 := \{a \in \mathbb{R}^n : \mathbf{1}^\top a = 0\}$, as in Corollary 3.3. For $a, b \in H_0$, (1) gives

$$\langle h(a), h(b) \rangle = a^\top (I + BB^\top) b = a^\top (R + J) b = a^\top R b, \quad \langle \mathbf{1}_{n+m}, h(a) \rangle = \mathbf{1}^\top a + \mathbf{1}^\top B^\top a = r^\top a,$$

where $a^\top J b = (\mathbf{1}^\top a)(\mathbf{1}^\top b) = 0$ was used. The projection is $h(c)$ with $c \in H_0$ determined by $c^\top R a = r^\top a$ for all $a \in H_0$, that is, by $Rc - r = \mu \mathbf{1}$ for some $\mu \in \mathbb{R}$. Hence $c_i = 1 + \mu/r_i$, and $\sum_i c_i = 0$ forces $\mu = -n / \sum_i (1/r_i)$. Writing $S = \sum_i (1/r_i)$,

$$\mathcal{Y}(M)^2 = \|h(c)\|^2 = c^\top R c = \sum_i r_i + 2\mu n + \mu^2 S = \sum_i r_i - \frac{n^2}{S},$$

which is the first equality in (17), and $n(A - H)$ is a rewriting of it. For the third expression, use $(\sum_i r_i)S - n^2 = \sum_{i < j} (r_i - r_j)^2 / (r_i r_j)$ and divide by S . The last assertion is the equality case of the inequality $A \geq H$. $\square$

The right-hand side of (17) depends only on the multiset of degrees. Like the characteristic polynomial, therefore, the defect does not see the incidence pattern of M , and two matroids can differ in every line while sharing both invariants.

4 Enumeration and extremal examples

We first record the defect of the near-pencils, which occur repeatedly below.

Example 4.1. For $n \geq 4$, let N_n be the near-pencil, that is, the matroid with one line carrying $n - 1$ points and one further point off it. The points on the long line have degree 2 and the remaining point has degree $n - 1$, so (17) gives

$$\mathcal{Y}(N_n)^2 = \frac{(n-1)(n-3)^2}{n^2 - 2n + 3}, \quad (18)$$

which equals $3/11$ for $n = 4$.

Table 1: Exhaustive census of labeled simple rank-three matroids on $3 \leq n \leq 7$ points. The column “regular types” counts isomorphism classes among the line-regular matroids.

n	all labeled	line-regular	regular types	distinct $\mathcal{Y}^2$	max $\mathcal{Y}^2$	maximizers
3	1	1	1	1	0	$U_{3,3}$
4	5	1	1	2	3/11	4 near-pencils
5	31	1	1	4	8/9	5 near-pencils
6	352	41	3	7	5/3	6 near-pencils
7	8,389	31	2	22	48/19	7 near-pencils

By Lemma 2.2, a labeled simple rank-three matroid on $\{1, \dots, n\}$ is the same thing as a family of long lines, so the matroids on a fixed small number of points can be enumerated directly. Encode a candidate long line, that is, a subset of $\{1, \dots, n\}$ of size between 3 and $n - 1$, by the bit mask of the pairs of points it contains. Run through the candidates in a fixed order and either discard a candidate or, if its mask is disjoint from the union of the masks already selected, adjoin it. At the end of a branch declare every pair not covered so far to be a line. Each labeled matroid is produced exactly once.

The enumeration uses exact integer and rational arithmetic. For every generated matroid, the program checks that each pair of points lies on exactly one line and evaluates (17) as an exact fraction. For the five examples in Table 2, it also checks the incidence identity, the congruence, the characteristic polynomial, the rank, the nullity and the Smith normal form by exact symbolic computation. The inertia is checked separately using numerical eigenvalues; its exact justification is the congruence proved in Theorem 3.2. Table 1 records the enumeration results. The labeled totals agree with the published counts of Dukes [5], providing an independent check on those totals.

The line-regular isomorphism classes can be listed explicitly. For $n = 3, 4, 5$, the only type is the uniform matroid $U_{3,n}$. For $n = 6$, there are three types: $U_{3,6}$; the matroid with two disjoint three-point lines and all remaining lines of size two, in which every point has degree 4; and the graphic matroid $M(K_4)$, in which every point has degree 3. For $n = 7$, the two types are $U_{3,7}$ and the Fano plane.

For $4 \leq n \leq 7$, the maximum defect is attained by exactly n labeled matroids. These are the near-pencils, one for each choice of the point off the long line. The isomorphism test and the supporting computational files are described in Appendix A.

One might read Table 1 as evidence that near-pencils maximize the defect for every n . The following family shows that this fails already at $n = 8$.

Definition 4.2. For $n \geq 4$ and $1 \leq s \leq n - 3$ let $M_{n,s}$ be the simple rank-three matroid with one line of $n - s$ points and with every other line of size two, so that s points lie off the long line. Thus $M_{n,1} = N_n$.

The $n - s$ points on the long line have degree $s + 1$ and each of the remaining points has degree $n - 1$, so (17) applies.

Theorem 4.3. For $1 \leq s \leq n - 3$,

$$\mathcal{Y}(M_{n,s})^2 = \frac{s(n-s)(n-s-2)^2}{n^2 - ns - n + s^2 + 2s}. \quad (19)$$

Moreover, for every $n \geq 5$, $\mathcal{Y}(M_{n,2})^2 > \mathcal{Y}(M_{n,1})^2$ holds if and only if $n \geq 8$, and at $n = 8$

$$\mathcal{Y}(M_{8,2})^2 = 4 > \frac{175}{51} = \mathcal{Y}(M_{8,1})^2.$$

Proof. The degree sequence of $M_{n,s}$ is $(s+1)^{n-s}, (n-1)^s$, so

$$\sum_i r_i = (n-s)(s+1) + s(n-1), \quad \sum_i \frac{1}{r_i} = \frac{n-s}{s+1} + \frac{s}{n-1} = \frac{n^2 - ns - n + s^2 + 2s}{(s+1)(n-1)}.$$

Substituting into (17) and simplifying gives (19). Taking $s = 1$ and $s = 2$ in (19) and subtracting,

$$\mathcal{Y}(M_{n,2})^2 - \mathcal{Y}(M_{n,1})^2 = \frac{P(n)}{(n^2 - 3n + 8)(n^2 - 2n + 3)},$$

where $P(n) = n^5 - 14n^4 + 66n^3 - 142n^2 + 173n - 120$ and both factors in the denominator are positive. For $n = 5, 6, 7$ one computes $P(n) = -180, -306, -36$, while for $n \geq 8$ the substitution $q = n - 8 \geq 0$ gives

$$P(n) = q^5 + 26q^4 + 258q^3 + 1186q^2 + 2381q + 1392 > 0. \quad \square$$

Near-pencil extremality, valid throughout Table 1, therefore fails at $n = 8$. Within the family of Definition 4.2 the optimal s is not eventually equal to 2 either; it grows linearly in n with an irrational slope.

Theorem 4.4. *For each $n \geq 4$ let $s_n \in \{1, \dots, n-3\}$ maximize $\mathcal{Y}(M_{n,s})^2$. Then $s_n/n \rightarrow \alpha_0$, where α_0 is the unique root in $(0, 1)$ of*

$$2\alpha^3 - 2\alpha^2 + 4\alpha - 1 = 0, \quad \alpha_0 = 0.2778800911\dots, \quad (20)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \max_{1 \leq s \leq n-3} \mathcal{Y}(M_{n,s})^2 = \frac{\alpha_0(1-\alpha_0)^3}{\alpha_0^2 - \alpha_0 + 1} = 0.1309047286\dots \quad (21)$$

Proof. Put $\alpha = s/n$. Dividing (19) by n^2 gives

$$\frac{\mathcal{Y}(M_{n,s})^2}{n^2} = \frac{\alpha(1-\alpha)(1-\alpha-2/n)^2}{\alpha^2 - \alpha + 1 + (2\alpha - 1)/n}.$$

For $n \geq 4$ and $0 \leq \alpha \leq 1$, the denominator satisfies

$$\alpha^2 - \alpha + 1 + \frac{2\alpha - 1}{n} = \left(\alpha - \frac{1}{2} + \frac{1}{n}\right)^2 + \frac{3}{4} - \frac{1}{n^2} \geq \frac{11}{16}.$$

The numerator and denominator converge uniformly on $[0, 1]$, and the denominators are bounded away from zero. Hence the displayed rational functions converge uniformly to

$$f(\alpha) = \frac{\alpha(1-\alpha)^3}{\alpha^2 - \alpha + 1}.$$

Differentiating,

$$f'(\alpha) = -\frac{(1-\alpha)^2(2\alpha^3 - 2\alpha^2 + 4\alpha - 1)}{(\alpha^2 - \alpha + 1)^2}.$$

The cubic factor is strictly increasing, since its derivative $6\alpha^2 - 4\alpha + 4$ has negative discriminant, and it is negative at $\alpha = 0$ and positive at $\alpha = 1$. Hence f increases and then decreases on $[0, 1]$ and has a unique maximizer α_0 there, characterized by (20); the cubic has no rational root, so α_0 is irrational. Uniform convergence on the grid $\{s/n\}$, whose mesh tends to 0, gives $s_n/n \rightarrow \alpha_0$ and (21). $\square$

Remark 4.5. Theorems 4.3 and 4.4 optimize inside one explicit family. The maximum of $\mathcal{Y}(M)$ over all simple rank-three matroids on n points is not determined here; see Problem 1.

Table 2: Five representative simple rank-three matroids. A nonzero constant curvature is attainable exactly in the line-regular rows. Superscripts in the degree and line-size columns indicate multiplicities.

matroid	n	m	degrees	$\mathcal{Y}^2$	line sizes	nonzero constant?
$U_{3,4}$	4	6	3^4	0	2^6	yes
near-pencil N_4	4	4	$2^3, 3$	3/11	$2^3, 3$	no
$M_{8,2}$	8	14	$3^6, 7^2$	4	$2^{13}, 6$	no
graphic $M(K_4)$	6	7	3^6	0	$2^3, 3^4$	yes
Fano plane	7	7	3^7	0	3^7	yes

Table 2 collects the invariants of five matroids, which we describe in turn.

For the uniform matroid $U_{3,n}$ every line has two points, $m = \binom{n}{2}$ and every degree equals $n - 1$, so every constant is a curvature by Theorem 3.9. The quadratic factor in the proof of Corollary 3.13 is $t^2 + (n - 1)t - n = (t - 1)(t + n)$, so

$$\text{Spec}(L_{U_{3,n}}) = \{1, -n, 0^{(n-1)}, (-(n-1))^{(n-1)}, (-1)^{\binom{n}{2}-n}\}.$$

The near-pencil N_4 has degree sequence $(2, 2, 2, 3)$, so no nonzero constant is a curvature, and by (18) the least squared residual for unit curvature is 3/11. Here $m = n = 4$ and (14) gives

$$\chi_{N_4}(t) = t^3(t + 2)^2(t^3 + 5t^2 + 2t - 11).$$

By Descartes' rule of signs the cubic factor has exactly one positive root, as the inertia in Corollary 3.4 requires, and that root lies between 1 and 2 because the cubic takes the values -3 and 21 there.

The matroid $M_{8,2}$ has one six-point line and two points off it, hence degree sequence $(3^6, 7^2)$ and $\mathcal{Y}^2 = 4$ by (17). By Theorem 4.3 it is the smallest member of the family of Definition 4.2 whose defect exceeds that of the near-pencil on the same number of points.

In the graphic matroid $M(K_4)$ the points are the six edges of K_4 and the lines are the four triangles and the three pairs of opposite edges. Every edge lies in two triangles and one pair of opposite edges, so every degree is 3 and every constant is a curvature, although the line sizes are not all equal. For $\lambda \neq 0$ formula (13) assigns the value $-\lambda/3$ to the rays of the three two-point lines and $-\lambda$ to the rays of the four triangles.

The Fano plane has seven points and seven three-point lines, and every degree is 3, so $\mathcal{Y} = 0$. Its eigenvalues are $(-3 \pm \sqrt{37})/2$ together with -3 and 0 , each of multiplicity six, and these account for all fourteen.

5 Conclusion

For a simple matroid of rank three the identity $BB^\top = J + R - I$ converts the divisor matrix into the congruence (6), which gives its kernel, image, inertia and Smith normal form. A separate Schur-complement calculation yields the characteristic polynomial of L_M . A ray weight is a divisor if and only if it is balanced, so a nonzero constant curvature exists if and only if every point of M lies on the same number of lines. When it does not, the least squared residual is n times the gap between the arithmetic and the harmonic mean of the degree sequence.

The enumeration of Section 4 evaluates the proved defect formula on all 8,778 labeled simple rank-three matroids on at most seven points. It shows that, for $4 \leq n \leq 7$, the near-pencils, and only they, maximize the defect.

Several features of the Yamabe problem are absent from this model. The equation is linear, whereas the Yamabe equation is nonlinear and variational. The word constant refers to equal

weights in the primitive ray basis of the fine fan, and is not invariant under refinement of the fan. By Remark 3.11 all piecewise linear changes of potential are admitted, so there is a single additive class and no analogue of a conformal class. Finally, $\mathcal{K}(\varphi)$ is defined through the divisor operator and is not derived from a metric.

Three problems remain open.

Problem 1. Determine $\max_M \mathcal{V}(M)^2$ over all simple rank-three matroids on n points. Is a matroid $M_{n,s}$ of Definition 4.2 always extremal, or do configurations with several long lines do better for large n ?

Problem 2. Find a natural proper subset of the space of piecewise linear potentials, for instance one cut out by an ample class, by a symmetry of M , by a condition on supports or by an energy bound, for which the reference potential of Remark 3.11 becomes relevant. Is the corresponding prescription problem solvable, and does it acquire a variational structure?

Problem 3. For $\text{rk } M \geq 4$, write the divisor map explicitly in flag coordinates, find factorizations analogous to (6), and compute, with respect to a specified inner product, distances from natural prescribed codimension-one weight vectors, not necessarily balanced, to its image.

One further question is suggested by Theorem 3.12. In rank three the spectrum of L_M retains only the number of lines and the multiset of degrees, and forgets the incidence pattern; the same is true of the defect. It would be interesting to know which operators with a comparable tropical meaning separate non-isomorphic matroids with the same degree sequence. The Chow ring of a matroid carries much finer structure, by the work of Adiprasito, Huh and Katz [1], and is a natural place to look for such operators.

Appendix A Computational reproducibility

The script `verify_rank3.py` requires Python 3.10 or later and SymPy. It uses labels $\{0, \dots, n-1\}$ rather than $\{1, \dots, n\}$; this change does not affect the results. The computations can be reproduced by running:

```
python verify_rank3.py
```

Enumeration. For each $3 \leq n \leq 7$, every subset of size between 3 and $n-1$ is a candidate long line. Its contained point-pairs are encoded by a bit mask. The program recursively excludes or includes each candidate, allowing inclusion only when its mask is disjoint from all previously selected masks. At each terminal branch, uncovered pairs are added as two-point lines. Lemma 2.2 shows that this procedure produces each labeled simple rank-three matroid exactly once. The pair-cover condition is checked for every output, and the defect formula is evaluated using exact rational arithmetic.

Isomorphism classes. For each line-regular matroid, the function `canonical_long_lines` examines all $n!$ permutations of the point labels. Under each permutation, the points of each long line are sorted, and the long lines are ordered by size and then lexicographically. The least resulting tuple is taken as the canonical form. Two matroids have the same canonical form if and only if they are isomorphic, because the long lines determine all remaining two-point lines. Counting distinct canonical forms gives the “regular types” column of Table 1.

Verification scope. For all 8,778 enumerated objects, the script checks the pair-cover condition, computes degrees and evaluates $\mathcal{V}(M)^2$ from (17). This evaluation is not an independent least-squares verification of the defect formula. Exact symbolic matrix checks are performed for the five representative examples in Table 2. A separate numerical inertia check evaluates eigenvalues to 40-digit precision and uses a sign threshold of 10^{-12} . For the one-long-line family, the script checks every $4 \leq n \leq 50$ and $1 \leq s \leq n-3$, comparing the degree-based defect with (19) and recording all maximizing values of s . These finite checks support, but do not replace, the general proofs in the main text.

Supporting files. The accompanying computational materials are:

- `verify_rank3.py`: enumeration and verification code;
- `enumeration_summary.csv`: labeled totals, regular counts, isomorphism counts and extrema;
- `defect_distribution.csv`: exact defect values and their labeled multiplicities;
- `examples_summary.csv`: invariants of the five representative examples;
- `one_long_line_family.csv`: family extrema and near-pencil comparisons through $n = 50$;
- `verification_report.txt`: a run summary including the SHA-256 hash of the script.

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This project grew out of my interest in matroid theory and tropical geometry. After learning about Bergman fans and the classical Yamabe problem, I became curious about whether a similar curvature prescription problem could be studied in a tropical setting. Following discussions with my mathematics teacher and research advisor Ruiwen An, I decided to focus on rank-three matroid fans, where the divisor operator has a clear combinatorial structure.

Ruiwen An is my mathematics teacher at Shanghai Pinghe School and worked as my research advisor. She provided all guidance with no compensation. She offered help with the project's mathematical background and theoretical framework, and gave advice on paper organization, academic writing, and presentation. I was fully responsible for defining the specific research problem, completing all mathematical derivations and proofs, running computer calculations and enumerations, verifying results, and writing the final manuscript.

A major challenge was finding a form of the divisor operator that could be directly analyzed. I solved this by expressing the operator through the point–line incidence matrix of a rank-three matroid. This method became the core tool for studying the kernel, image, inertia, Smith normal form, constant-curvature condition, and defect. All rank-three calculations and computational work in this paper were done independently by me, and I personally checked all final arguments and results.

With my advisor's permission, I used ChatGPT as an auxiliary tool while writing and revising the paper. I used it in multiple sessions between June 21 and September 15, 2026. Exported records show the recorded sessions used ChatGPT 5.6, though I did not keep the exact model version for every session. ChatGPT helped with mathematical consistency checks, calculation verification, code debugging, literature search leads, language editing, structural suggestions, LaTeX formatting, citation checking, and submission format review. Specifically, it helped spot possible sign or indexing errors, cross-check calculations with exact linear algebra, point out statements that needed more justification, and improve the clarity and structure of the paper.

All AI responses were treated only as suggestions, and were never accepted as proofs or authoritative sources. I independently verified all mathematical arguments, reran all relevant computations, and checked all references against their original sources before including them. The complete AI chat records are submitted to the organizers as supplementary materials. I take full responsibility for all final proofs, calculations, conclusions, and contribution descriptions in this paper.