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Hilbert–Samuel Functions and Multiplicity of Plane Curves

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Abstract

This report studies the Hilbert–Samuel function of a plane curve at the origin through two problems. The first problem relates the differences of these dimensions to the multiplicity r of f , giving $h_f(n) = \min\{n + 1, r\}$, and illustrates the formula with explicit computations. The second problem compares the same three polynomials of multiplicity 2. Each gives the same dimension sequence $1, 3, 5, 7, \dots$, despite defining different curves. These computations show the limitations of the one-curve Hilbert–Samuel function as it displays the independent direction at the origin for a specific dimension.

Keywords: Hilbert–Samuel function; plane curves; multiplicity; quotient spaces; truncation.

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1 Introduction

This project studies the Hilbert–Samuel function of a single plane curve at the origin. Through proofs and computations, we investigate how the multiplicity of a polynomial determines its Hilbert–Samuel function and whether this function can distinguish different curves.

In Problem 1, we prove the one-curve formula and compute examples for

$$f = x, \quad f = y^2 - x^4, \quad f = x^2 + y^5.$$

In Problem 2, we compare

$$f = xy, \quad f = y^2 - x^3, \quad f = y^2 - x^4, \quad f = x^2 + y^5.$$

These polynomials all have multiplicity 2 at the origin. We show that they have the same Hilbert–Samuel function, despite defining different curves.

2 Definitions and Setup

Let k be a field, and let

$$R = k[x, y], \quad \mathfrak{m} = (x, y).$$

The ideal $\mathfrak{m}$ consists of polynomials that vanish at the origin. For each integer $n \geq 0$,

$$\mathfrak{m}^n = \text{span}_k \{x^i y^j \mid i, j \geq 0, i + j \geq n\}.$$

For a nonzero polynomial f satisfying $f(0, 0) = 0$, its multiplicity at the origin, denoted by $\text{mult}_0(f)$, is the smallest total degree of a term with a nonzero coefficient in f .

2.1 Hilbert–Samuel Functions

For an ideal $I \subseteq \mathfrak{m}$, we define

$$H_I(n) = \dim_k \left(\frac{R}{I + \mathfrak{m}^{n+1}} \right), \quad n \geq 0.$$

This counts the independent polynomial classes remaining after imposing the relations in I and discarding terms of total degree greater than n .

For a single curve $f = 0$, we write

$$H_f(n) = \dim_k \left(\frac{R}{(f) + \mathfrak{m}^{n+1}} \right).$$

We also define

$$h_f(n) = H_f(n) - H_f(n-1),$$

with $H_f(-1) = 0$. Thus $h_f(n)$ counts the increase in dimension from one truncation degree to the next.

3 Main Method

We compute $H_f(n)$ by first keeping only monomials of total degree at most n , and then imposing the relations generated by f . The dimension of the resulting quotient gives $H_f(n)$.

In Problem 1, we use the lowest-degree nonzero part of f to prove

$$h_f(n) = \min\{n + 1, \text{mult}_0(f)\}.$$

We then check this formula through explicit examples.

In Problem 2, we apply the formula to four polynomials of multiplicity 2 and compare their Hilbert–Samuel functions.

3.1 Truncation and Relation Matrices

Let

$$V_n = \text{span}_k\{x^i y^j \mid i, j \geq 0, i + j \leq n\}.$$

Truncation removes all terms of total degree greater than n , so

$$R/\mathfrak{m}^{n+1} \cong V_n$$

as vector spaces, and

$$\dim_k V_n = \binom{n+2}{2}.$$

Let $r = \text{mult}_0(f)$. For $n \geq r$, we truncate the multiples

$$x^a y^b f, \quad a, b \geq 0, \quad a + b \leq n - r.$$

Their coefficient vectors, written in a fixed monomial basis of V_n , form the rows of a relation matrix $M_f(n)$. These rows span the relations imposed by (f) after truncation.

Therefore,

$$H_f(n) = \binom{n+2}{2} - \text{rank } M_f(n).$$

When $n < r$, there are no nonzero relations after truncation, so

$$H_f(n) = \binom{n+2}{2}.$$

4 Problem 1

4.1 Statement

For a nonzero polynomial $f \in \mathfrak{m}$ of multiplicity r and every integer $n \geq 0$,

$$H_f(n) = \begin{cases} \binom{n+2}{2}, & 0 \leq n < r, \\ \binom{n+2}{2} - \binom{n-r+2}{2}, & n \geq r, \end{cases} \quad h_f(n) = \min\{n + 1, r\}.$$

4.2 Proof

Proof. Let (f) be an ideal of the function $f(x, y)$, where $f \in R$ and $R = k[x, y]$, $\mathfrak{m} = (x, y)$ and $\text{Let } f(0, 0) = 0$

We let $H_f(n)$ be the $\dim_k R/((f) + \mathfrak{m}^{n+1})$

$$\mathfrak{m}^{n+1} = \text{span}_k\{x^i y^j \mid i, j \geq 0, i + j \geq n + 1\}$$

And to simplify this $\mathfrak{m}^{n+1}$, we look at the example of

$$W = \left\{ \begin{pmatrix} 0 \\ 0 \\ a \\ b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

Here $V = \mathbb{R}^4$ and W is a subspace of V ; the quotient space is V/W . The map $[u] \mapsto (u_1, u_2)$ identifies this quotient with $\mathbb{R}^2$.

Let W be the Quotient Space in $\mathbb{R}^4$ and $u, v \in \mathbb{R}^4$

$$V/(W) = \{[u_i] \mid u_i \in \mathbb{R}^4\}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$u - v \in W \iff u_1 = v_1, u_2 = v_2$$

$$\implies u \sim v \implies [u] = [v]$$

Thus we can simplify $V/(W)$ because $\forall [u_i]$, where the first two elements are the same are equal.

$$V/(W) \cong \mathbb{R}^2$$

Where now the Dimension of $V/(W)$ is 2 instead of 4, we can apply the same logic to our quotient

$$R/(\mathfrak{m}^{n+1})$$

where we can take the basis of

$$\mathfrak{m}^{n+1} = \text{span}\{x^i y^j \mid i + j \geq n + 1\}$$

so we can deter that

$$\dim R/(\mathfrak{m}^{n+1}) = \dim \text{span}\{x^i y^j \mid i + j \leq n\}$$

And then we impose the ideal (f) on the Quotient Space to study the direction of the curve of the function f in the origin, by doing so, we shrink the degree of f down to at most n degree and study the changes degree by degree whilst the quotient reduces the set to it a span of it's basis to reveal every distinct direction of f

$$\text{Let } r = \text{mult}_0(f)$$

$$f = f_r + f_{r+1} + \dots + f_{\deg f}$$

We then define $h_f(n) = H_f(n) - H_f(n-1)$, where we ask ourselves how many new independent terms are created and have survived.

We will separate this into two cases: when $\text{mult}_0 f > n$, or when $\text{mult}_0 f \leq n$

If $\text{mult}_0 f > n$:

Imposing (f) on quotient space $\mathfrak{m}^{n+1}$ or $\mathfrak{m}^{n+1}$ doesn't change the quotient, because of that the basis for $\mathfrak{m}^{n+1}$ and $\mathfrak{m}^{n+1}$ would compose the ideal (f) .

Hence

$$h_f(n) = \dim R/(\mathfrak{m}^{n+1}) - \dim R/(\mathfrak{m}^n)$$

and as we recall

$$\dim R/(\mathfrak{m}^{n+1}) = \binom{n+2}{2}$$

$$h_f(n) = \binom{n+2}{2} - \binom{n+1}{2} = \frac{(n+2)(n+1)}{2} - \frac{(n+1)n}{2} = n+1$$

Case 2: $\text{mult}_0 f \leq n$

Here we know that since $\text{mult}_0 f \leq n$, the basis of $\mathfrak{m}^{n+1}$ is no longer sufficient to cover (f) , so we need the $\text{mult}_0 f$, and since r here is the smallest degree in f . Here then we can infer that the amount of degrees that are newly imposed to the quotient space is

$$\dim R/(\mathfrak{m}^{n-r+1})$$

So the remaining independent dimensions would be

$$\begin{aligned} H_f(n) &= \dim R/(\mathfrak{m}^{n+1}) - \dim R/(\mathfrak{m}^{n-r+1}) = \binom{n+2}{2} - \binom{n-r+2}{2} \\ &= \frac{(n+2)(n+1)}{2} - \frac{(n-r+2)(n-r+1)}{2} = \frac{2nr - r^2 + 3r}{2} \\ H_f(n-1) &= \binom{n+1}{2} - \binom{n-r+1}{2} \\ &= \frac{(n+1)(n)}{2} - \frac{(n-r+1)(n-r)}{2} = \frac{2nr - r^2 + r}{2} \\ h_f(n) &= \frac{2nr - r^2 + 3r}{2} - \frac{2nr - r^2 + r}{2} = \frac{2r}{2} = r \end{aligned}$$

There for we can conclude that

$$h_f(n) = \begin{cases} n+1, & n < r, \\ r, & n \geq r. \end{cases}$$

Hence,

$$h_f(n) = \min\{n+1, r\}$$

□

4.3 Justification of the relation count

Write f_r for the nonzero homogeneous part of f of degree r . If a nonzero multiplier q has lowest-degree part q_s , then the lowest-degree part of qf is $q_s f_r$. This product is nonzero because $k[x, y]$ is an integral domain. Consequently,

$$\text{mult}_0(qf) = s + r.$$

For $n \geq r$, the linear map

$$V_{n-r} \longrightarrow V_n, \quad q \longmapsto \tau_n(qf),$$

where τ_n discards terms of degree greater than n , is injective: a nonzero $q \in V_{n-r}$ gives $s+r \leq n$, so its product does not vanish after truncation. Its image is exactly the relation space generated by f , since terms of a multiplier of degree greater than $n-r$ contribute nothing after truncation. Thus this relation space has dimension $\binom{n-r+2}{2}$.

For $n < r$, both relevant quotients have no surviving relations from f . Subtracting their dimensions gives $h_f(n) = n+1$, with the case $n = 0$ using $H_f(-1) = 0$. For $n \geq r$, the relation count above gives $h_f(n) = r$; at $n = r$, the preceding relation count is zero. This also verifies the boundary between the two cases.

4.4 Computation and examples

In the following computations, a monomial inside a span denotes its class in the stated quotient. A list of basis classes specifies the dimension; it does not list every element of the quotient. The calculations hold over any field k , including finite fields.

Now we will test this proof over the finite fields with various functions in the $R, k[x, y]$

$$f = x$$

$$f = y^2 - x^4$$

$$f = x^2 + y^5$$

Example 4.1. $f = x^2 + y^5$

$$\text{mult}_0(f) = 2$$

Let $n=0$, $H_f(0) = \dim_k R/((f) + \mathfrak{m}^1)$, the remaining set can be represented by $\text{span}\{[1]\}$, where the dimension is 1 as there is only a single coset in the set.

Then let $n=1$, $H_f(1) = \dim_k R/((f) + \mathfrak{m}^2)$, the remaining set can be represented by $\text{span}\{1, x, y\}$, where the dimension is 3 by the three coset in the basis.

Then let $n=2$, $H_f(2) = \dim_k R/((f) + \mathfrak{m}^3)$, the remaining set can be represented by $\text{span}\{1, x, y, y^2, xy\}$, where the dimension is 5 by the five coset in the basis.

$$n=3, \dim \text{span}\{1, x, y, xy, y^2, xy^2, y^3\} = 7$$

$$n=4, \dim \text{span}\{1, x, y, xy, y^2, xy^2, y^3, xy^3, y^4\} = 9$$

$$n=5, \dim \text{span}\{1, x, y, xy, y^2, xy^2, y^3, xy^3, y^4, xy^4, y^5\} = 11$$

$$n=6, \dim \text{span}\{1, x, y, xy, y^2, xy^2, y^3, xy^3, y^4, xy^4, y^5, xy^5, y^6\} = 13$$

And to cross check it we can plug it into $h_f(n)$ where the difference is 1, 2, 2, 2, 2, 2, 2 respectively

Example 4.2. $f = x$

$$\text{mult}_0(f) = 1$$

Let $n=0$, $H_f(0) = \dim_k R/((f) + \mathfrak{m}^1)$, the remaining set can be represented by $\text{span}\{1\}$, where the dimension is 1 by the one coset in the basis.

Then let $n=1$, $H_f(1) = \dim_k R/((f) + \mathfrak{m}^2)$, the remaining set can be represented by $\text{span}\{1, y\}$, where the dimension is 2 by the two coset in the basis.

Then let $n=2$, $H_f(2) = \dim_k R/((f) + \mathfrak{m}^3)$, the remaining set can be represented by $\text{span}\{1, y, y^2\}$, where the dimension is 3 by the three coset in the basis.

$$n=3, \dim \text{span}\{1, y, y^2, y^3\} = 4$$

$$n=4, \dim \text{span}\{1, y, y^2, y^3, y^4\} = 5$$

$$n=5, \dim \text{span}\{1, y, y^2, y^3, y^4, y^5\} = 6$$

$$n=6, \dim \text{span}\{1, y, y^2, y^3, y^4, y^5, y^6\} = 7$$

And to cross check it we can plug it into $h_f(n)$ where the difference is 1, 1, 1, 1, 1, 1, 1 respectively.

Example 4.3. $f = y^2 - x^4$

$$\text{mult}_0(f) = 2$$

Let $n=0$, $H_f(0) = \dim_k R/((f) + \mathfrak{m}^1)$, the remaining set can be represented by $\text{span}\{1\}$, where the dimension is 1 by the one coset in the basis.

Then let $n=1$, $H_f(1) = \dim_k R/((f) + \mathfrak{m}^2)$, the remaining set can be represented by $\text{span}\{1, x, y\}$, where the dimension is 3 by the three coset in the basis.

Then let $n=2$, $H_f(2) = \dim_k R/((f) + \mathfrak{m}^3)$, the remaining set can be represented by $\text{span}\{1, x, y, x^2, xy\}$, where the dimension is 5 by the five coset in the basis.

$$n=3, \dim \text{span}\{1, x, y, x^2, xy, x^3, x^2y\} = 7$$

$$n=4, \dim \text{span}\{1, x, y, x^2, xy, x^3, x^2y, x^4, x^3y\} = 9$$

$$n=5, \dim \text{span}\{1, x, y, x^2, xy, x^3, x^2y, x^4, x^3y, x^5, x^4y\} = 11$$

$$n=6, \dim \text{span}\{1, x, y, x^2, xy, x^3, x^2y, x^4, x^3y, x^5, x^4y, x^6, x^5y\} = 13$$

And to cross check it we can plug it into $h_f(n)$ where the difference is 1, 2, 2, 2, 2, 2, 2 respectively.

4.5 Conclusion For Problem 1

The examples agree with the formula $h_f(n) = \min\{n+1, r\}$. For $f = x$, the multiplicity is 1, so $H_f(n) = n+1$. For $f = y^2 - x^4$ and $f = x^2 + y^5$, the multiplicity is 2, so $H_f(n) = 2n+1$.

n	0	1	2	3	4	5	6
$H_x(n)$	1	2	3	4	5	6	7
$H_{y^2-x^4}(n)$	1	3	5	7	9	11	13
$H_{x^2+y^5}(n)$	1	3	5	7	9	11	13

Thus the lowest total degree of a nonzero term determines the entire one-curve Hilbert–Samuel function. In particular, the two multiplicity-2 examples already suggest the comparison in Problem 2.

5 Problem 2

5.1 Proof

Proof. We now compare the one-curve Hilbert–Samuel functions of

$$f_1 = xy,$$

$$f_2 = y^2 - x^3,$$

$$f_3 = y^2 - x^4,$$

and

$$f_4 = x^2 + y^5.$$

Recall from Problem 1 that

$$h_f(n) = \min\{n+1, \text{mult}_0(f)\}.$$

Thus the one-curve Hilbert–Samuel function only depends on the multiplicity of the function at the origin.

For

$$f_1 = xy,$$

the lowest degree nonzero term is xy , which has degree 2. Hence

$$\text{mult}_0(xy) = 2.$$

For

$$f_2 = y^2 - x^3,$$

the lowest degree nonzero term is y^2 , which has degree 2. Hence

$$\text{mult}_0(y^2 - x^3) = 2.$$

For

$$f_3 = y^2 - x^4,$$

the lowest degree nonzero term is y^2 , which has degree 2. Hence

$$\text{mult}_0(y^2 - x^4) = 2.$$

For

$$f_4 = x^2 + y^5,$$

the lowest degree nonzero term is x^2 , which has degree 2. Hence

$$\text{mult}_0(x^2 + y^5) = 2.$$

Therefore all four functions have

$$\text{mult}_0(f) = 2.$$

From the one-curve formula,

$$h_f(n) = \begin{cases} n+1, & n < 2, \\ 2, & n \geq 2. \end{cases}$$

Hence for all four curves,

$$h_f = (1, 2, 2, 2, 2, \dots).$$

Since

$$H_f(n) = h_f(0) + h_f(1) + \dots + h_f(n),$$

we obtain

$$H_f(0) = 1,$$

$$H_f(1) = 1 + 2 = 3,$$

$$H_f(2) = 1 + 2 + 2 = 5,$$

$$H_f(3) = 1 + 2 + 2 + 2 = 7,$$

and in general

$$H_f(n) = 2n + 1.$$

Therefore

$$H_f = (1, 3, 5, 7, 9, 11, 13, \dots).$$

Even though the four curves are defined by different functions and have different geometry, the one-curve Hilbert–Samuel function cannot distinguish them because it only detects the multiplicity at the origin.

There for we can conclude that since

$$\text{mult}_0(xy) = \text{mult}_0(y^2 - x^3) = \text{mult}_0(y^2 - x^4) = \text{mult}_0(x^2 + y^5) = 2,$$

they must all have the same one-curve Hilbert–Samuel function.

□

5.2 Computation and examples

Example 5.1. $f = xy$

$$\text{mult}_0(f) = 2$$

Let $n=0$, $H_f(0) = 1$.

Then let $n=1$, $H_f(1) = 3$.

Then let $n=2$, $H_f(2) = 5$.

$n=3$, $H_f(3) = 7$

$n=4$, $H_f(4) = 9$

$n=5$, $H_f(5) = 11$

$n=6$, $H_f(6) = 13$

And to cross check it we can plug it into $h_f(n)$ where the difference is

$$1, 2, 2, 2, 2, 2, 2$$

respectively.

Example 5.2. $f = y^2 - x^3$

$$\text{mult}_0(f) = 2$$

Let $n=0$, $H_f(0) = 1$.

Then let $n=1$, $H_f(1) = 3$.

Then let $n=2$, $H_f(2) = 5$.

$n=3$, $H_f(3) = 7$

$n=4$, $H_f(4) = 9$

$n=5$, $H_f(5) = 11$

$n=6$, $H_f(6) = 13$

And to cross check it we can plug it into $h_f(n)$ where the difference is

$$1, 2, 2, 2, 2, 2, 2$$

respectively.

Example 5.3. $f = y^2 - x^4$

$$\text{mult}_0(f) = 2$$

Let $n=0$, $H_f(0) = 1$.

Then let $n=1$, $H_f(1) = 3$.

Then let $n=2$, $H_f(2) = 5$.

$n=3$, $H_f(3) = 7$

$n=4$, $H_f(4) = 9$

$n=5$, $H_f(5) = 11$

$n=6$, $H_f(6) = 13$

And to cross check it we can plug it into $h_f(n)$ where the difference is

$$1, 2, 2, 2, 2, 2, 2$$

respectively.

Example 5.4. $f = x^2 + y^5$

$$\text{mult}_0(f) = 2$$

Let $n=0$, $H_f(0) = 1$.

Then let $n=1$, $H_f(1) = 3$.

Then let $n=2$, $H_f(2) = 5$.

$n=3$, $H_f(3) = 7$

$n=4$, $H_f(4) = 9$

$n=5$, $H_f(5) = 11$

$n=6$, $H_f(6) = 13$

And to cross check it we can plug it into $h_f(n)$ where the difference is

$$1, 2, 2, 2, 2, 2$$

respectively.

5.3 Conclusion For Problem 2

Even though

$$xy = 0,$$

$$y^2 - x^3 = 0,$$

$$y^2 - x^4 = 0,$$

and

$$x^2 + y^5 = 0$$

describe different curves, each function has its first nonzero homogeneous term in degree 2.

Therefore

$$\text{mult}_0(f) = 2$$

for all four functions.

The one-curve Hilbert–Samuel function therefore gives

$$h_f(n) = \min\{n + 1, 2\},$$

so

$$h_f = (1, 2, 2, 2, \dots)$$

for every one of the four curves.

Hence the one-curve Hilbert–Samuel function detects the multiplicity of the curve at the origin, but does not necessarily distinguish curves with different geometric shapes.

6 Discussion and Limitations

The examples show that the one-curve Hilbert–Samuel function depends only on the multiplicity at the origin. In particular, the four multiplicity-2 curves in Problem 2 have identical functions. Thus this function alone does not determine the geometric shape of a curve. The report considers nonzero polynomials vanishing at the origin and does not address intersections of two curves.

7 Conclusion and Next Steps

For a plane curve of multiplicity r , the successive differences satisfy $h_f(n) = \min\{n + 1, r\}$. The examples in Problem 1 illustrate this formula, while Problem 2 demonstrates that different curves can share the same Hilbert–Samuel function. A possible next step is to study additional invariants that distinguish such curves.