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CARDINAL STRATIFICATION OF FILTER CATEGORIES AND ADJOINTS OF EMBEDDING FUNCTORS

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ABSTRACT. For a fixed mapping $f : X \rightarrow Y$, the inverse image and image constructions on filters form the classical adjunction $f^{-1} \dashv f_*$. Under cardinal stratification, the adjunction $f^{-1}|_\kappa \dashv f_*|_\kappa$ survives, and the natural question is how these adjunctions compare across different cardinal levels. This is captured by the embedding functor

$$\iota_\lambda^{(\kappa)} : \text{Filter}_\kappa(X) \hookrightarrow \text{Filter}_\lambda(X).$$

We determine when $\iota_\lambda^{(\kappa)}$ admits a left or right adjoint. Both adjoints exist when X is finite, or $\kappa = \lambda$, or $\kappa \geq 2^{|X|}$. In the remaining infinite case, the right adjoint fails to exist whenever $\kappa < 2^{|X|}$, while for the left adjoint we prove non-existence when $\kappa < |X|$; the case $|X| \leq \kappa < 2^{|X|}$ is left open and is discussed in the final section. The proofs use two pointwise criteria, κ -descending and κ -ascending approximability, together with independent family and partition constructions. The obstruction is related to the failure of κ -generated filters to be closed under arbitrary intersections and generated unions.

KEYWORDS. filter, cardinal stratification, embedding functor, adjunction, approximability

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1. INTRODUCTION

Given a mapping $f : X \rightarrow Y$, the inverse image and image constructions on filters form an adjunction (Fact2.4)

$$f^{-1} \dashv f_*.$$

The present work builds upon this adjunction as its foundational observation.

A natural question is what happens when we restrict attention to filters that admit a small generating family. For a cardinal κ , call a filter κ -generated if it has a generating family of size at most κ , and write $\text{Filter}_\kappa(X)$ for the resulting poset. The adjunction restricts to every cardinal level,

$$f^{-1}|_\kappa \dashv f_*|_\kappa,$$

so the adjunction survives at each fixed level (Theorem 3.1). The question we study here is different: how do these adjunctions compare across different cardinal levels?

This comparison is captured by the embedding functor

$$\iota_\lambda^{(\kappa)} : \text{Filter}_\kappa(X) \hookrightarrow \text{Filter}_\lambda(X),$$

for $\kappa \leq \lambda$. We ask whether $\iota_\lambda^{(\kappa)}$ admits a left or right adjoint, i.e., whether there is a canonical way to move information between the κ -level and the λ -level. In the posetal setting, this is equivalent to asking whether every λ -generated filter has a largest κ -generated subfilter, or a smallest κ -generated superfilter.

Both adjoints exist in the trivial cases: when X is finite, when $\kappa = \lambda$, or when $\kappa \geq 2^{|X|}$. In the remaining infinite case, the right adjoint admits a complete description: it exists precisely when $\kappa \geq 2^{|X|}$, and fails to exist whenever $\kappa < 2^{|X|}$. For the left adjoint, we prove non-existence when $\kappa < |X|$. The case $|X| \leq \kappa < 2^{|X|}$ is left open and is discussed in the final section.

The proof uses two pointwise criteria, κ -descending (Proposition 4.3) and κ -ascending approximability (Proposition 4.4), which are equivalent to the existence

of the right and left adjoints, respectively. We also make use of the classical independent family construction of Fichtenholz–Kantorovich–Hausdorff (Corollary 4.2) and a partition argument (Example 4.1). The obstruction we identify is related to the failure of κ -generated filters to be closed under arbitrary intersections (Theorem 4.2) and generated unions (Example 4.1).

We now outline the structure of the paper. Section 2 reviews the basic definitions of filters and the language of Galois connections. Section 3 introduces κ -generated filters and the embedding functor $\iota_\lambda^{(\kappa)}$, and establishes the adjunction at each fixed cardinal level. Section 4 proves the main results on the existence of left and right adjoints. Section 5 discusses the remaining open case, the transition from set towers to filter towers, and some connections to other categories.

2. PRELIMINARIES AND MOTIVATION

We begin by fixing the basic object of study. Throughout this paper, we use $\mathcal{P}(X)$ to represent the family consisting of every subset of X . The filter goes back to Cartan [Car37]; we recall it here only to fix notation and to emphasize the slight departure from the usual convention adopted in this note.

Definition 2.1. *Let X be an arbitrary set.*

- (1) *Take a non-empty family $\mathcal{F} \subseteq \mathcal{P}(X)$. If both of the conditions listed below are satisfied*

- (a) $V_1 \cap V_2 \in \mathcal{F}, \quad \forall (V_1, V_2) \in \mathcal{F}^2;$
- (b) $V \subseteq W \Rightarrow W \in \mathcal{F}, \quad \forall (V, W) \in \mathcal{F} \times \mathcal{P}(X),$

*then we call $\mathcal{F}$ **filter** on X .*

- (2) *Given two filters $\mathcal{F} \subseteq \mathcal{G}$. In this situation, $\mathcal{F}$ qualifies as a subfilter of $\mathcal{G}$, while $\mathcal{G}$ is termed a superfilter of $\mathcal{F}$.*

Remark 2.1. *The classical definition of (narrower) filter $\mathcal{F}$ on a set X , originated by Cartan [Car37], additionally requires the axiom $\emptyset \notin \mathcal{F}$.*

When we consider the classical results such as ultrafilters or convergence, we usually use the classical definition of narrower filter (see [Myn20]). But in this note, we adopt the broader definition of filters than Cartan's, which makes inverse-image constructions canonical and allows the filter assignments induced by maps to form a Galois connection without additional properness assumptions such as surjectivity (see Definition 2.2 and Fact2.4).

When $\emptyset \in \mathcal{F}$, the equality $\mathcal{F} = \mathcal{P}(X)$ must hold. In this case, $\mathcal{F}$ is called degenerate in this note. Otherwise, $\mathcal{F}$ is non-degenerate.

Notation 2.1. For some family $\mathcal{S} \subseteq \mathcal{P}(X)$, if

$$\mathcal{F} = \{U \in \mathcal{P}(X) : \exists n \in \mathbb{N}_{>0}, \exists (U_1, \dots, U_n) \in \mathcal{S}^n, U_1 \cap \dots \cap U_n \subseteq U\},$$

we denote $\mathcal{F}$ by $\langle \mathcal{S} \rangle$.

We write the principal filter as

$$\uparrow A := \langle \{A\} \rangle.$$

Definition 2.2. Fix a mapping $f : X \rightarrow Y$.

(1) Given a filter $\mathcal{F}$ on X , the **(direct) image filter** induced by f , is defined by

$$f_*(\mathcal{F}) := \{B \in \mathcal{P}(Y) : f^{-1}(B) \in \mathcal{F}\}.$$

(2) Given a filter $\mathcal{G}$ on Y , the **inverse image filter** induced by f , is defined by

$$f^{-1}(\mathcal{G}) := \{U \in \mathcal{P}(X) : \exists V \in \mathcal{G}, f^{-1}(V) \subseteq U\}.$$

We begin by recalling some elementary and well-known properties of filters. The following three facts are standard and their proofs are straightforward verifications, hence omitted (see [Bou07] and [Mas26] for details).

Fact 2.1. ([Mas26, Lemme 2.4.3]) Given a mapping $f : X \rightarrow Y$ together with filters $\mathcal{F}$ on X and $\mathcal{G}$ on Y , $f_*(\mathcal{F})$ and $f^{-1}(\mathcal{G})$ form filters on Y and on X , respectively.

Fact 2.2. ([Bou07, Ch. I, §6, n° 1]) When X is infinite, the collection $\{U \in \mathcal{P}(X) : X \setminus U \text{ is finite (i.e., } U \text{ is cofinite)}\}$ forms a filter on X . This particular filter is known as the **Fréchet filter**.

Fact 2.3. ([Bou07, Ch. I, §6, n° 2]) Consider a non-empty family of filters $\{\mathcal{F}_i\}_{i \in I}$ on X . Then their intersection $\bigcap_{i \in I} \mathcal{F}_i$ is again a filter on X .

However, the union of two filters, taken in the usual set-theoretic way, does not always produce a filter.

e.g., let $X = \{0, 1\}$. Define two principal filters

$$\mathcal{F} := \uparrow \{0\} = \{\{0\}, \{0, 1\}\}, \quad \mathcal{G} := \uparrow \{1\} = \{\{1\}, \{0, 1\}\}.$$

Their set-theoretic union is

$$\mathcal{F} \cup \mathcal{G} = \{\{0\}, \{1\}, \{0, 1\}\}.$$

By the definition of filter,

$$\{0\} \cap \{1\} = \emptyset \in \mathcal{F} \cup \mathcal{G}.$$

But $\emptyset \notin \mathcal{F} \cup \mathcal{G}$, contradiction. Hence, $\mathcal{F} \cup \mathcal{G}$ is not a filter.

Notation 2.2. Given an arbitrary family of filters with index set I , the generated union is denoted by

$$\bigvee_{i \in I} \mathcal{G}_i := \left\langle \bigcup_{i \in I} \mathcal{G}_i \right\rangle.$$

This verification is purely formal: one checks that the two defining conditions of a filter are preserved under the two constructions.

We have recalled the basic closure properties of filters. The next standard fact is the categorical heart of the image–inverse image correspondence (see [Mas26] for details). The author independently observed this correspondence before consulting the literature.

Fact 2.4. (*Image–Inverse Image Correspondence*, [Mas26, Théorème 2.4.1, Approche concrète]) Given a mapping $f : X \rightarrow Y$ together with filters $\mathcal{F}$ on X and $\mathcal{G}$ on Y ,

$$f^{-1}(\mathcal{G}) \subseteq \mathcal{F} \iff \mathcal{G} \subseteq f_*(\mathcal{F}).$$

Remark 2.2. Fact 2.4 seems to be related to Galois correspondence.

Define

$$\Phi(K) := \text{Gal}(E/K), \quad \Psi(H) := E^H,$$

where K ranges over fields satisfying $F \subseteq K \subseteq E$, and H ranges over subgroups satisfying $H \leq \text{Gal}(E/F)$.

The fundamental theorem gives mutually inverse bijections Φ and Ψ , with

$$\Phi(K) \subseteq H \iff K \supseteq \Psi(H).$$

If we define the opposite order

$$K_1 \leq_{\text{op}} K_2 \iff K_2 \subseteq K_1, \quad \forall F \subseteq K_2 \subseteq K_1 \subseteq E,$$

then we have

$$\Phi(K) \subseteq H \iff K \leq_{\text{op}} \Psi(H).$$

To place this observation in a broader framework, next, we turn to the language of category theory.

Our main objects are the following. At this point, a structural shift in perspective is necessary. Instead of focusing on individual filters, we regard them as objects in a posetal category.

Definition 2.3. Given a set X , we form a posetal category $\text{Filter}(X)$ whose objects are the filters on X , and morphisms are

$$\mathcal{F} \rightarrow \mathcal{G} \iff \mathcal{F} \subseteq \mathcal{G}.$$

Remark 2.3. Actually, $\text{Filter}(X)$ is a small category, since the collection of all filters on X is contained in $\mathcal{P}(\mathcal{P}(X))$.

In the posetal setting, the abstract notion of adjunction reduces to the condition in order theory, which is the form we shall use throughout.

Definition 2.4. Given two posetal categories P and Q together with two order-preserving mappings $F : P \rightarrow Q$ and $G : Q \rightarrow P$. Suppose F and G satisfy the equivalence

$$F(p) \leq q \iff p \leq G(q), \quad \forall (p, q) \in P \times Q.$$

In this situation, $F \dashv G$ forms a **Galois connection (or adjunction)**. Here F serves as the **left adjoint**, and G as the **right adjoint**.

Remark 2.4. Mac Lane's definition uses opposite categories ($L : P \rightarrow Q^{op}$), reflecting the classical order-reversing intuition in Galois theory [ML98, Ch. IV, §5]. Our formulation adopts the modern covariant convention, which is simpler and avoids explicit op notation.

Remark 2.5. Define covariant functors induced by f

$$f_* : \text{Filter}(X) \rightarrow \text{Filter}(Y), \quad \mathcal{F} \mapsto f_*(\mathcal{F}),$$

$$f^{-1} : \text{Filter}(Y) \rightarrow \text{Filter}(X), \quad \mathcal{G} \mapsto f^{-1}(\mathcal{G}).$$

In Remark 2.2, it is immediate to get the two Galois connections $f^{-1} \dashv f_*$ and $\Phi \dashv \Psi$.

In other words, the covariant functors induced by a mapping $f : X \rightarrow Y$ between the filter categories always admit the adjoints described above. More precisely, the adjunction $f^{-1} \dashv f_*$ always exists without any additional hypotheses on f .

3. INITIAL OBSERVATIONS

3.1. THE ADJUNCTION AT A FIXED CARDINAL LEVEL

A natural next step is to ask whether these results persist when we restrict attention to filters with a small generating set — that is, when we work in the categories such as $\text{Filter}_\kappa(X)$ defined below.

Definition 3.1. Fix a set X together with $\mathcal{F} \in \text{Filter}(X)$, and a cardinal $\kappa \geq 1$. If $\exists$ family $\mathcal{S} \subseteq \mathcal{P}(X)$, such that

1. $|\mathcal{S}| \leq \kappa$,

2. $\mathcal{F} = \langle \mathcal{S} \rangle$,

then $\mathcal{F}$ is called κ -generated.

Denote $\text{Filter}_\kappa(X)$ the category of all κ -generated filters on X , endowed with inclusion posetal structure.

Throughout the remainder of this note, we will always assume that the cardinals satisfy $\lambda \geq \kappa \geq 1$.

Lemma 3.1. For cardinal κ satisfying $\kappa < \aleph_0$, $\text{Filter}_\kappa(X) = \text{Filter}_1(X)$.

Proof. For any $\mathcal{F} = \langle \mathcal{S} \rangle \in \text{Filter}_\kappa(X)$, where $|\mathcal{S}| \leq \kappa$, $\mathcal{F} = \uparrow \{ \bigcap_{S \in \mathcal{S}} S \} \in \text{Filter}_1(X)$.

Hence $\text{Filter}_\kappa(X) \subseteq \text{Filter}_1(X)$, and we can prove this Lemma by Lemma 3.4. $\square$

Definition 3.2. Fix a set X . $\mathcal{B} \subseteq \mathcal{P}(X)$ is non-empty. If $\forall (A, B) \in \mathcal{B}^2$, $\exists C \in \mathcal{B}$ satisfying $C \subseteq A \cap B$, then $\mathcal{B}$ is termed a **filter base**.

In the literature, a filter is often called κ -generated if it has a generating family of size at most κ . Some prefer the definition using filter bases. Actually, for infinite cardinals, the two notions coincide by the elementary lemma below.

Lemma 3.2. (Equivalence of generating families and filter bases) Fix an infinite cardinal κ . The two conditions listed below turn out to be equivalent.

- (1) $\exists \mathcal{S}, \mathcal{F} = \langle \mathcal{S} \rangle$, where $|\mathcal{S}| \leq \kappa$.
- (2) $\exists \mathcal{B}, \mathcal{F} = \{ B \in \mathcal{P}(X) : \exists A \in \mathcal{B}, A \subseteq B \}$ with filter base $\mathcal{B}$ satisfying $|\mathcal{B}| \leq \kappa$.

Proof. If $\mathcal{F}$ is finitely generated, then the result can be immediately get by Lemma 3.1.

Next, suppose that $\mathcal{F}$ is infinitely generated. (2) $\Rightarrow$ (1) is immediate, since every filter base is itself a generating family.

For (1) $\Rightarrow$ (2), suppose $\mathcal{F} = \langle \mathcal{S} \rangle$ with $|\mathcal{S}| \leq \kappa$. Define

$$\mathcal{B} := \left\{ \bigcap_{i=1}^n S_i : n < \omega, (S_1, \dots, S_n) \in \mathcal{S}^n \right\}.$$

Then $\mathcal{F} = \langle \mathcal{B} \rangle$.

Choose any $(A, B) \in \mathcal{F}^2$. A can be expressed as $\bigcap_{i=1}^m S_i$ where $S_i \in \mathcal{B}$ for any $i \in \{1, \dots, m\}$. The same holds for $B = \bigcap_{j=1}^n T_j$ where $T_j \in \mathcal{B}$ for any $j \in \{1, \dots, n\}$. Their intersection is

$$A \cap B = \bigcap_{i=1}^m S_i \cap \bigcap_{j=1}^n T_j \in \mathcal{B}.$$

Hence the family $\mathcal{B}$ qualifies as a filter base. This gives

$$\mathcal{F} = \{B \in \mathcal{P}(X) : \exists A \in \mathcal{B}, A \subseteq B\}.$$

Because κ is infinite, we obtain the bound

$$|\mathcal{B}| \leq \sum_{n < \aleph_0} |\mathcal{S}|^n = \aleph_0 \cdot |\mathcal{S}| \leq \kappa.$$

□

We aim to show that these functors induced by f and restricted to the subcategories of κ -generated filters are well-defined.

Lemma 3.3. (*Well-definedness*) *For cardinal κ , the well-definedness of the functors $f_*|_\kappa$ and $f^{-1}|_\kappa$ is guaranteed, where*

$$f_*|_\kappa : \text{Filter}_\kappa(X) \rightarrow \text{Filter}_\kappa(Y), \quad \mathcal{F} \mapsto f_*(\mathcal{F}),$$

and

$$f^{-1}|_\kappa : \text{Filter}_\kappa(Y) \rightarrow \text{Filter}_\kappa(X), \quad \mathcal{G} \mapsto f^{-1}(\mathcal{G}).$$

Proof. 1. Take any $\mathcal{F} \in \text{Filter}_\kappa(X)$. By definition, $\exists \mathcal{S} \subseteq \mathcal{P}(X)$ satisfying $\mathcal{F} = \langle \mathcal{S} \rangle$ with $|\mathcal{S}| \leq \kappa$.

Our goal is to produce a generating family for $f_*(\mathcal{F})$ of size at most κ . The natural candidate is obtained by applying f to finite intersections of elements of $\mathcal{S}$.

Consider

$$\mathcal{S}' := \left\{ f \left(\bigcap_{i=1}^n S_i \right) : n < \aleph_0, S_i \in \mathcal{S} \right\} \subseteq \mathcal{P}(Y).$$

We first estimate the cardinality of $\mathcal{S}'$. The argument separates the finite and infinite cases.

If $1 \leq \kappa < \aleph_0$, then $\exists A \in \mathcal{P}(X)$ such that $\mathcal{F} = \uparrow A$. At this time, $\mathcal{S}' = \{f(A)\}$. Thus $|\mathcal{S}'| = 1 \leq \kappa$.

If $\kappa \geq \aleph_0$, then

$$|\mathcal{S}|^n \leq \kappa^n = \kappa, \quad \forall n < \aleph_0.$$

Thus,

$$|\mathcal{S}'| \leq \sum_{n < \omega} |\mathcal{S}|^n \leq \sum_{n < \omega} \kappa = \kappa \cdot \aleph_0 = \kappa.$$

Consequently, $|\mathcal{S}'| \leq \kappa$ holds for every cardinal $\kappa \geq 1$.

We claim $f_*(\mathcal{F}) = \langle \mathcal{S}' \rangle$. Take arbitrary $B \in \mathcal{P}(Y)$, we have

$$\begin{aligned} B \in f_*(\mathcal{F}) &\iff f^{-1}(B) \in \mathcal{F} \\ &\iff \exists (S_1, \dots, S_n) \in \mathcal{S}^n, \bigcap_{i=1}^n S_i \subseteq f^{-1}(B) \\ &\iff \exists (S_1, \dots, S_n) \in \mathcal{S}^n, \left(\bigcap_{i=1}^n S_i \right) \subseteq B \\ &\iff B \in \langle \mathcal{S}' \rangle. \end{aligned}$$

Thus $f_*(\mathcal{F})$ is generated by $\mathcal{S}'$, and since $|\mathcal{S}'| \leq \kappa$, $f_*(\mathcal{F}) \in \text{Filter}_\kappa(Y)$. Therefore, the restricted functor $f_*|_\kappa$ is well-defined.

2. Take any $\mathcal{G} \in \text{Filter}_\kappa(Y)$. Then $\exists \mathcal{T} \subseteq \mathcal{P}(Y)$ satisfying

$$|\mathcal{T}| \leq \kappa \quad \text{and} \quad \mathcal{G} = \langle \mathcal{T} \rangle.$$

Define

$$\mathcal{T}' := \{f^{-1}(T) : T \in \mathcal{T}\}.$$

Immediately, $|\mathcal{T}'| \leq |\mathcal{T}| \leq \kappa$.

Our next goal is to establish the equality $f^{-1}(\mathcal{G}) = \langle \mathcal{T}' \rangle$. Take any $U \in \mathcal{P}(X)$. Assume $U \in f^{-1}(\mathcal{G})$. This means $\exists V \in \mathcal{G}$ satisfying $f^{-1}(V) \subseteq U$.

Since $V \in \mathcal{G} = \langle \mathcal{T} \rangle$, there exist $(T_1, \dots, T_m) \in \mathcal{T}^m$ such that $\bigcap_{j=1}^m T_j \subseteq V$. Then

$$\bigcap_{j=1}^m f^{-1}(T_j) = f^{-1}\left(\bigcap_{j=1}^m T_j\right) \subseteq f^{-1}(V) \subseteq U.$$

Hence $U \in \langle \mathcal{T}' \rangle$.

Conversely, if $U \in \langle \mathcal{T}' \rangle$, there exist $(T_1, \dots, T_m) \in \mathcal{T}^m$ such that

$$\bigcap_{j=1}^m f^{-1}(T_j) \subseteq U.$$

Let $V := \bigcap_{j=1}^m T_j \in \mathcal{G}$. Then $f^{-1}(V) \subseteq U$, which means $U \in f^{-1}(\mathcal{G})$.

Therefore,

$$f^{-1}(\mathcal{G}) = \langle \mathcal{T}' \rangle.$$

This shows that $f^{-1}(\mathcal{G}) \in \text{Filter}_\kappa(X)$. Consequently, the restricted functor $f^{-1}|_\kappa$ is well-defined.

□

Remark 3.1. In the proof of well-definedness of f_* , one might wonder why we defined $\mathcal{S}'$ as

$$\mathcal{S}' := \left\{ f \left(\bigcap_{i=1}^n S_i \right) : n < \aleph_0, S_i \in \mathcal{S} \right\},$$

rather than simply taking

$$\{f(S) : S \in \mathcal{S}\}.$$

The reason is that, if f is not injective, the image does not commute with intersections in general. Actually, in this case, we can only get

$$f \left(\bigcap_{i=1}^n S_i \right) \subsetneq \bigcap_{i=1}^n f(S_i).$$

Hence we have

$$\langle \{f(S) : S \in \mathcal{S}\} \rangle \subsetneq \langle \mathcal{S}' \rangle.$$

Having established that the induced functors $f^{-1}|_\kappa$ and $f_*|_\kappa$ are well-defined, we now confirm that the fundamental Galois connection is preserved on the restricted categories.

Theorem 3.1. $f^{-1}|_\kappa \dashv f_*|_\kappa$ forms a Galois connection between $\text{Filter}_\kappa(X)$ and $\text{Filter}_\kappa(Y)$.

Proof. By Proposition 2.4,

$$f^{-1}(\mathcal{G}) \subseteq \mathcal{F} \iff \mathcal{G} \subseteq f_*(\mathcal{F}).$$

According to Lemma 3.3, the inverse image filter and image filter of a κ -generated filter are also κ -generated. We only take $\mathcal{F}$ and $\mathcal{G}$ in $\text{Filter}_\kappa(X)$ and in $\text{Filter}_\kappa(Y)$, respectively.

Thus,

$$f^{-1}|_\kappa(\mathcal{G}) \subseteq \mathcal{F} \iff \mathcal{G} \subseteq f_*|_\kappa(\mathcal{F}).$$

Consequently, the pair $f^{-1}|_\kappa \dashv f_*|_\kappa$ forms a Galois connection. $\square$

3.2. INTERACTION BETWEEN CARDINAL LEVELS VIA EMBEDDING FUNCTORS

$\forall \kappa \geq 1$, the adjunction $f^{-1}|_\kappa \dashv f_*|_\kappa$ has been established. We now compare these adjunctions across different cardinal levels. The embedding functors provide the categorical tool.

Lemma 3.4. *If $\kappa \leq \lambda$, then $\text{Filter}_\kappa(X) \subseteq \text{Filter}_\lambda(X)$ as posetal subcategories. Additionally, the embedding functor*

$$\iota_X^{(\kappa, \lambda)} : \text{Filter}_\kappa(X) \hookrightarrow \text{Filter}_\lambda(X)$$

exists, $\forall \kappa \leq \lambda$.

(If fix the set X , write $\iota_X^{(\kappa, \lambda)}$ simply as $\iota_\lambda^{(\kappa)}$.)

Proof. Let $\mathcal{F} \in \text{Filter}_\kappa(X)$. Then $\exists \mathcal{S} \subseteq \mathcal{P}(X)$ with $|\mathcal{S}| \leq \kappa$ making $\mathcal{F} = \langle \mathcal{S} \rangle$. Since $\kappa \leq \lambda$, we have $|\mathcal{S}| \leq \kappa \leq \lambda$, hence $|\mathcal{S}| \leq \lambda$. Therefore $\mathcal{F} \in \text{Filter}_\lambda(X)$. Thus $\text{Filter}_\kappa(X) \subseteq \text{Filter}_\lambda(X)$.

Since $\text{Filter}_\kappa(X) \subseteq \text{Filter}_\lambda(X)$ on objects, whenever there is a morphism $\mathcal{F} \subseteq \mathcal{G}$ in $\text{Filter}_\kappa(X)$, the same inclusion holds in $\text{Filter}_\lambda(X)$. Hence the morphism sets are preserved by inclusion.

Define the embedding functor by

$$\iota_\lambda^{(\kappa)}(\mathcal{F}) := \mathcal{F}, \quad \iota_\lambda^{(\kappa)}(\mathcal{F} \subseteq \mathcal{G}) := (\mathcal{F} \subseteq \mathcal{G}).$$

Take arbitrary filter $\mathcal{F} \in \text{Filter}_\kappa(X)$. The identity morphism is the inclusion $\mathcal{F} \subseteq \mathcal{F}$. Applying $\iota_\lambda^{(\kappa)}$ to identity and composition, we get

$$\iota_\lambda^{(\kappa)}(\text{id}_\mathcal{F}) = \iota_\lambda^{(\kappa)}(\mathcal{F} \subseteq \mathcal{F}) = \iota_\lambda^{(\kappa)}(\mathcal{F}) \subseteq \iota_\lambda^{(\kappa)}(\mathcal{F}) = \text{id}_{\iota_\lambda^{(\kappa)}(\mathcal{F})},$$

and

$$\iota_\lambda^{(\kappa)}(\mathcal{G} \subseteq \mathcal{H} \circ \mathcal{F} \subseteq \mathcal{G}) = \iota_\lambda^{(\kappa)}(\mathcal{F} \subseteq \mathcal{H}) = \iota_\lambda^{(\kappa)}(\mathcal{G} \subseteq \mathcal{H}) \circ \iota_\lambda^{(\kappa)}(\mathcal{F} \subseteq \mathcal{G}),$$

respectively.

Therefore $\iota_\lambda^{(\kappa)}$ is a well-defined embedding functor. $\square$

To compare the adjunctions across different cardinal levels, we need to check whether the induced functors behave well with respect to the embeddings introduced above. For any $\kappa \leq \lambda$, both the image functor and the inverse image functor are expected to be compatible with these embeddings. In other words, applying the functor at the κ -level followed by the embedding into the λ -level should yield the same result as first embedding the corresponding filter and then applying the functor at the λ -level. More explicitly, this requires the commutativity of both the image and inverse image diagrams, which we verify in Proposition 3.1 below.

Proposition 3.1. *For the image functor, the following diagram is commutative:*

$$\begin{array}{ccc} \text{Filter}_\kappa(X) & \xrightarrow{f_*|_\kappa} & \text{Filter}_\kappa(Y) \\ \downarrow \iota_X^{(\kappa, \lambda)} & & \downarrow \iota_Y^{(\kappa, \lambda)} \\ \text{Filter}_\lambda(X) & \xrightarrow{f_*|_\lambda} & \text{Filter}_\lambda(Y) \end{array}$$

Analogously, for the inverse image functor, the following diagram is commutative:

$$\begin{array}{ccc} \text{Filter}_\kappa(Y) & \xrightarrow{f^{-1}|_\kappa} & \text{Filter}_\kappa(X) \\ \downarrow \iota_Y^{(\kappa, \lambda)} & & \downarrow \iota_X^{(\kappa, \lambda)} \\ \text{Filter}_\lambda(Y) & \xrightarrow{f^{-1}|_\lambda} & \text{Filter}_\lambda(X) \end{array}$$

Proof. We have

$$f_*|_\kappa(\mathcal{F}) = f_*|_\lambda(\mathcal{F}), \quad \forall \mathcal{F} \in \text{Filter}_\kappa(X).$$

Hence,

$$\iota_Y^{(\kappa, \lambda)}(f_*|_\kappa(\mathcal{F})) = f_*|_\lambda(\iota_X^{(\kappa, \lambda)}(\mathcal{F})), \quad \forall \mathcal{F} \in \text{Filter}_\kappa(X).$$

The proof for the second diagram is identical,

$$f^{-1}|_\kappa(\mathcal{G}) = f^{-1}|_\lambda(\mathcal{G}), \quad \forall \mathcal{G} \in \text{Filter}_\kappa(Y).$$

Hence,

$$\iota_X^{(\kappa, \lambda)}(f^{-1}|_\kappa(\mathcal{G})) = f^{-1}|_\lambda(\iota_Y^{(\kappa, \lambda)}(\mathcal{G})), \quad \forall \mathcal{G} \in \text{Filter}_\kappa(Y).$$

Therefore, both diagrams are commutative. $\square$

By the two commutative diagrams, we can get the proposition below.

Proposition 3.2. *If $\iota_Y^{(\kappa, \lambda)}$ has a right adjoint, denoted by $R_Y^{(\kappa, \lambda)}$ (fixing the set Y , cardinals κ and λ , write $R_Y^{(\kappa, \lambda)}$ simply as R), then*

$$R_Y^{(\kappa, \lambda)} \circ f_*|_\lambda \circ \iota_X^{(\kappa, \lambda)} = f_*|_\kappa.$$

Similarly, suppose that $\iota_X^{(\kappa, \lambda)}$ has a left adjoint, denoted by $L_X^{(\kappa, \lambda)}$ (fixing the set X , cardinals κ and λ , write $L_X^{(\kappa, \lambda)}$ simply as L), then

$$L_X^{(\kappa, \lambda)} \circ f^{-1}|_\lambda \circ \iota_Y^{(\kappa, \lambda)} = f^{-1}|_\kappa.$$

Proof. For the first statement, the commutativity of the diagram for the image functor in Proposition 3.1 gives

$$\iota_Y^{(\kappa, \lambda)} \circ f_*|_\kappa = f_*|_\lambda \circ \iota_X^{(\kappa, \lambda)}.$$

Since $\iota_Y^{(\kappa, \lambda)} \dashv R$, we have

$$R \circ \iota_Y^{(\kappa, \lambda)} = \text{id}_{\text{Filter}_\kappa(Y)}.$$

Hence,

$$R \circ f_*|_\lambda \circ \iota_X^{(\kappa, \lambda)} = R \circ \left(\iota_Y^{(\kappa, \lambda)} \circ f_*|_\kappa \right) = \left(R \circ \iota_Y^{(\kappa, \lambda)} \right) \circ f_*|_\kappa = f_*|_\kappa.$$

For the second statement, by the commutativity of the diagram for the inverse image functor in Proposition 3.1, we have

$$\iota_X^{(\kappa, \lambda)} \circ f^{-1}|_\kappa = f^{-1}|_\lambda \circ \iota_Y^{(\kappa, \lambda)}.$$

Since $L \dashv \iota_X^{(\kappa, \lambda)}$, we have

$$L \circ \iota_X^{(\kappa, \lambda)} = \text{id}_{\text{Filter}_\kappa(X)}.$$

Therefore,

$$L \circ f^{-1}|_\lambda \circ \iota_Y^{(\kappa, \lambda)} = L \circ \left(\iota_X^{(\kappa, \lambda)} \circ f^{-1}|_\kappa \right) = \left(L \circ \iota_X^{(\kappa, \lambda)} \right) \circ f^{-1}|_\kappa = f^{-1}|_\kappa.$$

□

Remark 3.2. The proposition shows that, whenever the relevant adjoints of the embedding exist, the higher-level functors $f^{-1}|_\lambda$ and $f_*|_\lambda$ can be “lowered” to the κ -level by composition with the adjoints, and the resulting functors are exactly the original ones $f^{-1}|_\kappa$ and $f_*|_\kappa$.

Conversely, if the required left adjoint $L_X^{(\kappa, \lambda)}$ or right adjoint $R_Y^{(\kappa, \lambda)}$ does not exist, then there is no adjoint-induced way to recover the κ -level functors from their λ -level counterparts by such a composite construction.

Hence, we need to discuss when the left adjoint $L_X^{(\kappa, \lambda)}$ or right adjoint $R_Y^{(\kappa, \lambda)}$ of the embeddings exist.

4. MAIN RESULTS

We begin with some elementary observations in this section.

4.1. BASIC CASES

Proposition 4.1. Take finite set X , and cardinals $\kappa \leq \lambda$. Consider the embedding functor

$$\iota_\lambda^{(\kappa)} : \text{Filter}_\kappa(X) \hookrightarrow \text{Filter}_\lambda(X).$$

Then

$$\text{Filter}_\kappa(X) = \text{Filter}_\lambda(X) = \text{Filter}(X),$$

and $\iota_\lambda^{(\kappa)}$ serves as an adjoint to itself on both sides.

Proof. Since X is finite, its power set $\mathcal{P}(X)$ is also finite. Consequently, every filter on X admits a finite generating family, which means that it must be principal. It follows that $\text{Filter}_\kappa(X) = \text{Filter}_\lambda(X) = \text{Filter}(X)$ for every κ and λ . In this situation, the embedding $\iota_\lambda^{(\kappa)}$ is simply the identity functor and therefore serves as an adjoint to itself on both sides. □

Next assume X is infinite.

Proposition 4.2. *Suppose X is infinite. If both κ and λ are finite, or if $\lambda = \kappa$, or if $\lambda, \kappa \geq 2^{|X|}$, then the embedding $\iota_\lambda^{(\kappa)}$ serves as an adjoint to itself on both sides.*

Proof. If κ and λ are both finite, then $\text{Filter}_\kappa(X) = \text{Filter}_\lambda(X) = \text{Filter}_1(X)$. Then $\iota_\lambda^{(\kappa)}$ is an identity.

If $\lambda = \kappa$, then $\text{Filter}_\kappa(X) = \text{Filter}_\lambda(X)$. Then $\iota_\lambda^{(\kappa)}$ is an identity.

If $\lambda, \kappa \geq 2^{|X|}$, take an arbitrary filter $\mathcal{F} \in \text{Filter}(X)$, the equality $\mathcal{F} = \langle \mathcal{F} \rangle$ holds.

Because $\mathcal{F} \subseteq \mathcal{P}(X)$,

$$|\mathcal{F}| \leq |\mathcal{P}(X)| = 2^{|X|} \leq \kappa$$

Then $\mathcal{F} \in \text{Filter}_\kappa(X)$. Hence, $\text{Filter}_\kappa(X) \subseteq \text{Filter}(X)$.

And since $\text{Filter}_\kappa(X) \subseteq \text{Filter}_\lambda(X) \subseteq \text{Filter}(X)$, then $\text{Filter}_\kappa(X) = \text{Filter}_\lambda(X) = \text{Filter}(X)$. Thus $\iota_\lambda^{(\kappa)}$ is an identity.

Hence in the three cases, the embedding $\iota_\lambda^{(\kappa)}$ serves as an adjoint to itself on both sides. $\square$

4.2. APPROXIMABILITY

The theory of locally presentable categories gives a standard approach to deciding the existence of adjoints to an embedding functor: if both the ambient category and the subcategory are locally presentable, then the embedding has a left adjoint precisely when the subcategory is closed under certain colimits.

We recall the standard notions that of λ -directed colimit, and that of λ -presentable objects.

Definition 4.1. *Fix regular λ , and a locally small category $\mathcal{C}$.*

(1) *For a poset $(I, \leq)$, if $\forall J \subseteq I$ with $|J| < \lambda$, $\exists u \in I$ such that $\forall j \in J, j \leq u$, then I is called λ -**directed**.*

(2) *For $K \in \text{ob}(\mathcal{C})$, if*

$$\text{Hom}_{\mathcal{C}}(K, \varinjlim_{i \in I} C_i) \cong \varinjlim_{i \in I} \text{Hom}_{\mathcal{C}}(K, C_i), \quad \forall \lambda\text{-directed diagram } \{C_i\}_{i \in I} \in \text{ob}(\mathcal{C})^I$$

*then K is called λ -**presentable**.*

*Simply, when some regular cardinal λ makes an object λ -presentable, then the object is called **presentable**.*

(3) *For the category $\mathcal{C}$, suppose that $\mathcal{C}$ satisfies*

- (a) $\varinjlim D \in \text{ob}(\mathcal{C})$, $\forall$ small diagram $D : I \rightarrow \mathcal{C}$, i.e., $\mathcal{C}$ is cocomplete;
(b) $\exists$ set $\mathcal{S} \subseteq \{K \in \text{ob}(\mathcal{C}) : K \text{ is } \lambda\text{-presentable}\}$, such that $\forall L \in \text{ob}(\mathcal{C})$, $\exists$ diagram $D : I \rightarrow \mathcal{S}$ with λ -directed index set I satisfying $L \cong \varinjlim D$.

Under this condition, we refer to such a category $\mathcal{C}$ as **locally λ -presentable**.

Simply, when regular cardinal λ makes a locally small category λ -presentable, we refer to such a category as **locally presentable**.

Remark 4.1. Fix small diagram $D : I \rightarrow \text{Filter}(X)$ with $D(i) = \mathcal{F}_i$. Then

$$\varprojlim D = \bigcap_{i \in I} \mathcal{F}_i, \quad \varinjlim D = \bigvee_{i \in I} \mathcal{F}_i.$$

For locally presentable categories, we can apply the following adjoint functor theorem. One can find the details of this proof in [AR94, Theorem 1.66].

Theorem 4.1. (Adjoint Functor Theorem, [AR94, Theorem 1.66]) Fix two categories $\mathcal{C}$ and $\mathcal{D}$, which are locally presentable, and a functor $F : \mathcal{C} \rightarrow \mathcal{D}$. Under these hypotheses, a left adjoint for F exists precisely when it satisfies conditions at the same time:

1. $\forall$ small diagram $D : I \rightarrow \mathcal{C}$, $F(\varinjlim D) \cong \varinjlim F \circ D$;
2. $\exists$ regular λ such that $\forall$ diagram $D' : I' \rightarrow \mathcal{C}$ with λ -directed index set I' , $F(\varinjlim D') \cong \varinjlim F \circ D'$.

Actually, the category $\text{Filter}_\kappa(X)$ is not always locally presentable. Thus the adjoint functor theorem sometimes fails.

Theorem 4.2. For two regular cardinals λ, κ , if $|X| \geq \lambda > \kappa$, $\text{Filter}_\kappa(X)$ is not λ -presentable.

Proof. We recall the construction similar to the Fréchet filter.

Let

$$I := \{A \in \mathcal{P}(X) : |A| < \lambda\},$$

ordered by inclusion. Since λ is regular, any subfamily with cardinal $< \lambda$ has a union with cardinal $< \lambda$, which is again in I , hence, I is λ -directed.

Define

$$\mathcal{F}_A := \uparrow(X \setminus A) \in \text{Filter}_\kappa(X), \quad \forall A \in I.$$

Consider the diagram $\{\mathcal{F}_A\}_{A \in I}$ with colimit

$$\begin{aligned} \mathcal{F} &:= \bigvee_{A \in I} \uparrow(X \setminus A) = \left\langle \bigcup_{A \in I} \uparrow(X \setminus A) \right\rangle \\ &= \{B \in \mathcal{P}(X) : \exists A \in I, X \setminus A \subseteq B\} = \{B \in \mathcal{P}(X) : |X \setminus B| < \lambda\}. \end{aligned}$$

We claim $\mathcal{F} \notin \text{Filter}_\kappa(X)$.

Suppose $\exists \mathcal{S} \subseteq \mathcal{P}(X)$ with $|\mathcal{S}| < \kappa < \lambda$ satisfying $\mathcal{F} = \langle \mathcal{S} \rangle$.

For every $S \in \mathcal{S}$, we have $S \in \mathcal{F}$, so

$$|X \setminus S| < \lambda.$$

Now define

$$D := \bigcup_{S \in \mathcal{S}} (X \setminus S).$$

Since λ is regular, we have $|D| < \lambda \leq |X|$. Therefore, $\exists x \in X \setminus D$.

Turn to the set $X \setminus \{x\} \in \mathcal{F}$.

Since $\mathcal{F} = \langle \mathcal{S} \rangle$, there exist finitely many generators $(S_1, \dots, S_n) \in \mathcal{S}^n$ making $\bigcap_{i=1}^n S_i \subseteq X \setminus \{x\}$.

By Taking complements,

$$\{x\} \subseteq \bigcup_{i=1}^n (X \setminus S_i) \subseteq D,$$

which implies $x \in D$, contradiction.

Thus no such generating family $\mathcal{S}$ exists.

Therefore, $\text{Filter}_\kappa(X)$ does not have all λ -directed colimits, and consequently it cannot be a λ -presentable category. □

It is immediate to get the following statement.

Corollary 4.1. *Fix infinite X with $|X| = \kappa$, the Fréchet filter $\mathcal{F}$ on X . Under these hypotheses,*

$$\mathcal{F} \in \text{Filter}_\kappa(X) \text{ but } \mathcal{F} \notin \text{Filter}_\alpha(X), \quad \forall \alpha < \kappa.$$

The locally presentable criterion is global and non-constructive, but tells us that the existence of adjoints is a matter of closure under λ -directed limits and colimits. To make this effective, however, we need a condition that can be checked locally. We now introduce the notion of κ -descending approximability and κ -ascending approximability, which are pointwise criteria equivalent to the existence of the adjoints.

Definition 4.2. (κ -Descending Approximability) *Fix cardinals $\kappa \leq \lambda$ and let $\mathcal{G}$ be a λ -generated filter. If $\mathcal{G}$ has a largest κ -generated subfilter $\mathcal{F}$, then we say that $\mathcal{G}$ is κ -descending approximable.*

The adjoint criteria established below rely on a standard uniqueness property of adjoints. For completeness, we state this fact as a lemma below. A proof appears in Mac Lane's monograph on category theory [ML98].

Lemma 4.1. [ML98, Ch. IV, §1, Corollary 1] Fix a functor G . If F and F' are two right adjoints (resp. left adjoints) of G , then they are naturally isomorphic.

Proposition 4.3. (Right Adjoint Criterion) Fix X and cardinals $\kappa \leq \lambda$. The embedding functor

$$\iota_\lambda^{(\kappa)} : \text{Filter}_\kappa(X) \hookrightarrow \text{Filter}_\lambda(X)$$

has a right adjoint

$$R : \text{Filter}_\lambda(X) \rightarrow \text{Filter}_\kappa(X)$$

if and only if every λ -generated filter $\mathcal{G}$ on X is κ -descending approximable.

Moreover, if it exists, the right adjoint is uniquely determined by

$$R(\mathcal{G}) = \bigvee \{ \mathcal{H} \in \text{Filter}_\kappa(X) : \mathcal{H} \subseteq \mathcal{G} \}.$$

Proof. ($\Rightarrow$) Suppose $\iota_\lambda^{(\kappa)}$ has right adjoint $R : \text{Filter}_\lambda(X) \rightarrow \text{Filter}_\kappa(X)$. Then by definition,

$$\iota_\lambda^{(\kappa)}(\mathcal{H}) \subseteq \mathcal{G} \iff \mathcal{H} \subseteq R(\mathcal{G}), \quad \forall (\mathcal{H}, \mathcal{G}) \in \text{Filter}_\kappa(X) \times \text{Filter}_\lambda(X).$$

Now take $\mathcal{H} = R(\mathcal{G})$. Indeed, $R(\mathcal{G}) = \iota_\lambda^{(\kappa)}(R(\mathcal{G}))$ is a κ -generated subfilter of $\mathcal{G}$.

Now let $\mathcal{H} \subseteq \mathcal{G}$ be any κ -generated subfilter. Then $\iota_\lambda^{(\kappa)}(\mathcal{H}) \subseteq \mathcal{G}$, so $\mathcal{H} \subseteq R(\mathcal{G})$. Hence $R(\mathcal{G})$ is the largest κ -generated subfilter of $\mathcal{G}$. Therefore $\mathcal{G}$ is κ -descending approximable.

($\Leftarrow$) Conversely, assume every λ -generated filter $\mathcal{G}$ on X is κ -descending approximable. For each $\mathcal{G} \in \text{Filter}_\lambda(X)$, it has a largest κ -generated subfilter $R(\mathcal{G})$.

Define

$$R : \text{Filter}_\lambda(X) \rightarrow \text{Filter}_\kappa(X), \quad \mathcal{G} \mapsto R(\mathcal{G}).$$

$\forall (\mathcal{G}_1, \mathcal{G}_2) \in \text{Filter}_\lambda(X) \times \text{Filter}_\lambda(X)$, if $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then any κ -generated subfilter of $\mathcal{G}_1$ is also a subfilter of $\mathcal{G}_2$, so $R(\mathcal{G}_1) \subseteq R(\mathcal{G}_2)$. Then R is a functor.

We verify that R is a right adjoint to $\iota_\lambda^{(\kappa)}$.

$\forall (\mathcal{H}, \mathcal{G}) \in \text{Filter}_\kappa(X) \times \text{Filter}_\lambda(X)$, If $\iota_\lambda^{(\kappa)}(\mathcal{H}) \subseteq \mathcal{G}$, then $\mathcal{H}$ is a κ -generated subfilter of $\mathcal{G}$, which forces $\mathcal{H} \subseteq R(\mathcal{G})$ by maximality of $R(\mathcal{G})$.

Conversely, if $\mathcal{H} \subseteq R(\mathcal{G})$, then since $R(\mathcal{G}) \subseteq \mathcal{G}$, we have $\iota_\lambda^{(\kappa)}(\mathcal{H}) = \mathcal{H} \subseteq \mathcal{G}$.

Thus $\iota_\lambda^{(\kappa)}(\mathcal{H}) \subseteq \mathcal{G} \iff \mathcal{H} \subseteq R(\mathcal{G})$, which is exactly $\iota_\lambda^{(\kappa)} \dashv R$.

The uniqueness follows from Lemma 4.1. □

Dually, we will define κ -ascending approximability.

Definition 4.3. (κ -Ascending Approximability) Fix cardinals $\kappa \leq \lambda$ and let $\mathcal{G}$ be a λ -generated filter. If $\mathcal{G}$ has a smallest κ -generated superfilter $\mathcal{F}$, then we say that $\mathcal{G}$ is κ -**ascending approximable**.

Proposition 4.4. (Left Adjoint Criterion) Fix X and cardinals $\kappa \leq \lambda$. The embedding functor

$$\iota_\lambda^{(\kappa)} : \text{Filter}_\kappa(X) \hookrightarrow \text{Filter}_\lambda(X)$$

has a left adjoint

$$L : \text{Filter}_\lambda(X) \rightarrow \text{Filter}_\kappa(X)$$

if and only if every λ -generated filter $\mathcal{G}$ on X is κ -ascending approximable.

Moreover, if it exists, the left adjoint is uniquely determined by

$$L(\mathcal{G}) = \bigcap \{ \mathcal{H} \in \text{Filter}_\kappa(X) : \mathcal{G} \subseteq \mathcal{H} \}.$$

Proof sketch. The argument for the left adjoint proceeds by duality. To obtain the proof, one simply reverses all arrows in the reasoning of Theorem 4.3. $\square$

By Theorem 4.2, $\bigvee \{ \mathcal{H} \in \text{Filter}_\kappa(X) : \mathcal{H} \subseteq \mathcal{G} \}$ may leave the subcategory $\text{Filter}_\kappa(X)$.

Here is an example to show that $\bigcap \{ \mathcal{H} \in \text{Filter}_\kappa(X) : \mathcal{G} \subseteq \mathcal{H} \}$ may leave the subcategory $\text{Filter}_\kappa(X)$.

Example 4.1. Take infinite κ , and view it as a set containing all the cardinals smaller than it.

Consider

$$X := \kappa \times \kappa.$$

Define

$$A_i := \{i\} \times \kappa, \quad \forall i < \kappa,$$

and

$$\mathcal{F}_i := \{ B \in \mathcal{P}(X) : A_i \setminus B \text{ is finite} \}, \quad \forall i < \kappa.$$

We claim that each $\mathcal{F}_i$ is κ -generated.

Fix $i < \kappa$. Consider

$$\mathcal{S}_i := \{ A_i \setminus C : C \subseteq A_i \text{ and } C \text{ is finite} \}.$$

Then

$$|\mathcal{S}_i| = |\{ C \subseteq A_i : C \text{ is finite} \}| = \left| \bigcup_{n < \aleph_0} \{ C \subseteq A_i : |C| = n \} \right| \leq \sum_{n < \aleph_0} |A_i| = \aleph_0 \cdot |A_i| = \kappa,$$

and

$$|\mathcal{S}_i| = |\{C \subseteq A_i : C \text{ is finite}\}| \geq |\{\{i\} \times \{j\} : j < \kappa\}| = \kappa.$$

Thus, by the Cantor–Bernstein Theorem, $|\mathcal{S}_i| = \kappa$.

We claim that $\langle \mathcal{S}_i \rangle = \mathcal{F}_i$.

Take any $A_i \setminus C \in \mathcal{S}_i$, where $C \subseteq A_i$ and C is finite. Then $A_i \setminus (A_i \setminus C) = C$, which is finite. Hence $A_i \setminus C \in \mathcal{F}_i$. Therefore

$$\mathcal{S}_i \subseteq \mathcal{F}_i,$$

which implies

$$\langle \mathcal{S}_i \rangle \subseteq \mathcal{F}_i.$$

Take any $B \in \mathcal{F}_i$, then $A_i \setminus B$ is finite. Let

$$C := A_i \setminus B.$$

Then $C \subseteq A_i$ and C is finite, so

$$A_i \cap B = A_i \setminus C \in \mathcal{S}_i.$$

Since $B \supseteq A_i \cap B$, $B \in \langle \mathcal{S}_i \rangle$. Therefore

$$\mathcal{F}_i \subseteq \langle \mathcal{S}_i \rangle.$$

Thus $\langle \mathcal{S}_i \rangle = \mathcal{F}_i$, hence, $\mathcal{F}$ is κ -generated.

Then we prove that the intersection

$$\mathcal{F} := \bigcap_{i < \kappa} \mathcal{F}_i.$$

is not κ -generated.

Assume that $\mathcal{F} = \langle \{X_\alpha : \alpha < \kappa\} \rangle$.

Define

$$B_\alpha := X \setminus X_\alpha, \quad \forall \alpha < \kappa.$$

For every $i < \kappa$, since $X_\alpha \in \mathcal{F}$, we have $X_\alpha \in \mathcal{F}_i$. Hence, for every $i < \kappa$, $A_i \setminus X_\alpha$ is finite, then $B_\alpha \cap A_i$ is finite.

Our goal is to construct a set $B \subseteq X$ satisfying $B \cap A_i$ is finite for every $i < \kappa$, but for every finite $E \subseteq \kappa$,

$$B \not\subseteq \bigcup_{\alpha \in E} B_\alpha.$$

In the same way as the proof for $|\mathcal{S}_i| = \kappa$, we can enumerate all finite subsets of κ as $\{E_\eta : \eta < \kappa\}$.

For each $\eta < \kappa$, since

$$\bigcup_{\alpha \in E_\eta} B_\alpha \cap A_\eta$$

is finite, and A_η is infinite, there is some

$$b_\eta \in A_\eta \setminus \bigcup_{\alpha \in E_\eta} B_\alpha.$$

Let

$$B := \{b_\eta : \eta < \kappa\}.$$

Since the family $\{A_i : i < \kappa\}$ be disjoint, then $B \cap A_i = \{b_i\}$, $\forall i < \kappa$, which is finite. Therefore

$$X \setminus B \in \mathcal{F}.$$

For every finite $E \subseteq \kappa$, $\exists \eta < \kappa$ such that $E_\eta = E$. Then

$$b_\eta \notin \bigcup_{\alpha \in E_\eta} B_\alpha,$$

so

$$B \not\subseteq \bigcup_{\alpha \in E_\eta} B_\alpha.$$

Equivalently,

$$X \setminus B \not\supseteq X \setminus \bigcup_{\alpha \in E_\eta} B_\alpha = X \setminus \bigcup_{\alpha \in E_\eta} (X \setminus X_\alpha) = \bigcap_{\alpha \in E_\eta} X_\alpha, \quad \forall \eta < \kappa.$$

Thus $X \setminus B \notin \mathcal{F}$, contradiction.

It follows that $\mathcal{F}$ is not κ -generated.

4.3. OTHER CASES

In this part, we suppose that X is infinite and $\kappa < \lambda$ are infinite with $\kappa < 2^{|X|}$.

4.3. EXISTENCE OF RIGHT ADJOINT

Our first task is to determine whether a right adjoint exists when $\kappa < |X|$.

Theorem 4.3. *If $\kappa < |X|$, then $\iota_\lambda^{(\kappa)}$ has no right adjoint.*

Proof. For the first statement, let us first prove the non-existence of the right adjoint.

Assume that $\iota_\lambda^{(\kappa)}$ has a right adjoint R . Then for every λ -generated filter $\mathcal{G}$, $R(\mathcal{G})$ must be the largest κ -generated subfilter of $\mathcal{G}$.

- (1) If $|X| \leq \lambda$, take $\mathcal{G}$ to be the Fréchet filter on X . By Corollary 4.1, $\mathcal{G} \in \text{Filter}_{|X|}(X) \subseteq \text{Filter}_\lambda(X)$ but $\mathcal{G} \notin \text{Filter}_\alpha(X), \forall \alpha < |X|$.

Let $\mathcal{F} = \langle \{F_\xi \in \mathcal{P}(X) : \xi < \kappa\} \rangle$ be an κ -generated subfilter of $\mathcal{G}$. Since $\mathcal{F} \subseteq \mathcal{G}$, then $\forall \xi < \kappa, X \setminus F_\xi$ is finite.

First, we will prove $\bigcap_{\xi < \kappa} F_\xi \neq \emptyset$. Assume $\bigcap_{\xi < \kappa} F_\xi = \emptyset$, then $X = \bigcup_{\xi < \kappa} (X \setminus F_\xi)$. By the finiteness of $X \setminus F_\xi$ for every $\xi < \kappa$,

$$\left| \bigcup_{\xi < \kappa} (X \setminus F_\xi) \right| \leq \kappa \cdot \aleph_0 = \kappa < |X|,$$

contradiction. Consequently,

$$\bigcap_{\xi < \kappa} F_\xi \neq \emptyset.$$

Pick an element $x \in \bigcap_{\xi < \kappa} F_\xi$, and turn to the set $X \setminus \{x\}$. Observe the finiteness of

$$X \setminus (X \setminus \{x\}) = \{x\},$$

so $X \setminus \{x\} \in \mathcal{G}$. Assume that $X \setminus \{x\} \in \mathcal{F}$, then the generating family provides finitely many $\xi_1, \dots, \xi_n < \kappa$ satisfying $F_{\xi_1} \cap \dots \cap F_{\xi_n} \subseteq X \setminus \{x\}$. But $x \in F_\xi$ for every $\xi < \kappa$, so $x \in F_{\xi_1} \cap \dots \cap F_{\xi_n} \subseteq X \setminus \{x\}$, contradiction.

Consequently,

$$X \setminus \{x\} \notin \mathcal{F}.$$

Construct a new filter

$$\mathcal{F}' := \langle \mathcal{F} \cup \{X \setminus \{x\}\} \rangle \in \text{Filter}_\kappa(X).$$

Immediately, $\mathcal{F}' \subseteq \mathcal{G}$ and $\mathcal{F} \subsetneq \mathcal{F}'$.

Hence $\mathcal{G}$ has no largest κ -generated subfilter, contradiction.

- (2) If $|X| > \lambda$, we can take $Y \subseteq X$ with $|Y| = \lambda$.

Define

$$\mathcal{G} := \{U \in \mathcal{P}(X) : Y \setminus U \text{ is finite}\} = \langle \{Y \setminus \{y\} : y \in Y\} \rangle.$$

Actually, observe that if we restrict $\mathcal{G}$ to the subset Y , $\mathcal{G}|_Y$ is the Fréchet filter on Y . By Corollary 4.1 applied to the set Y ,

$$\mathcal{G}|_Y \in \text{Filter}_\lambda(Y) \text{ and } \mathcal{G}|_Y \notin \text{Filter}_\alpha(Y), \quad \forall \alpha < \lambda.$$

Then

$$\mathcal{G} \in \text{Filter}_\lambda(X) \text{ and } \mathcal{G} \notin \text{Filter}_\alpha(X), \quad \forall \alpha < \lambda.$$

Now assume that $\mathcal{G}$ has a largest κ -generated subfilter $\mathcal{F} = \langle \{F_\xi : \xi < \kappa\} \rangle$. Then $\forall \xi < \kappa$, $Y \setminus F_\xi$ is finite.

We claim that

$$\bigcap_{\xi < \kappa} (Y \cap F_\xi) \neq \emptyset.$$

Indeed, assume that it holds, then $Y = \bigcup_{\xi < \kappa} (Y \setminus F_\xi)$. Since each $Y \setminus F_\xi$ is finite

$$|Y| = \left| \bigcup_{\xi < \kappa} (Y \setminus F_\xi) \right| \leq \kappa \cdot \aleph_0 = \kappa < \lambda = |Y|,$$

contradiction.

Thus, $\exists y \in \bigcap_{\xi < \kappa} (Y \cap F_\xi)$.

Now consider $X \setminus \{y\}$. Observe the finiteness of

$$Y \setminus (X \setminus \{y\}) = \{y\},$$

so $X \setminus \{y\} \in \mathcal{G}$.

Assume $X \setminus \{y\} \in \mathcal{F}$. Then $\exists$ finitely many $\xi_1, \dots, \xi_n < \kappa$ such that

$$F_{\xi_1} \cap \dots \cap F_{\xi_n} \subseteq X \setminus \{y\}.$$

But actually, $y \in \bigcap_{\xi < \kappa} (Y \cap F_\xi)$, we have $y \in F_{\xi_i}$ for every $i \in \{1, \dots, n\}$. Hence

$$y \in F_{\xi_1} \cap \dots \cap F_{\xi_n} \subseteq X \setminus \{y\},$$

contradiction. Therefore $X \setminus \{y\} \notin \mathcal{F}$.

Now define

$$\mathcal{F}' := \langle \mathcal{F} \cup \{X \setminus \{y\}\} \rangle,$$

which is still κ -generated. It is immediate to get $\mathcal{F}' \subseteq \mathcal{G}$ and $\mathcal{F} \subsetneq \mathcal{F}'$.

Hence $\mathcal{G}$ has no largest κ -generated subfilter, contradiction.

Therefore, by Proposition 4.3, $\iota_\lambda^{(\kappa)}$ has no right adjoint when $\kappa < |X|$. □

For the existence of right adjoint, now we are left with the case $|X| \leq \kappa < 2^{|X|}$. We begin by introducing a classic construction. This result is often attributed to Fichtenholz and Kantorovich [FK34]. In their original paper, the statement appears in §4, where the authors introduce the notion of a *noyau indépendant* (independent, defined as below). They use it to show that the dual space consisting of bounded measurable functions has cardinality $2^{2^{\aleph_0}}$. Hausdorff, in 1936, simplified their argument and generalized the result to arbitrary infinite sets [Hau36].

Definition 4.4. For an infinite set X , a family $\mathcal{Z} = \{Z_i\}_{i \in I} \subseteq \mathcal{P}(X)$ where $Z_i \neq Z_j$ for every distinct indices $(i, j) \in I^2$, is called **independent** if for any finite sets of distinct indices $(t_1, \dots, t_p, t'_1, \dots, t'_q) \in I^{p+q}$,

$$\bigcap_{i=1}^p Z_{t_i} \cap \left(C \setminus \bigcup_{j=1}^q Z_{t'_j} \right) \neq \emptyset.$$

Lemma 4.2. [Hau36, II] Fix infinite X . Then there exist an family $\mathcal{Z} \subseteq \mathcal{P}(X)$ with $|\mathcal{Z}| = 2^{|X|}$ which is independent.

Proof. Define

$$\mathcal{S} := \{S \in \mathcal{P}(X) : S \text{ is finite}\}.$$

Since X is infinite, by the process from Example 4.1, $|\mathcal{S}| = |X|$. Define

$$Y := \{(S, \mathcal{H}) : S \in \mathcal{S} \text{ and } \mathcal{H} \subseteq \mathcal{P}(S)\}.$$

Then $|Y| = \sum_{S \in \mathcal{S}} 2^{2^{|S|}} = |X|$.

Define

$$Z_A := \{(S, \mathcal{H}) \in Y : A \cap S \in \mathcal{H}\}, \quad \forall A \in \mathcal{P}(X).$$

It is immediate that $|\{Z_A : A \in \mathcal{P}(X)\}| = 2^{|X|}$.

We now show that $\{Z_A : A \in \mathcal{P}(X)\}$ is an independent family.

Take finitely many distinct subsets

$$(A_1, \dots, A_p, B_1, \dots, B_q) \in \mathcal{P}(X)^{p+q}.$$

For any indices $(i, j) \in \{1, \dots, p\} \times \{1, \dots, q\}$, since $A_i \neq B_j$, there is

$$s_{ij} \in (A_i \setminus B_j) \cup (B_j \setminus A_i).$$

Let

$$S := \{s_{ij} : 1 \leq i \leq p, 1 \leq j \leq q\},$$

and

$$\mathcal{H} := \{A_1 \cap S, \dots, A_p \cap S\}.$$

Then $(S, \mathcal{H}) \in Y$ and $A_i \cap S \in \mathcal{H}$, hence $(S, \mathcal{H}) \in Z_{A_i}$, for each i .

For each j , if $B_j \cap S \in \mathcal{H}$, then $B_j \cap S = A_i \cap S$ for some i , then for all $s_{ij} \in S$, $s_{ij} \in A_i$ if and only if $s_{ij} \in B_j$.

But we have $s_{ij} \in (A_i \setminus B_j) \cup (B_j \setminus A_i)$, then s_{ij} belongs to exactly one of A_i and B_j , contradiction. Hence $B_j \cap S \notin \mathcal{H}$, so $(S, \mathcal{H}) \notin Z_{B_j}$, i.e., $(S, \mathcal{H}) \in (Y \setminus Z_{B_j})$.

Therefore,

$$(S, \mathcal{H}) \in Z_{A_1} \cap \dots \cap Z_{A_p} \cap (Y \setminus Z_{B_1}) \cap \dots \cap (Y \setminus Z_{B_q}),$$

then

$$Z_{A_1} \cap \cdots \cap Z_{A_p} \cap (Y \setminus Z_{B_1}) \cap \cdots \cap (Y \setminus Z_{B_q}) \neq \emptyset.$$

So the family $\{Z_A : A \in \mathcal{P}(X)\} \subseteq \mathcal{P}(Y)$ is independent of size $2^{|X|}$.

The equality $|Y| = |X|$ ensures the existence of a bijection $\varphi : Y \rightarrow X$.

Define

$$Z'_A := \varphi(Z_A) \in \mathcal{P}(X), \quad \forall A \in \mathcal{P}(X).$$

Then for any finitely many distinct subsets

$$(A_1, \dots, A_p, B_1, \dots, B_q) \in \mathcal{P}(X)^{p+q},$$

take the element $(S, \mathcal{H})$ constructed as above. Because

$$(S, \mathcal{H}) \in Z_{A_1} \cap \cdots \cap Z_{A_p} \cap (Y \setminus Z_{B_1}) \cap \cdots \cap (Y \setminus Z_{B_q}),$$

we have

$$\varphi((S, \mathcal{H})) \in Z'_{A_1} \cap \cdots \cap Z'_{A_p} \cap (X \setminus Z'_{B_1}) \cap \cdots \cap (X \setminus Z'_{B_q}).$$

Consequently, $\mathcal{Z} := \{Z'_A : A \in \mathcal{P}(X)\} \subseteq \mathcal{P}(X)$ is independent and has cardinality $2^{|X|}$. □

Corollary 4.2. (*Fichtenholz–Kantorovich–Hausdorff*)/[FK34, §4]

Given an infinite set X and cardinal $\mu \leq 2^{|X|}$, one can find a family $\{A_\alpha \in \mathcal{P}(X) : \alpha < \mu\}$ satisfying $\forall$ finite $F \subseteq \mu$ and every $\beta \in \mu \setminus F$,

$$\bigcap_{\alpha \in F} A_\alpha \not\subseteq A_\beta.$$

Proof. By Lemma 4.2, we have an independent family $\mathcal{Z} = \{Z_t \in \mathcal{P}(X) : t < 2^{|X|}\} \subseteq \mathcal{P}(X)$ with cardinality $2^{|X|}$. Then for any distinct $t_1, \dots, t_p, t'$,

$$Z_{t_1} \cap \cdots \cap Z_{t_p} \cap (X \setminus Z_{t'}) \neq \emptyset.$$

Thus

$$Z_{t_1} \cap \cdots \cap Z_{t_p} \not\subseteq Z_{t'}.$$

Take any subfamily $\{A_\alpha : \alpha < \mu\} \subseteq \{Z_t : t < 2^{|X|}\}$ of size μ . Then the desired condition holds. □

Theorem 4.4. If $|X| \leq \kappa < 2^{|X|}$, then $\iota_\lambda^{(\kappa)}$ has no right adjoint.

Proof. The inequalities $\lambda \geq \kappa^+$ and $\kappa^+ \leq 2^{|X|}$ are immediate consequences of our hypotheses. By Corollary 4.2, $\exists$ a family $\{A_\alpha \in \mathcal{P}(X) : \alpha < \kappa^+\}$ satisfying $\forall$ finite $F \subseteq \kappa^+$ and every $\beta \in \kappa^+ \setminus F$,

$$\bigcap_{\alpha \in F} A_\alpha \not\subseteq A_\beta.$$

Consider the filter

$$\mathcal{G} := \langle \{A_\alpha : \alpha < \kappa^+\} \rangle \in \text{Filter}_{\kappa^+}(X) \subseteq \text{Filter}_\lambda(X).$$

We claim that $\mathcal{G}$ has no largest κ -generated subfilter.

Suppose that $\mathcal{G}$ has a largest κ -generated subfilter $\mathcal{F} := \langle \{F_\xi : \xi < \kappa\} \rangle$.

Since $\mathcal{F} \subseteq \mathcal{G}$, for each $\xi < \kappa$, $F_\xi \in \mathcal{G}$, then $\exists$ finite $S_\xi \subseteq \kappa^+$ such that

$$\bigcap_{\alpha \in S_\xi} A_\alpha \subseteq F_\xi.$$

Now define

$$T := \bigcup_{\xi < \kappa} S_\xi,$$

then

$$|T| \leq \kappa \cdot \aleph_0 = \kappa < \kappa^+ = |\kappa^+|.$$

Therefore, there exists some index $\beta \in \kappa^+ \setminus T$.

Consider the set $A_\beta \in \mathcal{G}$. Assume $A_\beta \in \mathcal{F}$. Then there exist finitely many $\xi_1, \dots, \xi_n < \kappa$ such that

$$F_{\xi_1} \cap \dots \cap F_{\xi_n} \subseteq A_\beta.$$

Then

$$\bigcap_{\alpha \in S_{\xi_1} \cup \dots \cup S_{\xi_n}} A_\alpha = \bigcap_{i=1}^n \bigcap_{\alpha \in S_{\xi_i}} A_\alpha \subseteq \bigcap_{i=1}^n F_{\xi_i} \subseteq A_\beta,$$

contradiction. Hence $A_\beta \notin \mathcal{F}$.

Now define

$$\mathcal{F}' := \langle \mathcal{F} \cup \{A_\beta\} \rangle,$$

which is again a κ -generated subfilter of $\mathcal{G}$, strictly larger than $\mathcal{F}$, contradiction.

Therefore $\mathcal{G}$ has no largest κ -generated subfilter. By Proposition 4.3, $\iota_\lambda^{(\kappa)}$ has no right adjoint. $\square$

4.3. EXISTENCE OF LEFT ADJOINT

Theorem 4.5. *If $\kappa < |X|$, then $\iota_\lambda^{(\kappa)}$ has no left adjoint.*

Proof. We first discuss the case $|X| \geq \lambda$. The construction is similar to Example 4.1. From $|X| \geq \lambda > \kappa$, it follows that one can pick a subset $Y \subseteq X$ with cardinal $|Y| = \lambda$. Since λ is infinite, the product

$$\lambda \cdot \kappa = \max(\lambda, \kappa) = \lambda.$$

Hence, $\exists$ bijection

$$f : Y \rightarrow \lambda \times \kappa.$$

Define

$$A_i := f^{-1}(\{i\} \times \kappa), \quad \forall i < \lambda.$$

Thus, we get the partition

$$Y = \bigcup_{i < \lambda} A_i, \quad |A_i| = \kappa, \quad \forall i < \lambda.$$

Define

$$\mathcal{F}_i := \{B \in \mathcal{P}(X) : A_i \setminus B \text{ is finite}\} = \langle \mathcal{S}_i \rangle, \quad \forall i < \lambda,$$

where $\mathcal{S}_i := \{A_i \setminus C : C \subseteq A_i, C \text{ is finite}\}$. Hence $\mathcal{F}_i$ is κ -generated.

Let

$$\mathcal{G} := \bigcap_{i < \lambda} \mathcal{F}_i.$$

By the same argument as in Example 4.1, we conclude that $\mathcal{G}$ is not κ -generated.

Assume that a left adjoint L exists. Then for each $i < \lambda$, $\mathcal{G} \subseteq \mathcal{F}_i$, so

$$L(\mathcal{G}) \subseteq \bigcap_{i < \lambda} \mathcal{F}_i = \mathcal{G}.$$

The inclusion $\mathcal{G} \subseteq L(\mathcal{G})$ holds as well. Combining this with the previous containment forces $L(\mathcal{G}) = \mathcal{G} \notin \text{Filter}_\kappa(X)$, which does not belong to $\text{Filter}_\kappa(X)$. By Proposition 4.4, no left adjoint exists.

Now we discuss the case $|X| < \lambda$.

Let $\mu := |X|$. Since $\kappa < \mu < \lambda$, there is a decomposition as a disjoint union

$$X = \bigcup_{i < \mu} A_i, \quad |A_i| = \kappa.$$

For each $i < \mu$, define $\mathcal{F}_i := \{B \in \mathcal{P}(X) : A_i \setminus B \text{ is finite}\}$.

Let

$$\mathcal{G} := \bigcap_{i < \mu} \mathcal{F}_i.$$

By the same argument as in Example 4.1, we conclude that $\mathcal{G}$ is not κ -generated.

Assume that a left adjoint L exists. The proof that $L(\mathcal{G}) = \mathcal{G} \notin \text{Filter}_\kappa(X)$ is identical to the first case with λ replaced by μ . By Proposition 4.4, no left adjoint exists. □

5. CONCLUSION AND OUTLOOK

The preceding sections have examined whether the embedding functor $\iota_\lambda^{(\kappa)}$ admits adjoints on either side. For the right adjoint, we obtain a complete result: $\iota_\lambda^{(\kappa)}$ has no right adjoint if and only if X is infinite, and $\kappa < \lambda$ are infinite with $\kappa < 2^{|X|}$. For the left adjoint, however, one case remains open: X is infinite and $\kappa < \lambda$ are infinite with $|X| \leq \kappa < 2^{|X|}$.

5.1. OPEN PROBLEMS

Question 5.1. *Fix infinite X , and let $\kappa < \lambda$ be infinite cardinals. When $|X| \leq \kappa < 2^{|X|}$, it is unknown whether $\iota_\lambda^{(\kappa)}$ has left adjoint.*

If we assume GCH, the situation can be simpler.

Question 5.2. *Fix infinite X , and let $\kappa = |X| < \lambda$ be infinite cardinals. It is unknown whether $\iota_\lambda^{(\kappa)}$ has left adjoint.*

By Proposition 4.4 and Theorem 4.5, a natural way to prove that $\iota_\lambda^{(\kappa)}$ has no left adjoint is to force $L(\mathcal{G}) = \mathcal{G}$ and ensure that $\mathcal{G}$ itself is not κ -generated. This suggests writing $\mathcal{G}$ as the intersection of a decreasing sequence of κ -generated filters

$$\mathcal{G} = \bigcap_{\alpha < \kappa^+} \mathcal{F}_\alpha,$$

where each $\mathcal{F}_\alpha$ is κ -generated and $\mathcal{F}_0 \supsetneq \mathcal{F}_1 \supsetneq \dots$. If such a sequence exists and its intersection is not κ -generated, we have

$$L(\mathcal{G}) \subseteq \bigcap_{\alpha < \kappa^+} \mathcal{F}_\alpha = \mathcal{G}.$$

Consequently, $L(\mathcal{G}) = \mathcal{G} \notin \text{Filter}_\kappa(X)$. This observation motivates the notion of a *filter tower*.

Definition 5.1. Fix infinite X , and let $\kappa = |X| < \lambda$ be infinite cardinals. A non-degenerate sequence $\{\mathcal{F}_\alpha \in \text{Filter}_\kappa(X) : \alpha < \kappa^+\}$ is called a **filter tower** if

1. $\mathcal{F}_{\alpha+1} \subsetneq \mathcal{F}_\alpha$ for all $\alpha < \kappa^+$, and for limit α , $\mathcal{F}_\alpha = \bigcap_{\beta < \alpha} \mathcal{F}_\beta$;
2. $\bigcap_{\alpha < \kappa^+} \mathcal{F}_\alpha \notin \text{Filter}_\kappa(X)$.

Question 5.3. How to construct such a filter tower on X remains unclear.

Constructing the filter tower directly is difficult. We can replace filters by their generating sets and work with a family whose members are subsets of X . This leads to the notion of a (set) tower, which are easier to construct than filter towers because the almost-inclusion relation $\subseteq^*$ (defined as below) is combinatorially simpler than the inclusion of filters, and because one can use trees, branches, and forcing to build them.

Notation 5.1. Fix two subsets A and B of κ , both of cardinality κ .

- (1) $A \subseteq^* B$ means that $|A \setminus B| < \kappa$;
- (2) $A \subsetneq^* B$ is used when both $A \subseteq^* B$ and $B \not\subseteq^* A$ hold.

Definition 5.2. Let $\mathcal{T}$ be a family whose members are subsets of κ , each of cardinality κ . If $\mathcal{T}$ satisfies

1. $\exists$ a subfamily $\{A_\alpha : \alpha < \theta\} \subseteq \mathcal{T}$ for some $\theta \leq 2^\kappa$, such that $A_\beta \subsetneq^* A_\alpha$ for every $\alpha < \beta < \theta$;
2. $\forall \mathcal{U} \subseteq \mathcal{T} : |\mathcal{U}| < \kappa, |\bigcap \mathcal{U}| = \kappa$;
3. $\nexists A \subseteq \kappa : |A| = \kappa$ such that $A \subseteq^* T, \forall T \in \mathcal{T}$.

then we call $\mathcal{T}$ a **tower**.

The **tower number** $\mathfrak{t}(\kappa)$ is the least length θ for which there exists a tower of length θ . If no such tower exists, $\mathfrak{t}(\kappa) := \infty$.

Indeed, under large cardinal assumptions, a model relatively consistent with ZFC is constructed by Honzik and Stejskalov [HS23] in which an inaccessible cardinal κ satisfies $\mathfrak{t}(\kappa) = \kappa^+$.

The remaining obstacle is the transition from a set tower to a filter tower. The natural attempt

$$\mathcal{F}_\alpha := \langle \{X \setminus A_\beta : \beta < \alpha\} \rangle$$

produces an increasing sequence of κ -generated filters, while using the tails $\{\beta \geq \alpha\}$ produces a decreasing sequence whose generating set has size κ^+ , so the resulting filters are not κ -generated. Bridging this gap remains an open problem. Maybe a new forcing construction is needed.

5.2. CONNECTIONS

The lack of adjoints admits a clean explanation in terms of reflection and coreflection. In module categories, directed colimits may destroy finite generation, and submodules may not be generated finitely. The same dichotomy appears here. κ -generated filters are not closed under intersections, which obstructs a left adjoint, and they are not closed under unions, which obstructs a right adjoint.

Logically, this is structurally analogous to the downward and upward Löwenheim–Skolem phenomena. The right adjoint plays the role of passing to a smaller subobject, and the left adjoint the role of passing to a larger extension [CK90].

As a final perspective, we expect the techniques developed in this paper to be applicable to other concrete categories where similar approximation problems arise. $\text{Filter}(X)$ is a complete lattice [Mas26]. Its objects are filters and the order is inclusion, so its structure is simple. However, the subcategory of κ -generated filters may be not closed under arbitrary intersections or under generated unions, which makes it rich enough to exhibit the essential difficulty. Consequently, it raises a natural and intriguing question: can we characterize those categories of algebraic structures where the subcategory of κ -generated objects is reflective or coreflective?

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The initial motivation came from a structural observation made while reviewing filter theory: the image-inverse image adjunction for filters, $f^{-1} \dashv f_*$, has the same categorical form as the Galois correspondence in field theory. Later, I researched and learned about the related concepts of adjunction.

I first observed that, under cardinal stratification, the adjunction $f^{-1}|_{\kappa} \dashv f_*|_{\kappa}$ survives at each fixed cardinal level. This naturally led to the question of how these adjunctions compare across different cardinal levels. To make this comparison precise, I formulated the definitions of $\text{Filter}_{\kappa}(X)$ and $\text{Filter}_{\lambda}(X)$, introduced the embedding functor $\iota_{\lambda}^{(\kappa)}$, and derived the explicit left and right adjoint criteria.

The independent family lemma is the one place where I later consulted existing literature. I originally obtained the corollary version myself while working through the left adjoint case. Later, I found that Hausdorff had already proved a stronger lemma [Hau36]. I read his proof and rewrote the argument in a way that fits the present paper.

The main unresolved challenge lies in the transition from a combinatorial tower to a filter tower. While the natural attempt $\mathcal{F}_{\alpha} := \langle \{X \setminus A_{\beta} : \beta < \alpha\} \rangle$ produces an increasing sequence of κ -generated filters, the corresponding tails $\{\beta \geq \alpha\}$ yield a decreasing sequence whose generating set has cardinality κ^+ , thereby failing to remain κ -generated. Bridging this gap is a promising direction for future research, potentially requiring new forcing constructions [HS23].

After the main results were obtained, I consulted my supervisor, Jibin Luo. Given the highly specialized nature of this topic, as a high school teacher, he guided me mainly on language polishing, structural refinement, and overall encouragement of exploring the unknown. As a high school student, I am deeply grateful for his unwavering support in helping me navigate the transition from the textbook to a broader mathematical context. No financial compensation was involved.

Any remaining errors are solely my own.

DECLARATION OF AI USAGE

In accordance with the academic integrity guidelines of the competition, I hereby declare the use of Artificial Intelligence (AI) tools in this project:

Tool Name and Version: DeepSeek, V4-Flash

Phase of Use 1: Literature Search (Approximately 5% of total research time, August to September)

I used AI-assisted search tools to locate the original 1936 paper by Hausdorff [Hau36] regarding the strong independent family lemma. Later, I used it to discover the recent work by Honzik and Stejskalová [HS23] to confirm that the transition problem from a set tower to a filter tower remains open.

Phase of Use 2: Language Polishing (Approximately 2% of total writing time, July to September)

I provided my own drafted text to the AI tool to receive suggestions for more natural academic expression. No mathematical arguments or results were generated by AI.

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