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论文题目: A class of non-weight modules of the Takiff  $N=1$  Heisenberg-Virasoro superalgebra

# A class of non-weight modules of the Takiff $N = 1$ Heisenberg-Virasoro superalgebra

Jeffrey Yang

**Abstract:** Let  $\mathfrak{h} = \mathbb{C}L_0$  be the degree-zero subalgebra of  $\mathcal{H}$ . We construct a new family of  $\mathbb{Z}_2$ -graded  $\mathcal{H}$ -modules  $\Psi(\lambda, \alpha, \beta)$ , parameterized by  $\lambda \in \mathbb{C}^*$  and  $\alpha, \beta \in \mathbb{C}$ , which are free of rank 2 over the universal enveloping algebra  $U(\mathfrak{h}) \cong \mathbb{C}[L_0]$ . We determine the simplicity and isomorphism classes of these modules:  $\Psi(\lambda, \alpha, \beta)$  is simple if and only if  $\alpha \neq 0$  or  $\beta \neq 0$ ; the degenerate module  $\Psi(\lambda, 0, 0)$  possesses a unique proper submodule  $\chi$ , with a one-dimensional trivial quotient, and  $\chi$  is isomorphic to a parity-twisted module  $\Pi(\Psi(\lambda, \frac{1}{2}, 0))$ ; moreover, two such modules are isomorphic precisely when their defining parameters coincide.

Building on these constructions, we provide a complete classification of all  $\mathcal{H}$ -modules that are free of rank 2 over  $U(\mathfrak{h})$ . The proof proceeds via a detailed analysis of the module structure over the Virasoro subalgebra and the actions of the remaining generators, showing that any such module is isomorphic, up to a parity change, to some  $\Psi(\lambda, \alpha, \beta)$ . As a consequence, our classification also determines all  $\mathcal{H}$ -modules that are free of rank 1 over the universal enveloping algebras  $U(\mathbb{C}L_0 \oplus \mathbb{C}G_0)$  and  $U(\mathbb{C}L_0 \oplus \mathbb{C}Q_0)$ , thereby extending known results on rank-one free modules over the Heisenberg-Virasoro algebra.

Finally, by applying the weighting functor introduced by Nilsson, we construct a class of intermediate series weight  $\mathcal{H}$ -modules from the modules  $\Psi(\lambda, \alpha, \beta)$ , and we show that the resulting modules are isomorphic to the intermediate series  $\mathcal{H}$ -modules  $A(\frac{1}{2} - \alpha, \beta, d)$ . This establishes a precise relationship between the non-weight modules studied here and the well-known weight modules in the intermediate series.

**Keywords:** Takiff  $N = 1$  Heisenberg-Virasoro superalgebra; non-weight module; simple module;  $U(\mathfrak{h})$ -free module; weighting functor; module of the intermediate series.

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# 1 Introduction

Let  $\mathbb{C}$  denote the field of complex numbers, and write  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  for the set of nonzero complex numbers. We use  $\mathbb{Z}$  for the set of integers. Throughout this paper, all vector superspaces, Lie superalgebras, and supermodules are defined over  $\mathbb{C}$ , and all modules are assumed to be  $\mathbb{Z}_2$ -graded.

Let  $L_0$  be the even generator with index zero in the Virasoro part of the algebra  $\mathcal{H}$ , and set

$$\mathfrak{h} = \mathbb{C}L_0.$$

Since  $\mathfrak{h}$  is one-dimensional and even,

$$U(\mathfrak{h}) \cong \mathbb{C}[L_0].$$

A module is called a weight module with respect to  $\mathfrak{h}$  if it is the direct sum of its  $L_0$ -eigenspaces. A module that does not have such a decomposition is simply called a non-weight module.

For a basic example, consider the polynomial space  $\mathbb{C}[x]$  and let  $L_0$  act by multiplication by  $x$ :

$$L_0 \cdot f(x) = xf(x).$$

If a nonzero polynomial  $f(x)$  were an eigenvector of  $L_0$  with eigenvalue  $\mu$ , then

$$xf(x) = \mu f(x), \quad \text{so} \quad (x - \mu)f(x) = 0.$$

This is impossible for a nonzero polynomial. Hence  $\mathbb{C}[x]$  is not a weight module with respect to  $\mathfrak{h}$ ; it is therefore a non-weight module.

It is natural to study modules that are free over

$$U(\mathfrak{h}) \cong \mathbb{C}[L_0].$$

The rank counts the number of polynomial generators, rather than the dimension over  $\mathbb{C}$ . For example,

$$M = \mathbb{C}[x]u \oplus \mathbb{C}[x]v$$

is free of rank two over  $\mathbb{C}[L_0]$  when  $L_0$  acts as multiplication by  $x$ . The module is nevertheless infinite-dimensional over  $\mathbb{C}$ . In the graded setting,  $u$  is even and  $v$  is odd.

Polynomial-type representations of this kind, often called  $U(\mathfrak{h})$ -free modules, were first introduced by Nilsson for finite-dimensional simple Lie algebras [5]. Later work by Tan and Zhao, Chen and Guo, Dilxat and Liu, and others extended the theory to infinite-dimensional Lie algebras and Lie superalgebras, including the Virasoro algebra, the  $W(2, 2)$  algebra, the Heisenberg–Virasoro algebra, and several types of super-Virasoro and  $N = 1$  Heisenberg–Virasoro superalgebras [2–4, 7–10].

The Takiff  $N = 1$  Heisenberg–Virasoro superalgebra  $\mathcal{H}$  has four familiar families of generators,

$$L_m, \quad H_m, \quad G_p, \quad Q_p,$$

together with four additional families,

$$W_m, \quad I_m, \quad F_p, \quad P_p.$$

The first four families comprise the basic part of the algebra, while the remaining four arise from the Takiff construction. The relations among these generators are given in Section 3, while the  $N = 1$  Heisenberg–Virasoro superalgebra is studied in [1].

A natural question is whether the additional generators  $W_m, I_m, F_p, P_p$  can act non-trivially on a module of small polynomial rank. In this paper, we study graded  $\mathcal{H}$ -modules that are free of finite rank over  $\mathbb{C}[L_0]$ . Our main result concerns rank two. We prove that, in this case, the additional generators necessarily act as zero. Thus the rank-two problem reduces to determining the possible actions of  $L_m, H_m, G_p, Q_p$ .

Up to parity change, every rank-two module is isomorphic to a module of the form

$$\Psi(\lambda, \alpha, \beta), \quad \lambda \in \mathbb{C}^*, \quad \alpha, \beta \in \mathbb{C}.$$

We determine the submodule structure, simplicity, and isomorphism classes of this family. In particular,

$$\Psi(\lambda, \alpha, \beta) \text{ is simple if and only if } (\alpha, \beta) \neq (0, 0).$$

Finally, we study the weighting functor introduced by Nilsson [6]. Applying this functor to  $\Psi(\lambda, \alpha, \beta)$  produces a family of weight modules  $A(a, b, c)$ . The parameter  $\lambda$ , which distinguishes non-isomorphic polynomial modules, disappears after applying the weighting functor. Moreover, the functor does not necessarily preserve simplicity: it can send a simple polynomial module to a reducible weight module, or a reducible polynomial module to a simple one.

## 2 Basic notations and definitions

This section fixes the conventions concerning Lie superalgebras and their representations. All gradings in this section are by  $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ , with addition understood modulo 2. We first recall that a vector superspace is a vector space  $V$  together with a decomposition

$$V = V_{\bar{0}} \oplus V_{\bar{1}}.$$

The two summands are called the even and odd parts of  $V$ . A nonzero element  $v \in V_{\bar{i}}$  is homogeneous of parity  $|v| = \bar{i}$ ; arbitrary elements are sums of homogeneous elements. For graded vector spaces  $V, W$ , and  $X$ , a bilinear map  $V \times W \rightarrow X$  is called even if it preserves the grading, in the sense that the image of  $V_{\bar{i}} \times W_{\bar{j}}$  is contained in  $X_{\bar{i}+\bar{j}}$ .

**Definition 2.1.** A Lie superalgebra is a vector superspace  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  equipped with an even bilinear bracket  $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  such that, for homogeneous  $x, y, z \in \mathfrak{g}$ ,

$$\begin{aligned} [x, y] &= -(-1)^{|x||y|}[y, x], \\ (-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] &= 0. \end{aligned}$$

The first identity is the graded skew-symmetry and the second is the super-Jacobi identity.

Thus  $[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subseteq \mathfrak{g}_{\bar{i}+\bar{j}}$ . The even part  $\mathfrak{g}_{\bar{0}}$  is an ordinary Lie algebra, while the bracket of two odd elements is symmetric in the ordinary sense. An ordinary Lie algebra is recovered by taking  $\mathfrak{g}_{\bar{1}} = 0$ .

For a Lie superalgebra  $\mathfrak{g}$ , its universal enveloping algebra  $U(\mathfrak{g})$  is the associative graded algebra generated by  $\mathfrak{g}$  with relations

$$xy - (-1)^{|x||y|}yx = [x, y]$$

for homogeneous  $x, y \in \mathfrak{g}$ . Consequently, a graded  $\mathfrak{g}$ -module can equivalently be viewed as a graded left  $U(\mathfrak{g})$ -module.

**Definition 2.2.** Let  $\mathfrak{g}$  be a Lie superalgebra. A (left)  $\mathfrak{g}$ -module is a vector superspace  $M = M_{\bar{0}} \oplus M_{\bar{1}}$  with an even action  $\mathfrak{g} \times M \rightarrow M$ . Equivalently, for homogeneous  $x \in \mathfrak{g}_{\bar{i}}$  and  $v \in M_{\bar{j}}$ ,

$$x \cdot v \in M_{\bar{i}+\bar{j}},$$

and the action satisfies

$$x \cdot (y \cdot v) - (-1)^{|x||y|}y \cdot (x \cdot v) = [x, y] \cdot v$$

for all homogeneous  $x, y \in \mathfrak{g}$  and  $v \in M$ . The identities for nonhomogeneous elements follow by linearity.

A submodule is a graded subspace stable under the action of every element of  $\mathfrak{g}$ . In particular, simplicity means that the only graded submodules are 0 and the whole module. This graded convention is the one used throughout the article.

The parity-change functor, denoted by  $\Pi$ , exchanges the two homogeneous components:

$$(\Pi M)_{\bar{i}} = M_{\bar{i}+\bar{1}}.$$

If  $\pi : M \rightarrow \Pi M$  denotes the canonical parity-reversing identification, the standard induced action is

$$x \cdot \pi(v) = (-1)^{|x|}\pi(x \cdot v)$$

for homogeneous  $x$  and  $v$ . Thus  $\Pi M$  has the same underlying vector space and action up to this conventional sign, but its even and odd parts are interchanged.

Let  $\mathfrak{t} \subseteq \mathfrak{g}_0$  be an abelian subalgebra. For a functional  $\mu \in \mathfrak{t}^*$ , the corresponding weight space of a  $\mathfrak{g}$ -module  $M$  is

$$M_\mu = \{v \in M \mid h \cdot v = \mu(h)v \text{ for every } h \in \mathfrak{t}\}.$$

Since  $\mathfrak{t}$  is even, every  $M_\mu$  is naturally  $\mathbb{Z}_2$ -graded. The support is the set

$$\text{Supp}_{\mathfrak{t}}(M) = \{\mu \in \mathfrak{t}^* \mid M_\mu \neq 0\}.$$

**Definition 2.3.** *The module  $M$  is called a weight module (with respect to  $\mathfrak{t}$ ) if*

$$M = \bigoplus_{\mu \in \mathfrak{t}^*} M_\mu.$$

*Equivalently, the commuting operators from  $\mathfrak{t}$  act simultaneously diagonalizably.*

The weight spaces need not be one-dimensional; when they are all finite-dimensional, one often uses the additional term Harish–Chandra modules. Some authors also use “generalized weight module” for a decomposition into generalized eigenspaces; that variant is not needed here.

More generally, if a homogeneous element  $x \in \mathfrak{g}$  satisfies  $[h, x] = \nu(h)x$  for every  $h \in \mathfrak{t}$ , then  $xM_\mu \subseteq M_{\mu+\nu}$ . This elementary weight-shift rule explains the role of the integer subscripts on many generators in graded Lie algebras.

### 3 Takiff $N=1$ Heisenberg-Virasoro superalgebra

In this section, we present the definition and basic structural properties of the Takiff  $N = 1$  Heisenberg-Virasoro superalgebra  $\mathcal{H}$ .

The **Takiff  $N = 1$  Heisenberg-Virasoro superalgebra**  $\mathcal{H}$  is an infinite-dimensional  $\mathbb{Z}_2$ -graded Lie superalgebra with homogeneous basis

$$\{L_m, W_m, H_m, I_m, G_p, F_p, Q_p, P_p \mid m, p \in \mathbb{Z}\},$$

whose nontrivial supercommutation relations are given as follows:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{m+n}, & [L_m, W_n] &= (n - m)W_{m+n}, \\ [L_m, H_n] &= nH_{m+n}, & [L_m, I_n] &= nI_{m+n}, \\ [W_m, H_n] &= nI_{m+n}, & [L_m, F_p] &= \left(p - \frac{m}{2}\right) F_{m+p}, \\ [L_m, G_p] &= \left(p - \frac{m}{2}\right) G_{m+p}, & [L_m, P_p] &= \left(p + \frac{m}{2}\right) P_{m+p}, \\ [L_m, Q_p] &= \left(p + \frac{m}{2}\right) Q_{m+p}, & [W_m, Q_p] &= \left(p + \frac{m}{2}\right) P_{m+p}, \\ [W_m, G_p] &= \left(p - \frac{m}{2}\right) F_{m+p}, & [F_p, H_n] &= -nP_{p+n}, \\ [G_p, H_n] &= -nQ_{p+n}, & [G_p, F_q] &= -2W_{p+q}, \\ [G_p, I_n] &= -nP_{p+n}, & [G_p, P_q] &= I_{p+q}, \\ [G_p, G_q] &= -2L_{p+q}, & [G_p, Q_q] &= H_{p+q}, \\ [F_p, Q_q] &= I_{p+q}, \end{aligned}$$

where  $m, n, p, q \in \mathbb{Z}$ . All remaining brackets vanish identically.

The superalgebra  $\mathcal{H}$  possesses the standard  $\mathbb{Z}_2$ -gradation  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ , where

$$\mathcal{H}_0 = \{L_m, W_m, H_m, I_m \mid m \in \mathbb{Z}\}, \quad \mathcal{H}_1 = \{G_p, F_p, Q_p, P_p \mid p \in \mathbb{Z}\}.$$

Note that  $\mathcal{H}_0$  is exactly the **BMS-Kac-Moody algebra**. Moreover, several important subalgebras can be naturally identified:

- $\mathcal{W} = \text{span}\{L_m, W_m \mid m \in \mathbb{Z}\} \cong W(2, 2)$  algebra,
- $\mathcal{I} = \text{span}\{L_m, H_m \mid m \in \mathbb{Z}\}$  is the Heisenberg-Virasoro algebra,
- $\mathcal{R} = \text{span}\{L_m, G_p \mid m, p \in \mathbb{Z}\}$  is the  $N = 1$  superconformal algebra,
- $\mathcal{N} = \text{span}\{L_m, H_m, G_p, Q_p \mid m, p \in \mathbb{Z}\}$  is the classical  $N = 1$  Heisenberg-Virasoro superalgebra.

The Takiff deformation superalgebra  $\mathcal{H}$  can be realized as the second-order truncated tensor product  $\mathcal{N} \otimes (\mathbb{C}[t]/t^2\mathbb{C}[t])$ , as illustrated in [1]. Related representation results on higher-rank non-weight modules over  $N = 1$  superconformal algebras can be found in [3].

In the present paper, the relevant even subalgebra is  $\mathfrak{h} = \mathbb{C}L_0$ . Thus, a scalar  $\mu \in \mathbb{C}$  is a weight when

$$M_\mu = \{v \in M \mid L_0 \cdot v = \mu v\},$$

and the weight module condition is  $M = \bigoplus_{\mu \in \mathbb{C}} M_\mu$ .

A module is called **non-weight** (relative to  $\mathfrak{t}$ ) if it does not admit the preceding direct-sum decomposition. It may have no nonzero weight vectors, or it may have some weight vectors without being spanned by them. Thus “non-weight” refers to the action of the chosen even subalgebra, not to the parity grading and not to the dimension of the module.

The main non-weight modules in this article are free over the enveloping algebra of  $\mathfrak{h}$ . Since  $\mathfrak{h} = \mathbb{C}L_0$  is one-dimensional and even,

$$U(\mathfrak{h}) \cong \mathbb{C}[L_0].$$

A module  $M$  is rank- $r$   $U(\mathfrak{h})$ -free if homogeneous generators  $u_1, \dots, u_r$  can be chosen so that

$$M = \bigoplus_{k=1}^r \mathbb{C}[L_0]u_k.$$

Under this identification,  $L_0$  acts by multiplication by the polynomial variable. Hence a nonzero vector cannot be an  $L_0$ -eigenvector: if  $v = \sum_{k=1}^r f_k(L_0)u_k$  and  $L_0v = \mu v$ , then  $(L_0 - \mu)f_k(L_0) = 0$  for every  $k$ , which forces all  $f_k = 0$ . Nonzero finite-rank  $U(\mathfrak{h})$ -free modules are therefore basic examples of non-weight modules, including the rank-2 family studied below.

All  $\mathcal{H}$ -modules below are graded modules in the sense of Definition 2.2, and  $\Pi$  denotes the parity-change functor described in Section 2.

## 4 Non-weight modules $\Psi(\lambda, \alpha, \beta)$

We first recall the classical family of non-weight Virasoro modules. For any  $\lambda \in \mathbb{C}^*$  and  $\alpha \in \mathbb{C}$ , define  $\Omega(\lambda, \alpha) = \mathbb{C}[L_0]$  as a vector space with Virasoro action

$$L_m f(L_0) = \lambda^m (L_0 - m\alpha) f(L_0 - m), \quad \forall m \in \mathbb{Z}, f(L_0) \in \mathbb{C}[L_0]. \quad (4.1)$$

The following fundamental properties are well-known (see [4, 7]).

**Theorem 4.1.** *Let  $\lambda \in \mathbb{C}^*$  and  $\alpha \in \mathbb{C}$ . Then*

- (1)  $\Omega(\lambda, \alpha)$  is a simple Virasoro module if and only if  $\alpha \neq 0$ ;
- (2) Every rank-one  $U(\mathfrak{h})$ -free Virasoro module is isomorphic to some  $\Omega(\lambda, \alpha)$ ;
- (3) The module  $\Omega(\lambda, 0)$  has a unique non-trivial proper submodule  $\Lambda = L_0 \Omega(\lambda, 0)_{\bar{0}}$ , and  $\Omega(\lambda, 0)/\Lambda$  is the one-dimensional trivial module;
- (4)  $\Lambda \cong \Pi(\Omega(\lambda, \frac{1}{2}))$  and hence is simple.

We now construct our main module family over  $\mathcal{H}$ .

**Proposition 4.2.** *For  $\lambda \in \mathbb{C}^*$  and  $\alpha, \beta \in \mathbb{C}$ , define the  $\mathbb{Z}_2$ -graded vector space*

$$\Psi(\lambda, \alpha, \beta) = \mathbb{C}[s] \mathbf{1}_{\bar{0}} \oplus \mathbb{C}[t] \mathbf{1}_{\bar{1}}.$$

*The action of  $\mathcal{H}$  on this module is given by*

$$\begin{aligned} L_m f(s) &= \lambda^m (s - m\alpha) f(s - m), \\ L_m f(t) &= \lambda^m (t - m(\alpha + \frac{1}{2})) f(t - m), \\ W_m f(s) &= W_m f(t) = 0, \\ H_m f(s) &= \lambda^m \beta f(s - m), \\ H_m f(t) &= \lambda^m \beta f(t - m), \\ I_m f(s) &= I_m f(t) = 0, \\ G_m f(s) &= \lambda^m f(t - m), \\ G_m f(t) &= -\lambda^m (s - 2m\alpha) f(s - m), \\ F_m f(s) &= F_m f(t) = 0, \\ Q_m f(t) &= \lambda^m \beta f(s - m), \\ Q_m f(s) &= P_m f(s) = P_m f(t) = 0, \end{aligned}$$

for all  $m \in \mathbb{Z}$ ,  $f(s) \in \mathbb{C}[s]$ ,  $f(t) \in \mathbb{C}[t]$ .

**Proof.** For any  $f(s) \in \mathbb{C}[s]$ , we verify

$$\begin{aligned} [L_m, L_n] f(s) &= \lambda^{m+n} ((s - m\alpha)(s - n - n\alpha) \\ &\quad - (s - n\alpha)(s - n - m\alpha)) f(s - m - n) \\ &= (n - m) \lambda^{m+n} (s - (m + n)\alpha) f(s - m - n) \\ &= (n - m) L_{m+n} f(s). \end{aligned}$$

On  $f(t)$  the same computation holds with  $\alpha$  replaced by  $\alpha + \frac{1}{2}$ . Therefore the Virasoro relations are satisfied.

Now we verify

$$[L_m, H_n] = nH_{m+n}.$$

On  $f(s)$ ,

$$\begin{aligned} L_m H_n f(s) - H_n L_m f(s) &= n\beta \lambda^{m+n} f(s - m - n) \\ &= nH_{m+n} f(s). \end{aligned}$$

The verification on  $f(t)$  is identical. Hence  $[L_m, H_n] = nH_{m+n}$ . All remaining equations are similar; for brevity, we omit the detailed calculations.

The actions defined in Proposition 4.2 preserve the  $\mathbb{Z}_2$ -grading and satisfy all nontrivial supercommutation relations of the Takiff  $N = 1$  Heisenberg–Virasoro superalgebra  $\mathcal{H}$ . Hence,

$$\Psi(\lambda, \alpha, \beta) = \mathbb{C}[s]\mathbf{1}_0 \oplus \mathbb{C}[t]\mathbf{1}_{\bar{1}}$$

is a  $\mathbb{Z}_2$ -graded  $\mathcal{H}$ -module.  $\square$

**Remark 4.3.** (1)  $\Psi(\lambda, \alpha, \beta)$  is naturally  $\mathbb{Z}_2$ -graded with even part  $\mathbb{C}[s]$  and odd part  $\mathbb{C}[t]$ ;

(2) When  $\beta = 0$ ,  $\Psi(\lambda, \alpha, 0)$  restricts to the  $N = 1$  superconformal non-weight module  $\Omega(\lambda, \alpha)$  studied in [9].

**Theorem 4.4.** Let  $\lambda \in \mathbb{C}^*$ ,  $\alpha, \beta \in \mathbb{C}$  and define  $\chi = s\Psi(\lambda, 0, 0)_{\bar{0}} \oplus \Psi(\lambda, 0, 0)_{\bar{1}}$ . Then

- (1)  $\Psi(\lambda, \alpha, \beta)$  is simple if and only if  $\alpha \neq 0$  or  $\beta \neq 0$ ;
- (2)  $\Psi(\lambda, 0, 0)$  admits a unique proper submodule  $\chi$ , and the quotient  $\Psi(\lambda, 0, 0)/\chi$  is a one-dimensional trivial module;
- (3)  $\chi \cong \Pi(\Psi(\lambda, \frac{1}{2}, 0))$ , so  $\chi$  is simple.

**Proof.** (1) If  $\alpha \neq 0$  or  $\beta \neq 0$ , the even component  $\Psi(\lambda, \alpha, \beta)_{\bar{0}}$  is simple as an  $\mathcal{H}_{\bar{0}}$ -module by [2, 4]. The odd superalgebra actions further guarantee the simplicity of the full supermodule.

(2) Now assume that  $\alpha = \beta = 0$ . The formulas in Proposition 4.2 show that

$$\chi = s\mathbb{C}[s]\mathbf{1}_0 \oplus \mathbb{C}[t]\mathbf{1}_{\bar{1}}$$

is a proper graded submodule. It is straightforward to verify that  $\chi$  is simple. By Theorem 4.1, the quotient  $\Psi(\lambda, 0, 0)/\chi$  is a one-dimensional trivial module.

(3) For the parity isomorphism, let  $x$  denote a common polynomial variable, and let  $u_0, u_1$  be the homogeneous generators of  $\Psi(\lambda, \frac{1}{2}, 0)$ . Under the signed parity convention of Section 2, define the  $\mathbb{C}[x]$ -linear map

$$\varphi : \Pi\Psi(\lambda, \frac{1}{2}, 0) \rightarrow \chi, \quad \varphi(\pi(u_1)) = x\mathbf{1}_0, \quad \varphi(\pi(u_0)) = \mathbf{1}_{\bar{1}}.$$

This is an even bijection onto  $\chi$ . By Theorem 4.1, multiplication by  $x$  intertwines the even  $L_m$ -actions. For the odd actions,

$$G_m \pi(u_1) = \lambda^m (x - m) \pi(u_0), \quad G_m \pi(u_0) = -\lambda^m \pi(u_1),$$

whose images are precisely  $G_m(x\mathbf{1}_0)$  and  $G_m\mathbf{1}_{\bar{1}}$ , respectively. The actions on the other generators vanish. Hence,  $\varphi$  is an  $\mathcal{H}$ -module isomorphism.  $\square$

**Remark 4.5.** By Theorem 4.4, we obtain the short exact sequence

$$0 \rightarrow \chi \xrightarrow{\xi_1} \Psi(\lambda, 0, 0) \xrightarrow{\xi_2} \mathbb{C} \rightarrow 0,$$

where  $\xi_1$  and  $\xi_2$  are the embedding and canonical maps, respectively, defined by

$$\xi_1 : f(s) + g(t) \rightarrow sf(s) + g(t), \quad \xi_2 : f(s) + g(t) \rightarrow \overline{f(s) + g(t)}.$$

**Theorem 4.6.** Two modules  $\Psi(\lambda, \alpha, \beta)$  and  $\Psi(\lambda', \alpha', \beta')$  are isomorphic if and only if  $\lambda = \lambda'$ ,  $\alpha = \alpha'$  and  $\beta = \beta'$ .

**Proof.** An even isomorphism commutes with  $L_0$  and is therefore  $\mathbb{C}[L_0]$ -linear on each homogeneous component. Any bijection between two rank-one polynomial modules is given by multiplication by a nonzero constant. Comparing the action of  $L_1$  on the even generator yields

$$\lambda(s - \alpha) = \lambda'(s - \alpha').$$

Comparing the coefficient of  $s$  and the constant term gives  $\lambda = \lambda'$  and  $\alpha = \alpha'$ . Comparing the action of  $H_0$  yields  $\beta = \beta'$ . Conversely, equality of the parameters implies identical action formulas.  $\square$

## 5 Classification results

In this section, we provide a complete classification of rank 2  $U(\mathfrak{h})$ -free modules over  $\mathcal{H}$ .

Let  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  be an  $\mathcal{H}$ -module free of rank 2 over  $U(\mathfrak{h})$ . Choose the homogeneous basis vectors  $\mathbf{1}_{\bar{0}} \in V_{\bar{0}}$  and  $\mathbf{1}_{\bar{1}} \in V_{\bar{1}}$  such that

$$V = \mathbb{C}[L_0]\mathbf{1}_{\bar{0}} \oplus \mathbb{C}[L_0]\mathbf{1}_{\bar{1}}.$$

Set  $s = L_0\mathbf{1}_{\bar{0}}$  and  $t = L_0\mathbf{1}_{\bar{1}}$ , so that  $V = \mathbb{C}[s] \oplus \mathbb{C}[t]$ . As a Virasoro module,  $V$  satisfies

$$L_m f(s) = \lambda^m (s - m\alpha) f(s - m), \quad L_m f(t) = \lambda_1^m (t - m\alpha_1) f(t - m), \quad (5.1)$$

where  $\lambda, \lambda_1 \in \mathbb{C}^*$  and  $\alpha, \alpha_1 \in \mathbb{C}$ .

**Lemma 5.1.** *For all  $m \in \mathbb{Z}$  and  $f(L_0) \in \mathbb{C}[L_0]$ , we have*

$$\begin{aligned} L_m f(L_0) &= f(L_0 - m) L_m, & W_m f(L_0) &= f(L_0 - m) W_m, \\ H_m f(L_0) &= f(L_0 - m) H_m, & I_m f(L_0) &= f(L_0 - m) I_m, \\ G_m f(L_0) &= f(L_0 - m) G_m, & F_m f(L_0) &= f(L_0 - m) F_m, \\ Q_m f(L_0) &= f(L_0 - m) Q_m, & P_m f(L_0) &= f(L_0 - m) P_m. \end{aligned}$$

**Proof.** We establish the identity  $L_m f(L_0) = f(L_0 - m) L_m$ ; the remaining identities follow analogously from the corresponding commutation relations.

From the commutation relation

$$[L_m, L_0] = -m L_m,$$

we obtain

$$L_m L_0 = (L_0 - m) L_m.$$

Iterating this identity, for any non-negative integer  $n$ ,

$$L_m L_0^n = (L_0 - m) L_m L_0^{n-1} = \cdots = (L_0 - m)^n L_m.$$

By induction on the degree of  $f(L_0)$ , we obtain

$$L_m f(L_0) = f(L_0 - m) L_m$$

as desired. The remaining relations are verified similarly using their respective commutators in the Takiff  $N = 1$  Heisenberg–Virasoro superalgebra.  $\square$

**Lemma 5.2.** *Let  $\lambda, \lambda_1 \in \mathbb{C}^*, \alpha, \alpha_1 \in \mathbb{C}, f(s) \in \mathbb{C}[s], g(t) \in \mathbb{C}[t]$ . We obtain  $\lambda = \lambda_1$  and one of the following two cases occurs.*

- (i)  $G_m \mathbf{1}_{\bar{0}} = \lambda^m \mathbf{1}_{\bar{1}}, G_m \mathbf{1}_{\bar{1}} = -\lambda^m (s - 2m\alpha), \alpha_1 = \alpha + \frac{1}{2};$
- (ii)  $G_m \mathbf{1}_{\bar{0}} = -\lambda^m (s + m(1 - 2\alpha)), G_m \mathbf{1}_{\bar{1}} = \lambda^m \mathbf{1}_{\bar{0}}, \alpha_1 = \alpha - \frac{1}{2}.$

**Proof.** Suppose that  $G_m \mathbf{1}_{\bar{0}} = f_m(t)$  and  $G_m \mathbf{1}_{\bar{1}} = g_m(s)$ . Using

$$G_m^2 \mathbf{1}_{\bar{0}} = f_m(s - m) g_m(s) \text{ and } L_{2m} \mathbf{1}_{\bar{0}} = \lambda^{2m} (s - 2m\alpha)$$

in  $[G_m, G_m] \mathbf{1}_{\bar{0}} = -2L_{2m} \mathbf{1}_{\bar{0}}$ , we obtain

$$f_m(s - m) g_m(s) = -\lambda^{2m} (s - 2m\alpha). \quad (5.2)$$

Setting  $m = 0$  in (5.2), we immediately obtain

$$f_0(s) = \gamma_0, \quad g_0(s) = -\frac{s}{\gamma_0} \quad \text{or} \quad f_0(s) = -\frac{s}{\gamma_0}, \quad g_0(s) = \gamma_0. \quad (5.3)$$

Taking the even generator as  $\gamma_0 1_{\bar{0}}$ , we may assume without loss of generality that  $\gamma_0 = 1$ . Then (5.3) can be rewritten as

$$f_0(s) = 1, \quad g_0(s) = -s \quad \text{or} \quad f_0(s) = -s, \quad g_0(s) = 1.$$

Choosing  $m = k \neq 0$  in (5.2), we have

$$f_k(s - k) = \gamma_k, \quad g_k(s) = -\frac{1}{\gamma_k} \lambda^{2k} (s - 2k\alpha) \quad \text{or} \quad f_k(s - k) = -\frac{1}{\gamma_k} \lambda^{2k} (s - 2k\alpha), \quad g_k(s) = \gamma_k. \quad (5.4)$$

Similarly, from  $G_k^2 1_{\bar{1}} = -L_{2k} 1_{\bar{1}}$ , we obtain  $g_k(t - k) f_k(t) = -\lambda_1^{2k} (t - 2k\alpha_1)$ . By (5.4), it follows that  $\lambda = \lambda_1$ ,  $\alpha_1 = \alpha + \frac{1}{2}$  or  $\lambda = \lambda_1$ ,  $\alpha_1 = \alpha - \frac{1}{2}$ . Hence, the following four cases arise:

- (1)  $f_0(s) = 1, g_0(s) = -s, f_k(s) = \gamma_k, g_k(s) = -\frac{\lambda^{2k}}{\gamma_k} (s - 2k\alpha), \alpha_1 = \alpha + \frac{1}{2};$
- (2)  $f_0(s) = 1, g_0(s) = -s, f_k(s) = -\frac{\lambda^{2k}}{\gamma_k} (s + k - 2k\alpha), g_k(s) = \gamma_k, \alpha_1 = \alpha - \frac{1}{2};$
- (3)  $f_0(s) = -s, g_0(s) = 1, f_k(s) = -\frac{\lambda^{2k}}{\gamma_k} (s + k - 2k\alpha), g_k(s) = \gamma_k, \alpha_1 = \alpha - \frac{1}{2};$
- (4)  $f_0(s) = -s, g_0(s) = 1, f_k(s) = \gamma_k, g_k(s) = -\frac{\lambda^{2k}}{\gamma_k} (s - 2k\alpha), \alpha_1 = \alpha + \frac{1}{2}.$

**Claim 1.** Cases (2) and (4) cannot occur.

First, consider Case (2). Then,

$$\begin{aligned} G_k 1_{\bar{0}} &= -\frac{2}{k} (L_k G_0 - G_0 L_k) 1_{\bar{0}} \\ &= -\frac{2}{k} (L_k 1_{\bar{1}} - G_0 \lambda^k (s - k\alpha) 1_{\bar{0}}) \\ &= -\lambda^k 1_{\bar{1}} = -\frac{\lambda^{2k}}{\gamma_k} (t + k - 2k\alpha), \end{aligned}$$

which is a contradiction. Similarly, in Case (4), we have

$$\begin{aligned} G_k 1_{\bar{0}} &= -\frac{2}{k} (L_k G_0 - G_0 L_k) 1_{\bar{0}} \\ &= -\frac{2}{k} (L_k 1_{\bar{1}} - G_0 \lambda^k (s - k\alpha) 1_{\bar{0}}) \\ &= \lambda^k (-3t + 2k(\alpha + \frac{1}{2})) 1_{\bar{1}} = \gamma_k 1_{\bar{1}}, \end{aligned}$$

which is again a contradiction. Therefore, only Cases (1) and (3) remain.

Now we consider (1). For  $m, k \in \mathbb{Z}$  with  $k \neq \frac{m}{2}$ , from

$$[L_m, G_k] 1_{\bar{0}} = (k - \frac{m}{2}) G_{m+k} 1_{\bar{0}}, \quad (5.5)$$

which implies  $\gamma_k \lambda^m = \gamma_{m+k}$ . Setting  $k = 0$  in above equation, we have  $\gamma_m = \lambda^m$ . By similar consideration over (3), we also have  $\gamma_m = \lambda^m$ . This completes the proof.  $\square$

Due to Lemma 5.2, up to a parity, we can assume  $\alpha_1 = \alpha + \frac{1}{2}$ ,  $G_m 1_{\bar{0}} = \lambda^m 1_{\bar{1}}$  and  $G_m 1_{\bar{1}} = -\lambda^m (s - 2m\alpha)$  without loss of generality.

**Lemma 5.3.** *Let  $m \in \mathbb{Z}$ ,  $\lambda \in \mathbb{C}^*$ ,  $\beta \in \mathbb{C}$ . Then*

(i)  $H_m \mathbf{1}_{\bar{0}} = \beta \lambda^m \mathbf{1}_{\bar{0}}$  and

$$I_m \mathbf{1}_{\bar{0}} = W_m \mathbf{1}_{\bar{0}} = F_m \mathbf{1}_{\bar{0}} = Q_m \mathbf{1}_{\bar{0}} = P_m \mathbf{1}_{\bar{0}} = 0;$$

(ii)  $H_m \mathbf{1}_{\bar{1}} = \beta \lambda^m \mathbf{1}_{\bar{1}}$ ,  $Q_m \mathbf{1}_{\bar{1}} = \beta \lambda^m \mathbf{1}_{\bar{0}}$ , and

$$I_m \mathbf{1}_{\bar{1}} = W_m \mathbf{1}_{\bar{1}} = F_m \mathbf{1}_{\bar{1}} = P_m \mathbf{1}_{\bar{1}} = 0.$$

**Proof.** To prove this, we suppose that

$$H_m \mathbf{1}_{\bar{0}} = h_m(s), I_m \mathbf{1}_{\bar{0}} = i_m(s), W_m \mathbf{1}_{\bar{0}} = w_m(s),$$

$$F_m \mathbf{1}_{\bar{0}} = f_m(t), Q_m \mathbf{1}_{\bar{0}} = q_m(t), P_m \mathbf{1}_{\bar{0}} = p_m(t),$$

where  $m \in \mathbb{Z}$ ,  $h_m(s), i_m(s), w_m(s) \in \mathbb{C}[s]$ ,  $f_m(t), q_m(t), p_m(t) \in \mathbb{C}[t]$ .

For any  $m \in \mathbb{Z}$ , suppose that  $h_m(s) = \sum_{i=0}^k h_{m,i} s^i$ , where  $h_{m,i} \in \mathbb{C}$  and  $h_{m,k} \neq 0$ . We now compute  $m H_0 \mathbf{1}_{\bar{0}} = [L_{-m}, H_m] \mathbf{1}_{\bar{0}}$  as follows:

$$\begin{aligned} [L_{-m}, H_m] \mathbf{1}_{\bar{0}} &= L_{-m} H_m \mathbf{1}_{\bar{0}} - H_m L_{-m} \mathbf{1}_{\bar{0}} \\ &= L_{-m} h_m(s) - H_m \lambda^{-m} (s - m\alpha) \\ &= \lambda^{-m} (s - m\alpha) h_m(s - m) - \lambda^{-m} (s - m + m\alpha) h_m(s) \\ &= \lambda^{-m} m(k+1) h_{m,k} s^k + \text{lower-degree terms w.r.t. } s. \end{aligned}$$

This implies that  $k = 0$  for all  $m \in \mathbb{Z}$ . Hence,  $H_m \mathbf{1}_{\bar{0}} = \beta_m \mathbf{1}_{\bar{0}}$ , where  $\beta_m \in \mathbb{C}$  for all  $m \in \mathbb{Z}$ . Using  $[L_m, H_n] \mathbf{1}_{\bar{0}} = n H_{m+n} \mathbf{1}_{\bar{0}}$ , we check

$$\beta_n = \lambda^{-m} \beta_{m+n} \quad (5.6)$$

for  $n \neq 0$ . Setting  $m = -n \neq 0$  in (5.6), we obtain

$$\beta_n = \lambda^n \beta_0 \quad (5.7)$$

for  $n \neq 0$ . In conclusion, we get  $\beta_n = \lambda^n \beta$  for any  $n \in \mathbb{Z}$ ,  $\beta = \beta_0 \in \mathbb{C}$ . By a similar argument, we obtain  $H_m \mathbf{1}_{\bar{1}} = \beta' \mathbf{1}_{\bar{1}}$ , where  $\beta' \in \mathbb{C}$ .

From  $[W_m, W_n] \mathbf{1}_{\bar{0}} = 0$  and  $[H_m, I_n] \mathbf{1}_{\bar{0}} = 0$ , we respectively obtain  $w_m(s) = w_m \in \mathbb{C}$  and  $i_m(s) = i_m \in \mathbb{C}$ . Then, using  $[L_m, W_n] \mathbf{1}_{\bar{0}} = (n - m) W_{m+n} \mathbf{1}_{\bar{0}}$ , we deduce that  $w_m(s) = 0$  for all  $m \in \mathbb{Z}$ . It follows from this and  $n I_{m+n} \mathbf{1}_{\bar{0}} = [W_m, H_n] \mathbf{1}_{\bar{0}}$  that  $i_m = 0$  for all  $m \in \mathbb{Z}$ . Similarly,  $W_m \mathbf{1}_{\bar{1}} = I_m \mathbf{1}_{\bar{1}} = 0$  for all  $m \in \mathbb{Z}$ .

For any  $m \in \mathbb{Z}$ , Lemma 5.2 yields

$$\begin{aligned} Q_m \mathbf{1}_{\bar{0}} &= -[G_{m-1}, H_1] \mathbf{1}_{\bar{0}} \\ &= \lambda^m (\beta' - \beta) \mathbf{1}_{\bar{1}}. \end{aligned}$$

Thus,  $Q_m^2 \mathbf{1}_{\bar{0}} = \lambda^m (\beta' - \beta) Q_m \mathbf{1}_{\bar{1}} = 0$ , implying that  $\beta = \beta'$  or  $Q_m \mathbf{1}_{\bar{1}} = 0$ . If  $\beta = \beta'$ , then  $Q_m \mathbf{1}_{\bar{0}} = q_m(t) = 0$ . By Lemma 5.2, for any  $m \in \mathbb{Z}$ , we have

$$Q_m \mathbf{1}_{\bar{1}} = -(G_{m-1} H_1 - H_1 G_{m-1}) \mathbf{1}_{\bar{1}} = \beta \lambda^m \mathbf{1}_{\bar{0}}.$$

If  $Q_m \mathbf{1}_{\bar{1}} = 0$ , then

$$\begin{aligned} 0 &= Q_m \mathbf{1}_{\bar{1}} = -(G_{m-1} H_1 - H_1 G_{m-1}) \mathbf{1}_{\bar{1}} \\ &= (\beta' - \beta)(s - 2(m-1)\alpha) \lambda^m \mathbf{1}_{\bar{0}} + \beta \lambda^m \mathbf{1}_{\bar{0}}. \end{aligned}$$

This implies that  $\beta = \beta' = 0$ .

Moreover, from  $(n + \frac{m}{2})P_{m+n}\mathbf{1}_{\bar{0}} = [W_m, Q_n]\mathbf{1}_{\bar{0}}$  and  $(n - \frac{m}{2})F_{m+n}\mathbf{1}_{\bar{0}} = [W_m, G_n]\mathbf{1}_{\bar{0}}$ , it follows that  $f_m(t) = p_m(t) = 0$  for  $m \in \mathbb{Z}$ . Finally, it is straightforward to verify that  $F_m\mathbf{1}_{\bar{1}} = P_m\mathbf{1}_{\bar{1}} = 0$  for all  $m \in \mathbb{Z}$ . The conclusion follows.  $\square$

**Theorem 5.4.** *Let  $\mathcal{H}$  be the Takiff  $N = 1$  Heisenberg-Virasoro superalgebra and let  $M$  be a rank 2  $U(\mathfrak{h})$ -free  $\mathcal{H}$ -module. Then  $M \cong \Psi(\lambda, \alpha, \beta)$  up to parity-change.*

**Proof.** The parity-rank argument at the beginning of this section yields one polynomial generator in each parity. The rank-one Witt classification yields (5.1), while Lemmas 5.1-5.3 determine the action of each generator on all polynomial vectors. Up to a parity change, these actions coincide precisely with those in Proposition 4.2. Conversely, the verification following Proposition 4.2 establishes that these formulas define  $\mathcal{H}$ -modules. Hence, every graded rank-two  $\mathbb{C}[L_0]$ -free module is obtained.  $\square$

**Remark 5.5.** *The above theorem also determines the classification of certain rank 1  $U(\mathfrak{g})$ -free  $\mathcal{H}$ -modules, where  $\mathfrak{g} = \mathbb{C}L_0 \oplus \mathbb{C}G_0$  or  $\mathfrak{g} = \mathbb{C}L_0 \oplus \mathbb{C}Q_0$ , by defining  $1_{\bar{1}} = G_0\mathbf{1}_{\bar{0}}$  or  $1_{\bar{1}} = Q_0\mathbf{1}_{\bar{0}}$ , respectively.*

## 6 Intermediate series modules

In this section, we shall realize the intermediate series  $\mathcal{H}$ -modules by using the weighting functor  $\mathcal{W}$  introduced in [6] and  $\Psi(\lambda, \alpha, \beta)$ .

By Proposition 4.2, the actions can be rewritten as follows.

**Definition 6.1.** For  $\lambda \in \mathbb{C}^*$ ,  $\alpha, \beta \in \mathbb{C}$ ,  $f(L_0) \in \mathbb{C}[L_0]$ , nontrivial  $\mathcal{H}$ -actions on  $V = \mathbb{C}[L_0]\mathbf{1}_{\bar{0}} \oplus \mathbb{C}[L_0]\mathbf{1}_{\bar{1}}$  are defined by

$$\begin{aligned} L_m f(L_0)\mathbf{1}_{\bar{0}} &= \lambda^m (L_0 - m\alpha) f(L_0 - m)\mathbf{1}_{\bar{0}}, \\ L_m f(L_0)\mathbf{1}_{\bar{1}} &= \lambda^m (L_0 - m(\alpha + \tfrac{1}{2})) f(L_0 - m)\mathbf{1}_{\bar{1}}, \\ W_m f(L_0)\mathbf{1}_{\bar{0}} &= W_m f(L_0)\mathbf{1}_{\bar{1}} = 0, \\ H_m f(L_0)\mathbf{1}_{\bar{0}} &= \lambda^m \beta f(L_0 - m)\mathbf{1}_{\bar{0}}, \\ H_m f(L_0)\mathbf{1}_{\bar{1}} &= \lambda^m \beta f(L_0 - m)\mathbf{1}_{\bar{1}}, \\ I_m f(L_0)\mathbf{1}_{\bar{0}} &= I_m f(L_0)\mathbf{1}_{\bar{1}} = 0, \\ G_m f(L_0)\mathbf{1}_{\bar{0}} &= \lambda^m f(L_0 - m)\mathbf{1}_{\bar{1}}, \\ G_m f(L_0)\mathbf{1}_{\bar{1}} &= -\lambda^m (L_0 - 2m\alpha) f(L_0 - m)\mathbf{1}_{\bar{0}}, \\ F_m f(L_0)\mathbf{1}_{\bar{0}} &= F_m f(L_0)\mathbf{1}_{\bar{1}} = 0, \\ Q_m f(L_0)\mathbf{1}_{\bar{1}} &= \lambda^m \beta f(L_0 - m)\mathbf{1}_{\bar{0}}, \\ Q_m f(L_0)\mathbf{1}_{\bar{0}} &= P_m f(L_0)\mathbf{1}_{\bar{0}} = P_m f(L_0)\mathbf{1}_{\bar{1}} = 0, \end{aligned}$$

where  $m \in \mathbb{Z}$ . Then  $V$  is an  $\mathcal{H}$ -module under the above actions, and is denoted by  $\Psi'(\lambda, \alpha, \beta)$ .

**Proposition 6.2.** Let  $a, b, c \in \mathbb{C}$ . The intermediate series modules  $A(a, b, c)$  over  $\mathcal{R}$  are modules with a basis  $\{v_n^+, v_n^- \mid n \in \mathbb{Z}\}$  satisfying the nontrivial actions as follows

$$\begin{aligned} L_m v_n^+ &= (c + n + m(a + \tfrac{1}{2})) v_{n+m}^+, \\ L_m v_n^- &= (c + n + ma) v_{n+m}^-, \\ H_m v_n^+ &= b v_{n+m}^+, \quad H_m v_n^- = b v_{n+m}^-, \\ G_m v_n^- &= -(c + n + 2ma) v_{n+m}^+, \\ G_m v_n^+ &= v_{m+n}^-, \quad Q_m v_n^- = b v_{m+n}^+, \end{aligned}$$

where  $m \in \mathbb{Z}$ . Furthermore,  $A(a, b, c)$  is reducible if and only if  $c \in \mathbb{Z}$ ,  $a = b = 0$  or  $c \in \mathbb{Z}$ ,  $a = \frac{1}{2}$ ,  $b = 0$ .

For  $n \in \mathbb{Z}$ ,  $d \in \mathbb{C}$ , let  $\mathcal{I}_{n+d}$  generated by  $L_0 - n - d$  be the maximal ideal of  $U(\mathfrak{h}) = \mathbb{C}[L_0]$ . For an  $\mathcal{H}$ -module  $M$ , set  $M_{n+d} = M/\mathcal{I}_{n+d}M$ . Denote  $\mathcal{W}(M) = \bigoplus_{n \in \mathbb{Z}} M_{n+d}$ . It follows from Proposition 8 in [6] that we can check the following construction.

**Proposition 6.3.** Let  $d \in \mathbb{C}$ . The vector space  $\mathcal{W}(M)$  becomes an  $\mathcal{H}$ -module under the

following nontrivial actions:

$$\begin{aligned}
L_m \cdot (v + \mathcal{I}_{n+d}M) &= L_mv + \mathcal{I}_{m+n+d}M, \\
W_m \cdot (v + \mathcal{I}_{n+d}M) &= W_mv + \mathcal{I}_{m+n+d}M, \\
H_m \cdot (v + \mathcal{I}_{n+d}M) &= H_mv + \mathcal{I}_{m+n+d}M, \\
I_m \cdot (v + \mathcal{I}_{n+d}M) &= I_mv + \mathcal{I}_{m+n+d}M, \\
G_m \cdot (v + \mathcal{I}_{n+d}M) &= G_mv + \mathcal{I}_{m+n+d}M, \\
F_m \cdot (v + \mathcal{I}_{n+d}M) &= F_mv + \mathcal{I}_{m+n+d}M, \\
Q_m \cdot (v + \mathcal{I}_{n+d}M) &= Q_mv + \mathcal{I}_{m+n+d}M, \\
P_m \cdot (v + \mathcal{I}_{n+d}M) &= P_mv + \mathcal{I}_{m+n+d}M,
\end{aligned}$$

where  $m \in \mathbb{Z}$ .

Now we present the main result of this section.

**Theorem 6.4.** Let  $\lambda \in \mathbb{C}^*$ ,  $d \in \mathbb{C}$  and  $\alpha, \beta \in \mathbb{C}$ . As  $\mathcal{H}$ -modules, we obtain

$$\mathcal{W}(\Psi'(\lambda, \alpha, \beta)) \cong A\left(\frac{1}{2} - \alpha, \beta, d\right).$$

**Proof.** Let  $\mathbf{1}_{\bar{0}}$  and  $\mathbf{1}_{\bar{1}}$  be two generators of  $\Psi'(\lambda, \alpha, \beta)$ . For  $n \in \mathbb{Z}, d \in \mathbb{C}$ , setting

$$\begin{aligned}
v_n^+ &= \mathbf{1}_{\bar{0}} + \mathcal{I}_{n+d}\Psi'(\lambda, \alpha, \beta) \in \Psi'(\lambda, \alpha, \beta)_{\bar{0}}, \\
v_n^- &= \mathbf{1}_{\bar{1}} + \mathcal{I}_{n+d}\Psi'(\lambda, \alpha, \beta) \in \Psi'(\lambda, \alpha, \beta)_{\bar{1}},
\end{aligned}$$

Then we have the non-trivial actions:

$$\begin{aligned}
L_mv_n^+ &= \lambda^m(L_0 - m\alpha)\mathbf{1}_{\bar{0}} + \mathcal{I}_{m+n+d}\Psi'(\lambda, \alpha, \beta) \\
&= (n + d + m(1 - \alpha))\lambda^mv_{n+m}^+, \\
L_mv_n^- &= \lambda^m(L_0 - m(\alpha + \frac{1}{2}))\mathbf{1}_{\bar{1}} + \mathcal{I}_{m+n+d}\Psi'(\lambda, \alpha, \beta) \\
&= (n + d + m(\frac{1}{2} - \alpha))\lambda^mv_{n+m}^-, \\
H_mv_n^+ &= \lambda^m\beta\mathbf{1}_{\bar{0}} + \mathcal{I}_{m+n+d}\Psi'(\lambda, \alpha, \beta) = \beta\lambda^mv_{n+m}^+, \\
H_mv_n^- &= \lambda^m\beta\mathbf{1}_{\bar{1}} + \mathcal{I}_{m+n+d}\Psi'(\lambda, \alpha, \beta) = \beta\lambda^mv_{n+m}^-, \\
G_mv_n^+ &= \lambda^m\mathbf{1}_{\bar{0}} + \mathcal{I}_{m+n+d}\Psi'(\lambda, \alpha, \beta) = \lambda^mv_{n+m}^-, \\
G_mv_n^- &= -\lambda^m(L_0 - 2m\alpha)\mathbf{1}_{\bar{1}} + \mathcal{I}_{m+n+d}\Psi'(\lambda, \alpha, \beta) \\
&= -(n + d + (1 - 2\alpha)m)\lambda^mv_{n+m}^+, \\
Q_mv_n^- &= \lambda^m\beta\mathbf{1}_{\bar{1}} + \mathcal{I}_{m+n+d}\Psi'(\lambda, \alpha, \beta) = \beta\lambda^mv_{n+m}^+,
\end{aligned}$$

for  $m, n \in \mathbb{Z}$ .

Now let  $\hat{v}_n^+ = \lambda^n v_n^+$ ,  $\hat{v}_n^- = \lambda^n v_n^-$ , one can see that

$$\begin{aligned}
L_m\hat{v}_n^+ &= (n + d + m(1 - \alpha))\hat{v}_{n+m}^+, \\
L_m\hat{v}_n^- &= (n + d + m(\frac{1}{2} - \alpha))\hat{v}_{n+m}^-, \\
H_m\hat{v}_n^+ &= \beta\hat{v}_{n+m}^+, \quad H_m\hat{v}_n^- = \beta\hat{v}_{n+m}^-, \\
G_m\hat{v}_n^- &= -(n + d + (1 - 2\alpha)m)\hat{v}_{n+m}^+, \\
G_m\hat{v}_n^+ &= \hat{v}_{n+m}^-, \quad Q_m\hat{v}_n^- = \beta\hat{v}_{n+m}^+.
\end{aligned}$$

Therefore, we confirm that  $\mathcal{W}(\Psi'(\lambda, \alpha, \beta)) \cong A(\frac{1}{2} - \alpha, \beta, d)$ . □

## 7 Direction of Further Research

In this paper, we studied free non-weight modules over the Takiff  $N = 1$  Heisenberg–Virasoro superalgebra. We constructed the family  $\Psi(\lambda, \alpha, \beta)$  of rank-two modules over  $U(\mathfrak{h}) = \mathbb{C}[L_0]$  and classified all rank-two free modules of this type, up to parity reversal. We also determined when these modules are simple and when two modules in this family are isomorphic. In addition, we applied Nilsson’s weighting functor to these non-weight modules and obtained related weight modules. This shows how one family of non-weight modules can lead to different weight representations. The results give an initial description of the finite-rank free modules in this setting, but several questions remain open.

First, a direct extension is to consider the Neveu–Schwarz type. In the Ramond case studied here, the odd generators are indexed by integers, and the relation  $G_0^2 = -L_0$  is an important step in the classification. In the Neveu–Schwarz case, the odd generators are indexed by half-integers, so  $G_0$  is no longer available. One may investigate whether a similar classification holds by using relations involving  $G_{\frac{1}{2}}$  and  $G_{-\frac{1}{2}}$ . This would help identify which parts of the present proof can be adapted and which require a different argument.

Second, one may consider a multivariable version of the problem. Each homogeneous component studied here is a polynomial space in one variable, and the operators act by shifting that variable and multiplying by a polynomial. A possible extension is to work with polynomial spaces

$$\mathbb{C}[x_1, \dots, x_d], \quad 2 \leq d \leq 4.$$

This requires choosing a suitable multivariable analogue of the algebra, with compatible relations for shifts in different variables. One could begin with two variables and determine how the relations constrain the possible operator formulas. After constructing such modules, it would also be interesting to restrict them to suitable subalgebras and compare the resulting representations with the one-variable modules studied here.

Finally, one may study modules of higher rank over  $U(\mathfrak{h}) = \mathbb{C}[L_0]$ . In the present Ramond setting, the relation  $G_0^2 = -L_0$  forces the even and odd parts of a finite free module to have equal rank. Thus, the next case after total rank two is total rank four, with rank two in each parity component. The operator coefficients then become polynomial matrices, so the arguments based on degrees and factors of scalar polynomials need to be reconsidered. A first goal would be to construct and compare rank-four examples with nonzero actions of the additional Takiff generators. One could then investigate their simplicity and isomorphism classes, with a complete classification as a longer-term goal.

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After I completed an earlier research project with Dr. Yangyang Chen, he introduced me to his friend, Dr. Haibo Chen. Under Dr. Haibo Chen's guidance, I began studying the basic theory of Lie superalgebras and their representations. I also studied examples of representations, including polynomial modules. Through this study and our discussions, I became interested in the classification of polynomial modules and began the project presented in this paper.

The research involved learning the background theory, studying the defining relations, finding formulas for module actions, and checking that these formulas satisfied the required relations. I carried out the main calculations and wrote Sections 2–5, including the construction of the modules and the classification of the rank-two case. Sections 1, 6, and 7 were written and revised under Dr. Haibo Chen's guidance. His guidance concerned the motivation and background of the research, the application to weight modules, and possible directions for further research.

The classification in Section 5 was one of the most difficult parts of the project. The actions of the generators were initially expressed using unknown polynomials, and the defining relations imposed conditions on these polynomials. The difficulty was to narrow down the possible forms and rule out the remaining cases. This required careful attention to the parity of the generators, signs, and shifts in the polynomial variables. Working through these calculations helped me understand how the relations could be used step by step to obtain a classification.

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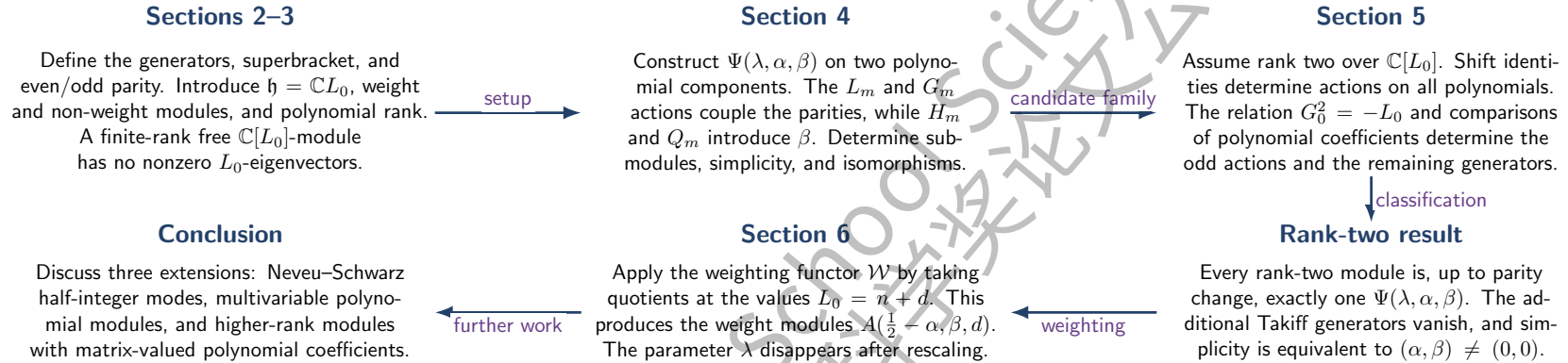
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# A class of non-weight modules of the Takiff $N = 1$ Heisenberg–Virasoro superalgebra

Construction, classification, simplicity, isomorphism, and the weighting functor



main argument and results