

参赛学生姓名：张馨茹、彭梓滢、万沐瑶

中学：合肥安生学校 Hefei Thomas School

省份：安徽省

国家/地区：中国

指导老师姓名：赵立璐、杨子辰

指导老师单位：中国科学技术大学

论文题目：On a problem of Erdős - Graham
about covering systems

ON A PROBLEM OF ERDŐS–GRAHAM ABOUT COVERING SYSTEMS

ZIHAO PENG, MUYAO WAN, AND QINGRU ZHANG

ABSTRACT. In this paper, we study the existence of covering systems whose moduli are all of the form $p - 1$ for some prime number $p \geq 5$. This problem was proposed by Erdős and Graham. We prove that no such covering system exists if all moduli are square-free; in fact, we prove a stronger statement that there is no covering system whose moduli are all even and square-free. Then we establish an explicit large-prime exclusion theorem in the general case. The proof uses the distortion method developed by Balister, Bollobás, Morris, Sahasrabudhe, and Tiba.

Keywords: Erdős–Graham problem, covering systems, the distortion method

CONTENTS

1. Introduction	1
1.1. Background	1
1.2. Our problem	2
1.3. Our results	2
2. The square-free case	2
3. The distortion method	3
4. Proof of Theorem 1.3	7
References	12

1. INTRODUCTION

1.1. Background. In 1950, Erdős [4] introduced the concept of covering system in connection with the Romanoff conjecture. A *covering system* is a finite collection $\mathcal{A}$ of arithmetic progressions of the form $a + d\mathbb{Z}$ with distinct moduli $d > 1$ such that every integer belongs to at least one arithmetic progression in $\mathcal{A}$. Erdős posed numerous questions and conjectures concerning the properties of these systems, many of which have been extensively studied. For example, Erdős [4] asked whether the smallest modulus of a covering system can be arbitrarily large, a question known as the minimum modulus problem. Another famous question, posed with Selfridge, asked whether there exists a covering system whose moduli are odd, the so-called odd modulus problem (also known as Erdős–Selfridge problem).

The first significant progress on the minimum modulus problem was made in 2007 by Filaseta, Ford, Konyagin, Pomerance and Yu [5], who proved the conjecture of Erdős and

Graham: if the distinct moduli $d_1, \dots, d_k$ all lie in the interval $[n, Cn]$ with $n > n_0(C)$ sufficiently large, then the uncovered set has density at least ε for some $\varepsilon = \varepsilon(C) > 0$. Later, Hough [6] resolved Erdős' minimum modulus problem, showing that the minimum modulus of a covering system is at most 10^{16} . By simplifying and generalising Hough's method into a systematic distortion method, Balister, Bollobás, Morris, Sahasrabudhe and Tiba [2] reduced the upper bound to 616000. Cummings, Filaseta and Trifonov [3] further refined the result by pushing the minimum-modulus bound down to 118 under the additional condition that all moduli are square-free.

Turning to the odd modulus problem, Balister, Bollobás, Morris, Sahasrabudhe and Tiba [1] proved in 2021 that there is no covering system with odd and square-free moduli. Moreover, Hough and Nielsen [7] proved in 2019 that among all covering systems, at least one modulus is divisible by either 2 or 3.

1.2. Our problem. The main aim of this paper is to study the $p-1$ modulus problem, which asks whether there exists a covering system in which every modulus is of the form $p-1$ for prime numbers $p \geq 5$. Despite the recent progress on the minimum modulus and odd modulus problems mentioned above, little is known about the existence of such a covering system.

1.3. Our results. In this paper we approach the $p-1$ modulus problem from two directions. Our first result concerns the square-free case. Specifically, we show that the answer to the $p-1$ modulus problem is negative if all of the moduli are assumed to be square-free.

Theorem 1.1. There does not exist a covering system whose moduli are all of the form $p-1$ with prime numbers $p \geq 3$ and are square-free.

As a stepping stone toward Theorem 1.1, we also prove the following auxiliary result, which is of independent interest.

Theorem 1.2. There does not exist a covering system whose moduli are all even and square-free.

Our second result provides an upper bound on the smallest prime that can appear in a $p-1$ covering system.

Theorem 1.3. There does not exist a covering system whose moduli are all of the form $p-1$ with all prime numbers $p \geq 383$.

2. THE SQUARE-FREE CASE

In this section, we study the square-free case. We begin with a simple observation.

Lemma 2.1. Let $\mathcal{A} = \{A_d = a_d + d\mathbb{Z} : d \in D\}$ be a covering system with $D \subseteq \mathbb{Z}_{\geq 2}$. If $k \geq 2$ is a positive integer such that $k \mid d$ for any $d \in D$, there exists a residue class $r \pmod{k}$ such that

$$\tilde{\mathcal{A}}_r = \left\{ \tilde{A}_d = \frac{a_d - r}{k} + \frac{d}{k}\mathbb{Z} : d \in D, a_d \equiv r \pmod{k} \right\}$$

is a covering system.

Proof. Since the moduli are all distinct, there exists at most one $d_0 \in D$ such that $d_0 = k$. Thus there exists a residue class r such that $r \not\equiv a_{d_0} \pmod{k}$. Because $\mathcal{A}$ covers $\mathbb{Z}$, clearly it covers $r + k\mathbb{Z}$ in particular. We claim that the subcollection

$$\mathcal{A}_r = \{A_d = a_d + d\mathbb{Z} : d \in D, a_d \equiv r \pmod{k}\}$$

covers $r + k\mathbb{Z}$. Indeed, for any integer $n \equiv r \pmod{k}$, because $\mathcal{A}$ covers $r + k\mathbb{Z}$, there exists $d \in D$ such that $n \equiv a_d \pmod{d}$. Since $k \mid d$ and $n \equiv r \pmod{k}$, we have $a_d \equiv r \pmod{k}$. This proves the claim. Because $\mathcal{A}_r$ covers $r + k\mathbb{Z}$, after an affine transformation, we see that $\tilde{\mathcal{A}}_r$ covers $\mathbb{Z}$. By our choice of r , the trivial arithmetic progression is not in $\tilde{\mathcal{A}}_r$. Hence, the collection $\tilde{\mathcal{A}}_r$ is indeed a covering system. $\square$

We need the following result of Balister, Bollobás, Morris, Sahasrabudhe, and Tiba to deduce a contradiction.

Theorem 2.2 (Balister, Bollobás, Morris, Sahasrabudhe, and Tiba). There is no covering system whose moduli are all odd and square-free.

Proof. See [1, Theorem 1.1]. $\square$

Now we are ready to prove the square-free case. First we show that there is no covering system whose moduli are all even and square-free.

Proof of Theorem 1.2. Suppose to the contrary that $\mathcal{A} = \{a_1 + d_1\mathbb{Z}, \dots, a_n + d_n\mathbb{Z}\}$ is a covering system such that $d_1, \dots, d_n$ are all even and square-free. Clearly, we can write $d_i = 2f_i$ for an odd square-free integer $f_i \geq 1$. Since the moduli are all even, we apply Lemma 2.1 with $k = 2$ to see that there exists a residue class $r \pmod{2}$ such that

$$\tilde{\mathcal{A}}_r = \left\{ \frac{a_i - r}{2} + f_i\mathbb{Z} : a_i + d_i\mathbb{Z} \in \mathcal{A}, a_i \equiv r \pmod{2} \right\}$$

is a covering system whose moduli are all odd and square-free. However, Theorem 2.2 asserts that such covering systems do not exist. Therefore, we arrive at a contradiction, which finishes the proof. $\square$

Since $p - 1$ are all even for any prime number $p \geq 3$, Theorem 1.2 immediately gives a negative answer to Erdős' problem in the square-free case.

Proof of Theorem 1.1. There is no covering system whose moduli are all square-free and are of the form $p - 1$ for prime numbers $p \geq 3$. $\square$

3. THE DISTORTION METHOD

In this section, we introduce the distortion method developed in [2, Section 2-3]. Let

$$\mathcal{A} = \{A_d = a_d + d\mathbb{Z} : d \in D\}$$

be a finite collection of arithmetic progressions with $D \subseteq \mathbb{Z}_{\geq 2}$. Fix $Q = \text{lcm}(D)$. Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots$ be the sequence of prime numbers listing in the ascending order.

By the unique factorization theorem, we have

$$Q = \prod_{j \geq 1} p_j^{\gamma_j}$$

with every $\gamma_j \geq 0$ and at most finitely many $\gamma_j \neq 0$. Let p_n be the largest prime number dividing Q . For each $0 \leq i \leq n$, we set

$$Q_i = \prod_{1 \leq j \leq i} p_j^{\gamma_j}.$$

In particular, we have $Q_0 = 1$ and $Q_n = Q$. We set

$$D_i = \{d \in D : d \mid Q_i\}, \quad N_i = D_i \setminus D_{i-1}, \quad B_i = \bigcup_{d \in N_i} A_d,$$

for $1 \leq i \leq n$. We denote by $\pi_M : \mathbb{Z} \rightarrow \mathbb{Z}_M$ the canonical projection given by $x \mapsto x \pmod{M}$. First, we show that we can pass the covering problem to $\mathbb{Z}_Q$.

Lemma 3.1. The collection $\mathcal{A}$ is a covering system if and only if the collection

$$\mathcal{A}_Q = \{\pi_Q(A_d) : d \in D\}$$

covers $\mathbb{Z}_Q$.

Proof. It is easy to verify that the set of integers covered by $\mathcal{A}$ is Q -cyclic, that is, n is covered by $\mathcal{A}$ if and only if $n + Q$ is covered by $\mathcal{A}$. The lemma follows. $\square$

We denote by

$$R = \mathbb{Z}_Q \setminus \bigcup_{d \in D} \pi_Q(A_d)$$

the uncovered set in $\mathbb{Z}_Q$. By Lemma 3.1, it is equivalent to studying whether the uncovered set R is non-empty or not. Suppose that we have a probability measure $\mathbb{P}_n$ on $\mathbb{Z}_Q$. By the union bound, we have

$$\mathbb{P}_n(R) = 1 - \mathbb{P}_n\left(\bigcup_{d \in D} \pi_Q(A_d)\right) = 1 - \mathbb{P}_n\left(\bigcup_{1 \leq i \leq n} \pi_Q(B_i)\right) \geq 1 - \sum_{1 \leq i \leq n} \mathbb{P}_n(B_i).$$

Therefore, if we can prove

$$\sum_{1 \leq i \leq n} \mathbb{P}_n(B_i) < 1,$$

then R is non-empty and $\mathcal{A}$ is not a covering system. For ease of the notations, we shall identify A_d and B_i with their images in $\mathbb{Z}_Q$ under the projection π_Q in the sequel.

Now we are going to construct a sequence of probability measures $\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n$ using the distortion method. Fix a sequence of constants $\delta_1, \dots, \delta_n \in [0, \frac{1}{2}]$. Let $\mathbb{P}_0$ be the uniform distribution on $\mathbb{Z}_{Q_0}$. Assuming that $\mathbb{P}_{i-1}$ is defined, we are going to construct $\mathbb{P}_i$ out of $\mathbb{P}_{i-1}$ inductively. First, we have the canonical projection

$$\pi_{Q_i, Q_{i-1}} : \mathbb{Z}_{Q_i} \rightarrow \mathbb{Z}_{Q_{i-1}},$$

which is given by $x \pmod{Q_i} \mapsto x \pmod{Q_{i-1}}$. By the Chinese remainder theorem, we can view $\mathbb{Z}_{Q_i}$ as $\mathbb{Z}_{Q_{i-1}} \times \mathbb{Z}_{p_i^{\gamma_i}}$. Therefore we can write an element in $\mathbb{Z}_{Q_i}$ as a pair (x, y) with $x \in \mathbb{Z}_{Q_{i-1}}$ and $y \in \mathbb{Z}_{p_i^{\gamma_i}}$. Hence, we have

$$\pi_{Q_i, Q_{i-1}}(x, y) = x.$$

For a map $f: A \rightarrow B$, the fiber of $b \in B$ is the set of $a \in A$ such that $f(a) = b$. For $x \in \mathbb{Z}_{Q_{i-1}}$, we denote by $F(x)$ the fiber of x under the canonical projection $\pi_{Q_i, Q_{i-1}}$. In other words, we have

$$F(x) = \{(x, y) : y \in \mathbb{Z}_{p_i^{\gamma_i}}\}.$$

Now we set

$$\alpha_i(x) = \frac{|\{y \in \mathbb{Z}_{p_i^{\gamma_i}} : (x, y) \in F(x) \cap B_i\}|}{p_i^{\gamma_i}}.$$

The probability measure $\mathbb{P}_i$ is defined as follows. If $\alpha_i(x) \leq \delta_i$, then we set

$$\mathbb{P}_i(x, y) = \frac{\mathbb{P}_{i-1}(x)}{p_i^{\gamma_i}} \times \begin{cases} 0, & \text{if } (x, y) \in B_i, \\ \frac{1}{1 - \alpha_i(x)}, & \text{if } (x, y) \notin B_i. \end{cases}$$

If $\alpha_i(x) > \delta_i$, then we set

$$\mathbb{P}_i(x, y) = \frac{\mathbb{P}_{i-1}(x)}{p_i^{\gamma_i}} \times \begin{cases} \frac{\alpha_i(x) - \delta_i}{\alpha_i(x)(1 - \delta_i)}, & \text{if } (x, y) \in B_i, \\ \frac{1}{1 - \delta_i}, & \text{if } (x, y) \notin B_i. \end{cases}$$

We note that the probability measure $\mathbb{P}_i$ that is defined on $\mathbb{Z}_{Q_i}$ gives rise to a probability measure on $\mathbb{Z}_Q$ by extending on each fiber uniformly. For simplicity, we again denote the extended probability measure by $\mathbb{P}_i$. To be more precise, if we write an element of $\mathbb{Z}_Q$ as a pair (x, y) with $x \in \mathbb{Z}_{Q_i}$ and $y \in \mathbb{Z}_{Q/Q_i}$ by the Chinese remainder theorem, then we have

$$\mathbb{P}_i(x, y) = \frac{Q_i}{Q} \mathbb{P}_i(x).$$

We denote by $\pi_{Q, Q_i}: \mathbb{Z}_Q \rightarrow \mathbb{Z}_{Q_i}$ the canonical projection $x \pmod{Q} \mapsto x \pmod{Q_i}$. If a subset $S \subseteq \mathbb{Z}_Q$ is a union of fibers under π_{Q, Q_i} , we say that S is a Q_i -measurable set.

Lemma 3.2. For a Q_{i-1} -measurable subset $S \subseteq \mathbb{Z}_Q$, we have

$$\mathbb{P}_i(S) = \mathbb{P}_{i-1}(S).$$

Proof. See [2, Lemma 2.1]. □

Clearly if $S \subseteq \mathbb{Z}_Q$ is a Q_i -measurable subset, then it is also a Q_{i+1} -measurable subset. Moreover, it is easy to check that $B_i \subseteq \mathbb{Z}_Q$ is Q_i -measurable. Therefore, Lemma 3.2 implies that

$$\mathbb{P}_n(B_i) = \mathbb{P}_{n-1}(B_i) = \cdots = \mathbb{P}_i(B_i).$$

So our goal of proving

$$\sum_{1 \leq i \leq n} \mathbb{P}_n(B_i) < 1$$

is equivalent to proving

$$\sum_{1 \leq i \leq n} \mathbb{P}_i(B_i) < 1.$$

We are going to use the moment method to bound the probabilities. So we define the expectation of a function $f: \mathbb{Z}_Q \rightarrow \mathbb{R}_{\geq 0}$ with respect to the probability measure $\mathbb{P}_i$ to be

$$\mathbb{E}_i(f) = \sum_{x \in \mathbb{Z}_Q} f(x) \mathbb{P}_i(x).$$

Recall that the function $\alpha_i(x)$ defined on $\mathbb{Z}_{Q_{i-1}}$ gives rise to a Q_{i-1} -measurable function on $\mathbb{Z}_Q$ by precomposing the canonical projection $\pi_{Q, Q_{i-1}}$. By abuse of notation, the composition is again denoted by the same symbol α_i . Next we are going to investigate the first moment and the second moment of α_i . More precisely, we set

$$M_i^{(1)} = \mathbb{E}_{i-1}(\alpha_i), \quad M_i^{(2)} = \mathbb{E}_{i-1}(\alpha_i^2).$$

It turns out that both moments give upper bounds for $\mathbb{P}_i(B_i)$.

Lemma 3.3. We have

$$\mathbb{P}_i(B_i) \leq M_i^{(1)}.$$

Furthermore, if $\delta_i \neq 0$, then we have

$$\mathbb{P}_i(B_i) \leq \frac{M_i^{(2)}}{4\delta_i(1 - \delta_i)}.$$

Proof. See [2, Theorem 3.1]. □

Now it is clear that we have to bound the growth of the first moment and the second moment. For a divisor $d \mid Q$, we set

$$\nu(d) = \prod_{p_j \mid d} \frac{1}{1 - \delta_j}.$$

Now we can state the upper bounds for the first moment and the second moment. In fact, we can bound the k -th moment for any $k \geq 1$.

Lemma 3.4. We have

$$\mathbb{E}_{i-1}(\alpha_i^k) \leq \sum_{m_1, \dots, m_k \mid Q_{i-1}} \sum_{\substack{j_1, \dots, j_k \geq 1 \\ m_1 p_i^{j_1}, \dots, m_k p_i^{j_k} \in N_i}} \frac{1}{p^{j_1 + \dots + j_k}} \frac{\nu(\text{lcm}(m_1, \dots, m_k))}{\text{lcm}(m_1, \dots, m_k)}.$$

Proof. See [2, Lemma 3.6]. □

Lemma 3.5. We have

$$M_i^{(1)} \leq \frac{1}{p_i - 1} \prod_{\substack{1 \leq j \leq i-1 \\ p_j | Q_{i-1}}} \left(1 + \frac{1}{(1 - \delta_j)(p_j - 1)} \right).$$

and

$$M_i^{(2)} \leq \frac{1}{(p_i - 1)^2} \prod_{\substack{1 \leq j \leq i-1 \\ p_j | Q_{i-1}}} \left(1 + \frac{3p_j - 1}{(1 - \delta_j)(p_j - 1)^2} \right).$$

Proof. See [2, Theorem 3.2]. $\square$

4. PROOF OF THEOREM 1.3

Let $\mathcal{A}$ be a covering system whose modulus are of the form $p - 1$ with all prime numbers $p - 1 \geq p_{\min} = 382$. Fix a parameter $N = 10000$ and choose the parameters δ_i as follows.

$$\delta_i = \begin{cases} 0 & \text{if } 1 \leq i \leq 100, \\ 0.3 & \text{if } 101 \leq i \leq N \\ 0.5 & \text{if } i \geq N + 1. \end{cases}$$

Recall that n is the largest integer so that p_n divides one of the moduli. Then we construct the probability measures $\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_n$ using the distortion method. By Dirichlet's theorem on the infinitude of prime numbers in arithmetic progressions, we can insert one more arithmetic progression into $\mathcal{A}$, whose modulus is of the form $p - 1$ and is divisible by a sufficiently large prime number. Therefore we may assume that $n \geq N$. To prove Theorem 1.3, we are going to show that

$$\sum_{1 \leq i \leq n} \mathbb{P}_i(B_i) < 1.$$

This task breaks down to bound

$$(4.1) \quad \sum_{1 \leq i \leq N} \mathbb{P}_i(B_i),$$

and

$$(4.2) \quad \sum_{N+1 \leq i \leq n} \mathbb{P}_i(B_i).$$

Remark 4.1. We choose the constants δ_i in the range $101 \leq i \leq N$ to minimize the contribution from

$$\sum_{101 \leq i \leq N} \mathbb{P}_i(B_i).$$

After trying different choices, we found that the constants $\delta_i = 0.3$ in this range achieve the minimal upper bound. See the following table.

δ	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\sum_{101 \leq i \leq n} \mathbb{P}_i(B_i)$	0.5009	0.3756	0.3200	0.2944	0.2865	0.2923	0.3121	0.3497

To bound (4.1), we apply the first moment bound and the second moment bound, together with the sparsity of prime numbers. First, we apply the first moment bound to control $\mathbb{P}_i(B_i)$ for $1 \leq i \leq 100$. By Lemma 3.3 and Lemma 3.4, we have

$$\sum_{i=1}^{100} \mathbb{P}_i(B_i) \leq \sum_{d \in D_{100}} \frac{1}{d} \leq \sum_{\substack{d \geq p_{\min} \\ d \in \mathcal{S}(p_{100}) \cap \mathcal{P}^*}} \frac{1}{d} \leq \sum_{\substack{d \geq p_{\min} \\ d \in \mathcal{S}(p_{100}) \\ d \text{ is even}}} \frac{1}{d} - \sum_{\substack{p_{\min} \leq d \leq 10^8 \\ d \in \mathcal{S}(p_{100}) \setminus \mathcal{P}^* \\ d \text{ is even}}} \frac{1}{d} \leq 0.712624,$$

where $\mathcal{S}(p_k)$ is the set of p_k -smooth numbers and

$$\mathcal{P}^* = \{p - 1 : p \geq 5 \text{ is a prime number}\}.$$

To bound $\mathbb{P}_i(B_i)$ for $101 \leq i \leq N$, we apply the second moment bound instead. Specifically, by Lemma 3.3 and Lemma 3.4 again, we have

$$\sum_{i=101}^N \mathbb{P}_i(B_i) \leq \sum_{i=101}^N \frac{1}{4\delta_i(1-\delta_i)} \sum_{m_1, m_2 | Q_{i-1}} \sum_{\substack{j_1, j_2 \geq 1 \\ m_1 p_i^{j_1}, m_2 p_i^{j_2} \in N_i}} \frac{1}{p^{j_1+j_2}} \frac{\nu(\text{lcm}(m_1, m_2))}{\text{lcm}(m_1, m_2)}.$$

To estimate the inner double sum, we first estimate the following double sum

$$S = \sum_{\substack{m_1, m_2 \in \mathcal{S}(p_{i-1}) \\ m_1, m_2 \text{ is even}}} \sum_{\substack{j_1, j_2 \geq 1 \\ m_1 p_i^{j_1}, m_2 p_i^{j_2} \geq p_{\min}}} \frac{1}{p^{j_1+j_2}} \frac{\nu(\text{lcm}(m_1, m_2))}{\text{lcm}(m_1, m_2)},$$

which can be calculated using a recursive formula given in [2, Proof of Theorem 8.1]. Then we subtract from S the following double sum

$$\sum_{\substack{m_1, m_2 \in \mathcal{S}(p_{i-1}) \\ m_1, m_2 \text{ is even}}} \sum_{\substack{j_1, j_2 \geq 1 \\ p_{\min} \leq m_1 p_i^{j_1}, m_2 p_i^{j_2} \leq 10^5 \\ m_1 p_i^{j_1} \notin \mathcal{P}^* \text{ or } m_2 p_i^{j_2} \notin \mathcal{P}^*}} \frac{1}{p^{j_1+j_2}} \frac{\nu(\text{lcm}(m_1, m_2))}{\text{lcm}(m_1, m_2)}.$$

This gives

$$\sum_{i=101}^N \mathbb{P}_i(B_i) \leq 0.244652.$$

Overall, with $N = 10000$ and the chosen parameters $\delta_1, \dots, \delta_N$, we have

$$\sum_{1 \leq i \leq N} \mathbb{P}_i(B_i) = \sum_{1 \leq i \leq 100} \mathbb{P}_i(B_i) + \sum_{101 \leq i \leq N} \mathbb{P}_i(B_i) \leq 0.712624 + 0.244652 = 0.957276.$$

To bound (4.2), we need to prove a generalization of [3, Lemma 4.7] for general moduli.

Lemma 4.2. Set

$$c_1 = -\log \log p_N + \frac{1}{\log^2 p_N}, \quad c_2 = 1 + \frac{3}{2 \log p_N},$$

and

$$M'_0 = \prod_{j=1}^N \left(1 + \frac{3p_j - 1}{(1 - \delta_j)(p_j - 1)^2} \right).$$

Furthermore, let

$$C_2 = \frac{p_{N+1}^2}{(p_{N+1} - 1)^2}.$$

Then

$$\sum_{k=N+1}^n \mathbb{P}_k(B_k) \leq 2e^{7c_1} c_2 C_2 M'_0 \frac{\phi(\log p_N)}{p_N},$$

with

$$\phi(x) = x^6 + 6x^5 + 30x^4 + 120x^3 + 360x^2 + 720x + 720.$$

Proof. Suppose that $k > N$. By Lemma 3.3 and Lemma 3.5, we have

$$\mathbb{P}_k(B_k) \leq \frac{1}{4\delta_k(1 - \delta_k)(p_k - 1)^2} \prod_{j < k} \left(1 + \frac{3p_j - 1}{(1 - \delta_j)(p_j - 1)^2} \right).$$

Since $\delta_j = 1/2$ for $j > N$, the above inequality simplifies to

$$\mathbb{P}_k(B_k) \leq \frac{1}{(p_k - 1)^2} \prod_{j < k} \left(1 + \frac{3p_j - 1}{(1 - \delta_j)(p_j - 1)^2} \right).$$

We decompose the product into the first N factors and the remaining tail:

$$\prod_{j < k} \left(1 + \frac{3p_j - 1}{(1 - \delta_j)(p_j - 1)^2} \right) = M'_0 \prod_{j=N+1}^{k-1} \left(1 + \frac{2(3p_j - 1)}{(p_j - 1)^2} \right).$$

For $p_j \geq 12$, we have

$$\frac{2(3p_j - 1)}{(p_j - 1)^2} \leq \frac{7}{p_j}.$$

This inequality is equivalent to

$$p_j^2 - 12p_j + 7 \geq 0.$$

Solving the quadratic equation proves the desired inequality. Since $N = 10000$, the inequality clearly holds. Consequently, we have

$$\sum_{j=N+1}^{k-1} \frac{2(3p_j - 1)}{(p_j - 1)^2} \leq 7 \sum_{j=N+1}^{k-1} \frac{1}{p_j}.$$

Applying this inequality $1 + x \leq e^x$, we obtain

$$\prod_{j=N+1}^{k-1} \left(1 + \frac{2(3p_j - 1)}{(p_j - 1)^2} \right) \leq \exp \left(\sum_{j=N+1}^{k-1} \frac{2(3p_j - 1)}{(p_j - 1)^2} \right) \leq \exp \left(7 \sum_{j=N+1}^{k-1} \frac{1}{p_j} \right).$$

By the Rosser–Schoenfeld bounds on the sum of reciprocals of primes [8, Theorem 5], for $x \geq 286$ we have

$$\log \log x + B - \frac{1}{2 \log^2 x} \leq \sum_{p \leq x} \frac{1}{p} \leq \log \log x + B + \frac{1}{2 \log^2 x},$$

where $B \approx 0.2614972128$. For the value of N used in the application, $p_N \geq 286$. Therefore,

$$\begin{aligned} \sum_{j=N+1}^{k-1} \frac{1}{p_j} &= \sum_{p \leq p_{k-1}} \frac{1}{p} - \sum_{p \leq p_N} \frac{1}{p} \\ &\leq \log \log p_{k-1} - \log \log p_N + \frac{1}{2 \log^2 p_{k-1}} + \frac{1}{2 \log^2 p_N} \\ &\leq \log \log p_k - \log \log p_N + \frac{1}{\log^2 p_N} \\ &= \log \log p_k + c_1. \end{aligned}$$

It follows that

$$\exp \left(7 \sum_{j=N+1}^{k-1} \frac{1}{p_j} \right) \leq e^{7c_1} \log^7 p_k.$$

Since $x/(x-1)$ is decreasing for $x > 1$, we have, for $k > N$,

$$\frac{1}{(p_k - 1)^2} \leq \frac{p_{N+1}^2}{(p_{N+1} - 1)^2} \frac{1}{p_k^2} = \frac{C_2}{p_k^2}.$$

Summing over $N+1 \leq k \leq n$, we obtain

$$\sum_{k=N+1}^n \mathbb{P}_k(B_k) \leq e^{7c_1} C_2 M'_0 \sum_{k \geq N+1} \frac{\log^7 p_k}{p_k^2}.$$

Using a similar argument in the proof of [3, Lemma 4.7], one can show that

$$\sum_{k \geq N+1} \frac{\log^7 p_k}{p_k^2} \leq 2c_2 \frac{\phi(\log p_N)}{p_N}.$$

Combining the estimates above, we obtain

$$\sum_{k=N+1}^n \mathbb{P}_k(B_k) \leq 2e^{7c_1} c_2 C_2 M'_0 \frac{\phi(\log p_N)}{p_N},$$

as desired. $\square$

Proof of Theorem 1.3. With $N = 10000$ and the parameter $\delta_1, \dots, \delta_n$ chosen in the beginning, we can deduce from Lemma 4.2 that

$$\sum_{N+1 \leq i \leq n} \mathbb{P}_i(B_i) \leq 0.041809,$$

by running a computer program. Together with the upper bound

$$\sum_{1 \leq i \leq N} \mathbb{P}_i(B_i) \leq 0.957276.$$

We see that

$$\sum_{1 \leq i \leq n} \mathbb{P}_i(B_i) \leq 0.957276 + 0.041809 = 0.999085 < 1.$$

So we prove that $\mathcal{A}$ is not a covering system. This completes the proof. $\square$

2026 S.-T. Yau High School Science Award
仅用于2026丘成桐中学科学奖论文公示

REFERENCES

- [1] P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe, and M. Tiba. The erdős-selfridge problem with square-free moduli. *Algebra & Number Theory*, 15(3):609–626, 2021.
- [2] P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe, and M. Tiba. On the Erdős covering problem: the density of the uncovered set. *Invent. Math.*, 228(1):377–414, 2022.
- [3] M. Cummings, M. Filaseta, and O. Trifonov. An upper bound for the minimum modulus in a covering system with squarefree moduli. *Acta Math. Hungar.*, 175(1):1–25, 2025.
- [4] P. Erdős. On integers of the form $2^k + p$ and some related problems. *Summa Brasiliensis Mathematicae*, 2:113–123, 1950.
- [5] M. Filaseta, K. Ford, S. Konyagin, C. Pomerance, and G. Yu. Sieving by large integers and covering systems of congruences. *J. Amer. Math. Soc.*, 20(2):495–517, 2007.
- [6] B. Hough. Solution of the minimum modulus problem for covering systems. *Ann. Math.*, 181(1):361–382, 2015.
- [7] R. D. Hough and P. P. Nielsen. Covering systems with restricted divisibility. *Duke Math. J.*, 168(17), 2019.
- [8] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6(1):64–94, 1962.

Email address: `zihaopeng0124@163.com`

HEFEI THOMAS SCHOOL, HEFEI, 230026, PEOPLE’S REPUBLIC OF CHINA

Email address: `winterandrr@gmail.com`

HEFEI THOMAS SCHOOL, HEFEI, 230026, PEOPLE’S REPUBLIC OF CHINA

Email address: `joyce20251223@163.com`

HEFEI THOMAS SCHOOL, HEFEI, 230026, PEOPLE’S REPUBLIC OF CHINA

致谢页

一、研究背景与指导说明

本文研究模数全部为 $p-1$ 形式的覆盖系统的存在性问题。指导老师为中国科学技术大学数学科学学院赵立璐老师及其博士后杨子辰老师。赵立璐老师负责了选题指导，确定了本文的研究问题和基本研究思路。杨子辰老师负责具体指导三名学生的研究，包括：推荐数论基础教材和选题相关文献，定期答疑，以及润色论文。除两位老师提供的以上帮助和第四节所述的 AI 工具外，本文的其余工作完全由三名学生独立完成，不存在他人代做的研究成果。

二、研究过程与分工

赵立璐老师确定了选题和基本研究方向，提出分为“无平方因子情形”与“一般情形排除大素数”两条路线研究 Erdős - Graham 问题。本项目为纯理论研究，无“数据获取”、“分析数据”以及“实验设计与实施”环节。具体研究过程和撰写论文的分工如下：

张馨茹：证明定理 1.3，撰写论文第 4 节，证明了引理 4.2，无平方因子的尾界推广到一般模数，并核对 (4.1) 的上界；撰写摘要；共同完成参考文献与排版校对。

彭梓涵：撰写论文第 3 节；证明了 (4.1) 的上界，并检查了第 4 节的其他计算结果，与张馨茹的计算交叉验证；共同完成参考文献与排版校对。

万沐瑶：证明定理 1.1 和定理 1.2，撰写论文第 1 节与第 2 节；共同完成参考文献整理与全文 LaTeX 排版以及符号编号校对。

三、困难与解决

困难：主要困难发生在阅读 Balister 等人论文的过程中。在尝试理解 Distortion Method 中的概率测度的构造以及各种上下界估计的依赖时，遇到了较大困难。

解决：通过分工精读，互相讲解，整理笔记，并借助 AI 讲解概念，最后克服了理解障碍。

四、AI 使用情况

AI 工具名称及版本：DeepSeek V4 网页版

具体使用环节及用途：文献检索、辅助论文学习、准备答辩环节以及英文写作论色。

时间与频率：集中在初期文献阅读以及论文润色阶段，频率较低。